Global properties of warped solutions in general relativity with an electromagnetic field and a cosmological constant

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We consider general relativity with a cosmological constant minimally coupled to an electromagnetic field and assume that a four-dimensional space-time manifold is the warped product of two surfaces with Lorentzian and Euclidean signature metrics. Einstein's equations imply that at least one of the surfaces must be of constant curvature. It means that the symmetry of the metric arises as the consequence of the equations of motion ("spontaneous symmetry emergence"). We give a classification of global solutions in two cases: (i) both surfaces are of a constant curvature and (ii) the Riemannian surface is of a constant curvature. The latter case includes spherically symmetric solutions [a sphere $\mathbb{S}^2$ with a $\mathbb{SO}(3)$-symmetry group], planar solutions [two-dimensional Euclidean space $\mathbb{R}^2$ with an $\mathbb{IO}(2)$-symmetry group], and hyperbolic solutions [a two-sheeted hyperboloid $\mathbb{H}^2$ with a $\mathbb{SO}(1,2)$-symmetry]. Totally, we get 37 topologically different solutions. There is a new one among them, which describes the changing topology of space in time already at the classical level.

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I. INTRODUCTION

There are many well-known exact solutions in general relativity (see, e.g., [1]). To give a physical interpretation of any solution to Einstein's equation, we must know not only the metric satisfying equations of general relativity but the global structure of space-time. By this, we mean a pair $(\mathbb{M}, g)$, where $\mathbb{M}$ is the four-dimensional space-time manifold and g is the metric on $\mathbb{M}$ such that the manifold $\mathbb{M}$ is maximally extended along geodesics: any geodesic line on $\mathbb{M}$ either can be continued to an infinite value of the canonical parameter in both directions, or it ends up at a singular point, where one of the geometric invariants becomes infinite. The famous example is the Kruskal–Szekeres extension [2,3] of the Schwarzschild solution. In this case, the space-time $\mathbb{M}$ is globally the topological product of a sphere (spherical symmetry) with the two-dimensional Lorentzian surface depicted by the well-known Carter–Penrose diagram. The knowledge of this global structure of space-time allows one to introduce the notion of black and white holes.

The famous Reissner–Nordström solution [4,5], which is the spherically symmetric solution of Einstein's equations with an electromagnetic field, is also known globally. There are three types of Carter–Penrose diagrams: the Reissner-Nordström black hole, extremal black hole, and naked singularity. The type of the Carter–Penrose diagram depends on the relation between mass and charge parameters. The spherically symmetric exact solution of Einstein's equations with an electromagnetic field and a cosmological constant is known locally but not analyzed in full detail globally. In this paper, in particular, we give a complete classification of global spherically symmetric solutions of Einstein's equations with an electromagnetic field and a cosmological constant, which depends on relations between three parameters: the mass, charge, and cosmological constant. We show that there are 16 different Carter–Penrose diagrams in the spherically symmetric case.

In fact, a more general classification is given. We do not assume that solutions have any symmetry from the very beginning. Instead, we require the space-time to be the warped product of two surfaces: $\mathbb{M} = \mathbb{U} \times \mathbb{V}$, where $\mathbb{U}$ and $\mathbb{V}$ are two two-dimensional surfaces with Lorentzian and Euclidean signature metrics, respectively. As a consequence of the equations of motion, at least one of the surfaces must be of a constant curvature. In this paper, we consider the cases when (i) both surfaces $\mathbb{U}$ and $\mathbb{V}$ are of a constant curvature and when (ii) only the surface $\mathbb{V}$ is of a constant curvature. In the latter case, there are three possibilities: $\mathbb{V}$ is the sphere $\mathbb{S}^2$ [the spherical $\mathbb{SO}(3)$

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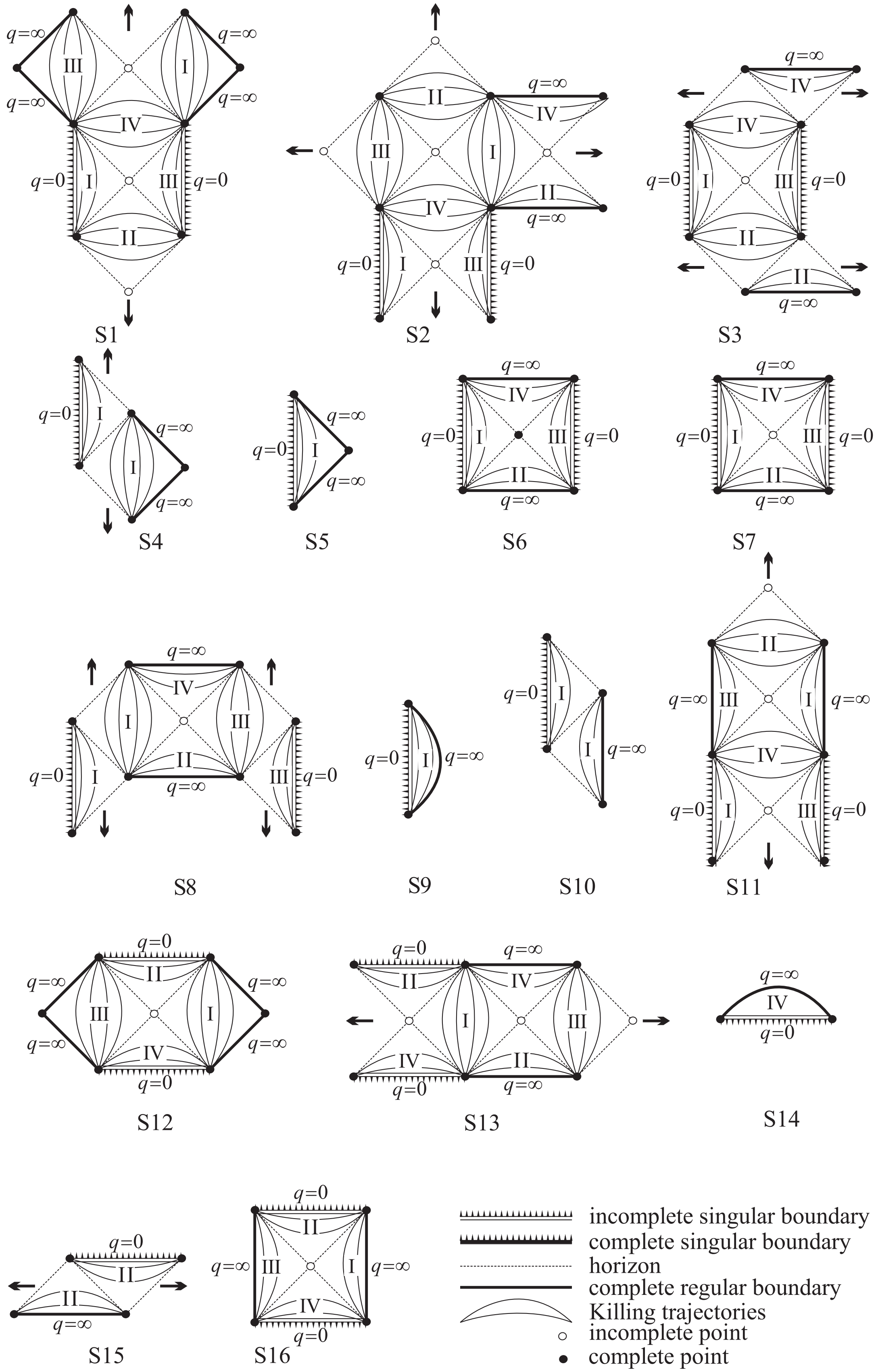

FIG. 1. The Carter–Penrose diagrams for spherically symmetric solutions of Einstein's equations with an electromagnetic field. Diagrams S1–S11 and S12–S16 correspond to metrics of a signature $(+---)$ and $(-+++)$, respectively.

symmetry], the Euclidean plane [the Poincare $\mathbb{ISO}(2)$ symmetry], and the two-sheeted hyperboloid $\mathbb{H}^2$ [the Lorentzian $\mathbb{SO}(1,2)$ symmetry]. We see that the symmetry of solutions is not assumed from the beginning but arise as the consequence of the equations of motions. This effect is called "spontaneous symmetry emergence". We classify all global solutions by drawing their Carter–Penrose diagrams for a surface $\mathbb{U}$ depending on the relations between the mass, charge, and cosmological constant. Totally, there are four different Carter–Penrose diagrams in case (i) and 33 globally different solutions in case (ii).

Moreover, we prove that there is the additional fourth Killing vector field in each case. This is a generalization of Birkhoff's theorem stating that any spherically symmetric solution of vacuum Einstein's equations must be static. The existence of an extra Killing vector field is proved for the $\mathbb{SO}(3)$, $\mathbb{ISO}(2)$, and $\mathbb{SO}(1,2)$ symmetry groups.

The global structure of space-times in general relativity and 2D gravity was analyzed, e.g., in [6–10]. In particular, the global planar and Lobachevsky plane solutions in general relativity were described in [11–13]. Though many solutions in the paper are known globally, some of them seem to be new. For example, the Carter–Penrose diagram S6 in Fig. 1 with a geodesically complete central point corresponding to the horizon of the third order is new to the best of our knowledge.

This paper follows the classification of global warped product solutions of general relativity with a cosmological constant (without electromagnetic field) given in [14]. The Carter–Penrose diagrams are constructed using the conformal block method described in [15].

As in [14], we assume that space-time $\mathbb{M}$ is the warped product of two surfaces: $\mathbb{M}=\mathbb{U}\times\mathbb{V}$, where $\mathbb{U}$ and $\mathbb{V}$ are surfaces with Lorentzian and Euclidean signature metrics, respectively. Local coordinates on $\mathbb{M}$ are denoted by $\hat{x}^i$, $i=0,1,2,3$, and coordinates on the surfaces by Greek letters from the beginning and middle of the alphabet,

$$(x^\alpha)\in\mathbb{U},\quad \alpha=0,1,\quad (y^\mu)\in\mathbb{V},\quad \mu=2,3.$$

That is $(\hat{x}^i):=(x^\alpha,y^\mu)$. Geometrical notions on four-dimensional space-time are marked by the hat to distinguish them from notions on the surfaces $\mathbb{U}$ and $\mathbb{V}$, which appear more often.

We do not assume any symmetry of solutions from the very beginning.

The four-dimensional metric of the warped product of two surfaces has a block diagonal form by definition,

$$\hat{g}_{ij}=\begin{pmatrix} k(y)g_{\alpha\beta}(x) & 0 \\ 0 & m(x)h_{\mu\nu}(y)\end{pmatrix}, \tag{1}$$

where $g_{\alpha\beta}(x)$ and $h_{\mu\nu}(y)$ are some metrics on the surfaces $\mathbb{U}$ and $\mathbb{V}$, respectively, $k(y)\neq 0$ and $m(x)\neq 0$ are scalar (dilaton) fields on $\mathbb{V}$ and $\mathbb{U}$. Without a loss of generality, the signatures of two-dimensional metrics $g_{\alpha\beta}$ and $h_{\mu\nu}$ are assumed to be $(+-)$ and $(++)$, respectively. In a rigourous sense, metric (1) is a doubly warped product. It reduces to a warped product in the usual sense for $k=\text{const}$ or $m=\text{const}$.

The Ricci tensor components for the metric (1) are

$$\begin{aligned}\hat{R}_{\alpha\beta}&=R_{\alpha\beta}+\frac{\nabla_\alpha\nabla_\beta m}{m}-\frac{\nabla_\alpha m\nabla_\beta m}{2m^2}+\frac{g_{\alpha\beta}\nabla^2 k}{2m}\\ \hat{R}_{\alpha\mu}&=\hat{R}_{\mu\alpha}=-\frac{\nabla_\alpha m\nabla_\mu k}{2mk}\\ \hat{R}_{\mu\nu}&=R_{\mu\nu}+\frac{\nabla_\mu\nabla_\nu k}{k}-\frac{\nabla_\mu k\nabla_\nu k}{2k^2}+\frac{h_{\mu\nu}\nabla^2 m}{2k},\end{aligned} \tag{2}$$

where, for brevity, we introduce the notation,

$$\nabla^2 m:=g^{\alpha\beta}\nabla_\alpha\nabla_\beta m,\qquad \nabla^2 k:=h^{\mu\nu}\nabla_\mu\nabla_\nu k. \tag{3}$$

Here and in what follows, the symbol ∇ denotes a covariant derivative with the corresponding Christoffel's symbols. The four-dimensional scalar curvature is

$$\hat{R}=\frac{1}{k}R^{(g)}+2\frac{\nabla^2 m}{km}-\frac{(\nabla m)^2}{2km^2}+\frac{1}{m}R^{(h)}+2\frac{\nabla^2 k}{km}-\frac{(\nabla k)^2}{2k^2 m}, \tag{4}$$

where

$$(\nabla m)^2:=g^{\alpha\beta}\partial_\alpha m\partial_\beta m,\qquad (\nabla k)^2:=h^{\mu\nu}\partial_\mu k\partial_\nu k. \tag{5}$$

Scalar curvatures of the surfaces $\mathbb{U}$ and $\mathbb{V}$ are denoted by $R^{(g)}$ and $R^{(h)}$, respectively.

II. A SOLUTION FOR AN ELECTROMAGNETIC FIELD

We assume that the electromagnetic field is minimally coupled to gravity. Then the action takes the form,

$$S=\int d^4x\sqrt{|\hat{g}|}\left(\hat{R}-2\Lambda-\frac{1}{4}\hat{F}^2\right), \tag{6}$$

where $\hat{R}$ is the scalar curvature for the metric $\hat{g}_{ij}$, $\hat{g}:=\det\hat{g}_{ij}$, Λ is a cosmological constant, and $\hat{F}^2$ is the square of the electromagnetic field strength,

$$\hat{F}^2:=\hat{F}_{ij}\hat{F}^{ij},\qquad \hat{F}_{ij}:=\partial_i\hat{A}_j-\partial_j\hat{A}_i.$$

Here, $\hat{A}_i$ are the components of the electromagnetic field potential. For brevity, gravitational and electromagnetic coupling constants are set to unity.

Variation of the action (6) with respect to the metric yields four-dimensional Einstein's equations,

$$\hat{R}_{ij} - \frac{1}{2}\hat{g}_{ij}\hat{R} + \hat{g}_{ij}\Lambda = -\frac{1}{2}\hat{T}_{\mathrm{EM}ij}, \tag{7}$$

where

$$\hat{T}_{\mathrm{EM}ij} := -\hat{F}_{ik}\hat{F}_j{}^k + \frac{1}{4}\hat{g}_{ij}\hat{F}^2 \tag{8}$$

is the electromagnetic field energy-momentum tensor. Variation of the action with respect to the electromagnetic field yields Maxwell's equations,

$$\partial_j(\sqrt{|\hat{g}|}\hat{F}^{ji}) = 0, \tag{9}$$

where

$$\hat{g} = k^2m^2gh, \qquad g := \det g_{\alpha\beta}, \qquad h := \det h_{\mu\nu}.$$

To simplify the problem, we assume that the four-dimensional electromagnetic potential consists of two parts,

$$\hat{A}_i = (A_\alpha(x), A_\mu(y)),$$

where $A_\alpha(x)$ and $A_\mu(y)$ are two-dimensional electromagnetic potentials on the surfaces $\mathbb{U}$ and $\mathbb{V}$, respectively. Then the electromagnetic field strength becomes a block diagonal,

$$\hat{F}_{ij} = \begin{pmatrix} F_{\alpha\beta} & 0 \\ 0 & F_{\mu\nu} \end{pmatrix}, \tag{10}$$

where

$$F_{\alpha\beta}(x) := \partial_\alpha A_\beta - \partial_\beta A_\alpha, \qquad F_{\mu\nu}(y) := \partial_\mu A_\nu - \partial_\nu A_\mu$$

are strength components for two-dimensional electromagnetic potentials.

In what follows, the raising of the greek indices from the beginning and middle of the greek alphabet is performed by using the inverse metrics $g^{\alpha\beta}$ and $h^{\mu\nu}$. Therefore,

$$\hat{F}^{\alpha\beta} = \frac{1}{k^2}F^{\alpha\beta}, \qquad \hat{F}^{\mu\nu} = \frac{1}{m^2}F^{\mu\nu},$$

where $k(y)$ and $m(x)$ are dilaton fields entering the four-dimensional metric (1). The square of the four-dimensional electromagnetic field strength is

$$\hat{F}^2 = \frac{1}{k^2}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{m^2}F_{\mu\nu}F^{\mu\nu}.$$

In the case under consideration, Maxwell's Eqs. (9) for $i = \alpha$ lead to the equality,

$$\frac{1}{|k|}\sqrt{|h|}\partial_\beta\left(|m|\sqrt{|g|}F^{\beta\alpha}\right) = 0.$$

A general solution to these equations has the form,

$$|m|\sqrt{|g|}F^{\alpha\beta} = 2\hat{\varepsilon}^{\alpha\beta}Q, \qquad Q = \mathrm{const}, \tag{11}$$

where $\hat{\varepsilon}^{\alpha\beta}$ is the totally antisymmetric second rank tensor density. The factor 2 is introduced in the right-hand side of the general solution for the simplification of subsequent formulas. This solution is rewritten as

$$F^{\alpha\beta} = \frac{2Q}{|m|}\varepsilon^{\alpha\beta}, \tag{12}$$

where $\varepsilon^{\alpha\beta} := \hat{\varepsilon}^{\alpha\beta}/\sqrt{|g|}$ is now the totally antisymmetric second rank tensor.

If $i = \mu$, then Maxwell's Eqs. (9) yield the equality,

$$\frac{1}{|m|}\sqrt{|g|}\partial_\mu(|k|\sqrt{h}F^{\mu\nu}) = 0.$$

Its general solution is

$$F^{\mu\nu} = \frac{2P}{|k|}\varepsilon^{\mu\nu}, \qquad P = \mathrm{const}. \tag{13}$$

Now the four-dimensional electromagnetic energy-momentum tensor (8) is easily calculated. It is a block diagonal,

$$\hat{T}_{ij} = \begin{pmatrix} \hat{T}_{\alpha\beta} & 0 \\ 0 & \hat{T}_{\mu\nu} \end{pmatrix}, \tag{14}$$

where

$$\hat{T}_{\alpha\beta} = \frac{2g_{\alpha\beta}}{km^2}(Q^2 + P^2), \qquad \hat{T}_{\mu\nu} = -\frac{2h_{\mu\nu}}{k^2m}(Q^2 + P^2).$$

Note that we do not need the electromagnetic potentials A_α and A_μ for the calculation of the energy-momentum tensor. It is sufficient to know strengthes (12) and (13).

Now we have to solve Einstein's Eqs. (7) with the right-hand side (14). Since energy-momentum tensor depends only on the sum $Q^2 + P^2$, we set $P = 0$ to simplify formulas. In the final answer, this constant is easily reconstructed by substitution $Q^2 \mapsto Q^2 + P^2$.

In what follows, we consider only the case $Q \neq 0$, because the case $Q = 0$ was considered in [14] in full detail.

III. EINSTEIN'S EQUATIONS

The right-hand side of Einstein's Eqs. (7) is defined by general solution of Maxwell's equations, which leads to the electromagnetic energy-momentum tensor (14). The trace of Einstein's equations can be easily solved with respect to the scalar curvature,

$$\hat{R} = 4\Lambda,$$

which does not depend on the electromagnetic field, because the trace of the electromagnetic field energy-momentum tensor equals zero. After elimination of the scalar curvature, Einstein's equations are simplified,

$$\hat{R}_{ij} - \hat{g}_{ij}\Lambda = -\frac{1}{2}\hat{T}_{\mathrm{EM}ij}. \tag{15}$$

For indices values $(ij) = (\alpha,\beta)$, $(\mu\nu)$, and (α,μ), these equations yield the following system of equations:

$$R_{\alpha\beta} + \frac{\nabla_\alpha\nabla_\beta m}{m} - \frac{\nabla_\alpha m\nabla_\beta m}{2m^2} + g_{\alpha\beta}\left(\frac{\nabla^2 k}{2m} - k\Lambda + \frac{Q^2}{m^2 k}\right) = 0, \tag{16}$$

$$R_{\mu\nu} + \frac{\nabla_\mu\nabla_\nu k}{k} - \frac{\nabla_\mu k\nabla_\nu k}{2k^2} + h_{\mu\nu}\left(\frac{\nabla^2 m}{2k} - m\Lambda - \frac{Q^2}{k^2 m}\right) = 0, \tag{17}$$

$$-\frac{\nabla_\alpha m\nabla_\mu k}{2mk} = 0, \tag{18}$$

where $R_{\alpha\beta}$ and $R_{\mu\nu}$ are Ricci tensors for the two-dimensional metrics $g_{\alpha\beta}$ and $h_{\mu\nu}$, respectively, ∇_α and ∇_μ are two-dimensional covariant derivatives with Christoffel's symbols on the surfaces $\mathbb{U}$ and $\mathbb{V}$, $\nabla^2 \coloneqq g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ or $\nabla^2 \coloneqq h^{\mu\nu}\nabla_\mu\nabla_\nu$, which is clear from the context. Sure, the equalities $\nabla_\alpha m = \partial_\alpha m$ and $\nabla_\mu k = \partial_\mu k$ hold, but we keep the symbol of the covariant derivative for uniformity.

For subsequent analysis of Einstein's equations, we extract the traces and traceless parts from Eqs. (16) and (17). Then, the full system of Einstein's equations takes the form,

$$\nabla_\alpha\nabla_\beta m - \frac{\nabla_\alpha m\nabla_\beta m}{2m} - \frac{1}{2}g_{\alpha\beta}\left(\nabla^2 m - \frac{(\nabla m)^2}{2m}\right) = 0, \tag{19}$$

$$\nabla_\mu\nabla_\nu k - \frac{\nabla_\mu k\nabla_\nu k}{2k} - \frac{1}{2}h_{\mu\nu}\left(\nabla^2 k - \frac{(\nabla k)^2}{2k}\right) = 0, \tag{20}$$

$$R^{(g)} + \frac{\nabla^2 m}{m} - \frac{(\nabla m)^2}{2m^2} + \frac{\nabla^2 k}{m} - 2k\Lambda + \frac{2Q^2}{m^2 k} = 0, \tag{21}$$

$$R^{(h)} + \frac{\nabla^2 k}{k} - \frac{(\nabla k)^2}{2k^2} + \frac{\nabla^2 m}{k} - 2m\Lambda - \frac{2Q^2}{k^2 m} = 0, \tag{22}$$

$$\nabla_\alpha m\nabla_\beta k = 0, \tag{23}$$

where $(\nabla m)^2 \coloneqq g^{\alpha\beta}\nabla_\alpha m\nabla_\beta m$, $(\nabla k)^2 \coloneqq g^{\mu\nu}\nabla_\mu k\nabla_\nu k$, $R^{(g)}$ and $R^{(h)}$ are scalar curvatures of the two-dimensional surfaces $\mathbb{U}$ and $\mathbb{V}$ for the metrics g and h, respectively. In the above formulas, we used equalities $R_{\alpha\beta} = \frac{1}{2}g_{\alpha\beta}R^{(g)}$ and $R_{\mu\nu} = \frac{1}{2}h_{\mu\nu}R^{(h)}$ valid in two dimensions.

The last Eq. (23), which corresponds to mixed values of the indices $(ij) = (\alpha\mu)$ in Einstein's equations, results in strong restrictions on solutions. Namely, as in the case without an electromagnetic field, there are only three cases,

$$\begin{array}{lll} \mathsf{A}: & k = \mathsf{const} \neq 0, & m = \mathsf{const} \neq 0, \\ \mathsf{B}: & k = \mathsf{const} \neq 0, & \nabla_\alpha m \neq 0, \\ \mathsf{C}: & \nabla_\mu k \neq 0, & m = \mathsf{const} \neq 0. \end{array} \tag{24}$$

We shall see in what follows, that this leads to "spontaneous symmetry emergence."

Now, we consider the first two cases in detail.

IV. THE PRODUCT OF CONSTANT CURVATURE SURFACES

The most symmetric solutions of Einstein's equations with an electromagnetic field in the form of the product of two constant curvature surfaces arise in case A (24), when both dilaton fields are constant. If k and m are constant, then Eqs. (19) and (20) are identically satisfied, and Eqs. (21) and (22) take the form,

$$\begin{aligned} R^{(g)} &= 2k\Lambda - \frac{2Q^2}{m^2 k} = -2K^{(g)}, \\ R^{(h)} &= 2m\Lambda + \frac{2Q^2}{k^2 m} = -2K^{(h)}, \end{aligned} \tag{25}$$

where

$$K^{(g)} \coloneqq -k\left(\Lambda - \frac{Q^2}{k^2 m^2}\right), \qquad K^{(h)} \coloneqq -m\left(\Lambda + \frac{Q^2}{k^2 m^2}\right)$$

are Gaussian curvatures of the surfaces $\mathbb{U}$ and $\mathbb{V}$, respectively. It means that both surfaces are of constant curvature in case A. The metric on each surface is invariant under a three-dimensional transformation group.

In stereographic coordinates on both surfaces, the metric of a four-dimensional space-time takes the form,

$$\begin{aligned} ds^2 &= k g_{\alpha\beta}dx^\alpha dx^\beta + m h_{\mu\nu}dy^\mu dy^\nu \\ &= k\frac{dt^2 - dx^2}{[1 + \frac{K^{(g)}}{4}(t^2 - x^2)]^2} + m\frac{dy^2 + dz^2}{[1 + \frac{K^{(h)}}{4}(y^2 + z^2)]^2}, \end{aligned} \tag{26}$$

where $(x^\alpha) \coloneqq (t, x)$ and $(y^\mu) \coloneqq (y, z)$.

We can put $k = \pm 1$ and $m = \pm 1$ by rescaling coordinates. One has also to redefine the constant of integration

$Q^2/(k^2m^2) \mapsto Q^2$. We choose $k = 1$ and $m = -1$ for the metric signature to be $(+---)$. Then, the Gaussian curvatures are

$$K^{(g)} = Q^2 - \Lambda, \qquad K^{(h)} = Q^2 + \Lambda. \tag{27}$$

There are four qualitatively different cases for topologically inequivalent global solutions depending on the relations between a cosmological constant and charge,

$$\begin{aligned}
\Lambda < -Q^2 &: \quad K^{(g)} > 0, \quad K^{(h)} < 0, \quad \mathbb{M} = \mathbb{L}^2 \times \mathbb{H}^2, \\
\Lambda = -Q^2 &: \quad K^{(g)} > 0, \quad K^{(h)} = 0, \quad \mathbb{M} = \mathbb{L}^2 \times \mathbb{R}^2, \\
-Q^2 < \Lambda < Q^2 &: \quad K^{(g)} > 0, \quad K^{(h)} > 0, \quad \mathbb{M} = \mathbb{L}^2 \times \mathbb{S}^2, \\
\Lambda = Q^2 &: \quad K^{(g)} = 0, \quad K^{(h)} > 0, \quad \mathbb{M} = \mathbb{R}^{1,1} \times \mathbb{S}^2, \\
\Lambda > Q^2 &: \quad K^{(g)} < 0, \quad K^{(h)} > 0, \quad \mathbb{M} = \mathbb{L}^2 \times \mathbb{S}^2,
\end{aligned} \tag{28}$$

where $\mathbb{L}^2$ is the one sheet hyperboloid (more precisely, its universal covering) embedded in the three-dimensional Minkowskian space $\mathbb{R}^{1,2}$, $\mathbb{H}^2$ is the Lobachevsky plane (the upper sheet of the two-sheeted hyperboloid embedded in $\mathbb{R}^{1,2}$), and $\mathbb{S}^2$ is the two-dimensional sphere. From a topological point of view, the third and fifth cases in Eq. (28) coincide. Therefore, there are only four topologically inequivalent global solutions of Einstein's equations in the form of a direct product of two constant curvature surfaces. Note that for $Q = 0$, there are only three topologically inequivalent solutions [14].

All solutions have exactly six Killing vector fields and belong to the type D in Petrov's classification.

The cases of other signatures of a four-dimensional metric for $k = \pm 1$ and $m = \pm 1$ are analyzed similarly. Qualitative properties of global solutions are the same.

We see that symmetry properties in this case are not imposed from the very beginning but arise as the result of a solution of equations of motion. This effect is called "spontaneous symmetry emergence."

V. SOLUTIONS WITH SPATIAL SYMMETRY

The dilaton field k is constant in the second case B (24). Without the loss of generality, we put $k = 1$. Then Einstein's equations (19)–(23) take the form,

$$\nabla_\alpha \nabla_\beta m - \frac{\nabla_\alpha m \nabla_\beta m}{2m} - \frac{1}{2} g_{\alpha\beta} \left[\nabla^2 m - \frac{(\nabla m)^2}{2m} \right] = 0, \tag{29}$$

$$R^{(h)} + \nabla^2 m - 2m\Lambda - \frac{2Q^2}{m} = 0, \tag{30}$$

$$R^{(g)} + \frac{\nabla^2 m}{m} - \frac{(\nabla m)^2}{2m^2} - 2\Lambda + \frac{2Q^2}{m^2} = 0. \tag{31}$$

Consider Eq. (30). The scalar curvature $R^{(h)}$ depends on the coordinates y on the surface $\mathbb{V}$, whereas all other terms depend on coordinates x on the surface $\mathbb{U}$. For this equation to be fulfilled, it is necessary that the equation $R^{(h)} = \mathsf{const}$ holds. It means that the surface $\mathbb{V}$ must be of a constant curvature as a consequence of Einstein equations. Therefore, the four-dimensional metric of space-time has at least three independent Killing vector fields. So, there is spontaneous symmetry emergence.

Let us put $R^{(h)} \coloneqq -2K^{(h)} = \mathsf{const}$. Then Eq. (30) is

$$\nabla^2 m - 2m\Lambda - 2K^{(h)} - \frac{2Q^2}{m} = 0. \tag{32}$$

Excluding the case A considered in the previous section, we proceed further assuming $\nabla_\alpha m \neq 0$ on the whole $\mathbb{U}$.

Proposition 5.1. Equation (32) is the first integral of Eqs. (29) and (31).

Proof. Differentiate Eq. (32) and use the equality,

$$[\nabla_\alpha, \nabla_\beta] A_\gamma = -R^{(g)}_{\alpha\beta\gamma}{}^{\delta} A_\delta,$$

valid for any covector field A_α, to change the order of the derivatives in the first term,

$$\begin{aligned}
\nabla_\alpha(32) = {} & \frac{\nabla^\beta \nabla_\alpha m \nabla_\beta m}{2m} + \frac{\nabla_\alpha m \nabla^2 m}{2m} - \frac{\nabla_\alpha m (\nabla m)^2}{2m^2} \\
& + \frac{1}{2}\nabla_\alpha \left(\nabla^2 m - \frac{(\nabla m)^2}{2m} \right) + \frac{1}{2}\nabla_\alpha m R^{(g)} - 2\nabla_\alpha m \Lambda \\
& + \nabla_\alpha m \frac{2Q^2}{m^2}.
\end{aligned}$$

Now exclude the derivatives $\nabla^\beta \nabla_\alpha m$ and $\nabla^2 m$ using Eqs. (29) and (30) in the first and fourth terms on the right-hand side. After rearranging the terms, the sum of the first and fourth terms takes the form,

$$\nabla_\alpha m \left(\frac{(\nabla m)^2}{4m^2} + \Lambda - \frac{Q^2}{m^2} \right).$$

Taking all the terms together, we obtain

$$\nabla_\alpha(32) = \frac{1}{2} \nabla_\alpha m (31). \tag{33}$$

Since $\nabla_\alpha m \neq 0$, it implies the statement of the proposition. ■

The proof of the proposition implies that it is sufficient to solve Eqs. (29) and (32), Eq. (31) being satisfied automatically.

To solve Eqs. (29) and (32) explicitly, we fix the conformal gauge for a metric $g_{\alpha\beta}$ on a Lorentzian surface $\mathbb{U}$,

$$g_{\alpha\beta} dx^\alpha dx^\beta = \Phi d\xi d\eta, \tag{34}$$

where $\Phi(\xi, \eta) \neq 0$ is the conformal factor depending on light cone coordinates $\xi \coloneqq \tau + \sigma$, $\eta \coloneqq \tau - \sigma$ on $\mathbb{U}$. The respective four-dimensional metric is

$$ds^2 = \Phi d\xi d\eta + m d\Omega^2, \tag{35}$$

where $d\Omega^2$ is the metric on the Riemannian surface of a constant curvature $\mathbb{V} = \mathbb{S}^2$, $\mathbb{R}^2$, or $\mathbb{H}^2$. The sign of the conformal factor Φ is not fixed for the present.

For $\Phi > 0$ and $m < 0$, the signature of the metric (35) is $(+---)$. If we change the sign of m, the signature of the metric becomes $(+-++)$. The same transformation of the signature can be achieved by changing the overall sign of the metric, $\hat{g}_{ij} \mapsto -\hat{g}_{ij}$, and interchanging the first two coordinates, $\tau \leftrightarrow \sigma$. Einstein's equations with a cosmological constant and an electromagnetic field (15) are not invariant with respect to these transformations with simultaneous changing of the sign of the cosmological constant, because the right-hand side changes its sign. Therefore, for $\Phi > 0$, we have to consider two cases,

$$m < 0 \Leftrightarrow \mathsf{sign}\hat{g}_{ij} = (+---) \quad \text{and}$$
$$m > 0 \Leftrightarrow \mathsf{sign}\hat{g}_{ij} = (-+++).$$

This is the difference for Einstein's equations without an electromagnetic field considered in [14]. If the change of the signature $(+---) \mapsto (-+++)$ is followed by the change of signs in front of $\hat{R}$, Λ, and $\hat{F}^2$ in the action (6), then both choices become equivalent. In our presentation, it is more natural to fix the sign of $\hat{F}^2$ in the action (6) and consider different signatures, because signs of dilaton fields are not fixed.

1. Metric signature $(+---)$

For $\Phi > 0$ and $m < 0$, we introduce convenient parametrization,

$$m \coloneqq -q^2, \qquad q(\xi,\eta) > 0. \tag{36}$$

Afterwards, we obtain the full system of equations,

$$-\partial^2_{\xi\xi} q + \frac{\partial_\xi \Phi \partial_\xi q}{\Phi} = 0, \tag{37}$$

$$-\partial^2_{\eta\eta} q + \frac{\partial_\eta \Phi \partial_\eta q}{\Phi} = 0, \tag{38}$$

$$-2\frac{\partial^2_{\xi\eta} q^2}{\Phi} - K^{(h)} + \Lambda q^2 + \frac{Q^2}{q^2} = 0. \tag{39}$$

The first two equations which do not depend on the electromagnetic field imply the following assertion.

Proposition 5.2. If $\partial_\xi q \partial_\eta q > 0$, then the function $q(\tau)$ depends only on the timelike coordinate $\tau \coloneqq \frac{1}{2}(\xi + \eta)$. If $\partial_\xi q \partial_\eta q < 0$, then the function $q(\sigma)$ depends only on the spacelike coordinate $\sigma \coloneqq \frac{1}{2}(\xi - \eta)$. And the following equality holds:

$$|\Phi| = |q'|, \tag{40}$$

where prime denotes differentiation on the argument (either τ or σ).

This proposition provides a general solution to Eqs. (37) and (38) up to conformal transformations. This statement is proved in [14,16].

Thus, we can always choose coordinates in such a way that q and Φ depend simultaneously on a timelike or spacelike coordinate,

$$\zeta \coloneqq \frac{1}{2}(\xi \pm \eta) \eqqcolon \begin{cases} \tau, & \partial_\xi q \partial_\eta q > 0, \\ \sigma, & \partial_\xi q \partial_\eta q < 0. \end{cases} \tag{41}$$

It means that the two-dimensional metric (34) and consequently, the four-dimensional metric (35) have the Killing vector ∂_σ or ∂_τ, as the consequence of Eqs. (37) and (38). We call these solutions homogeneous and static, respectively, though it is related to the fixed coordinate system. The existence of an additional Killing vector is the generalization of Birkhoff's theorem [17], stating that an arbitrary spherically symmetric solution of vacuum Einstein's equations must be static. (This statement was previously published in [18].) The generalization includes the addition of an electromagnetic field, and in addition, the existence of an extra Killing vector is proved not only for the spherically symmetric solution ($K^{(h)} = 1$), but also for solutions invariant with respect to $\mathbb{ISO}(2)$ ($K^{(h)} = 0$) and $\mathbb{SO}(1,2)$ ($K^{(h)} = -1$) transformation groups.

We are left to solve Eq. (39). In static, $q = q(\sigma)$, and homogeneous, $q = q(\tau)$, cases, Eq. (39) takes the form,

$$(q^2)'' = 2\left(K^{(h)} - \Lambda q^2 - \frac{Q^2}{q^2}\right)\Phi, \qquad q = q(\sigma), \tag{42}$$

$$(q^2)'' = -2\left(K^{(h)} - \Lambda q^2 - \frac{Q^2}{q^2}\right)\Phi, \qquad q = q(\tau). \tag{43}$$

To integrate the derived equations, one has to express Φ through q using Eq. (40) and removing the modulus signs.

We consider the static case $q = q(\sigma)$, $\Phi > 0$ and $q' > 0$ in detail. Then Eq. (42) together with Eq. (40) reduces to

$$(q^2)'' = 2\left(K^{(h)} - \Lambda q^2 - \frac{Q^2}{q^2}\right)q'.$$

It can be easily integrated

$$(q^2)' = 2\left(K^{(h)} q - \frac{\Lambda q^3}{3} - 2M + \frac{Q^2}{q}\right),$$

where $M = \mathsf{const}$ is an integration constant, which coincides with the mass in the Schwarzschild solution. Differentiating the left-hand side and dividing it by $2q > 0$, we obtain the equation,

$$q' = K^{(h)} - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3}.$$

Since $q' = \Phi$ in the case under consideration, it implies the expression for the conformal factor through the variable q,

$$\Phi(q) = K^{(h)} - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3}. \qquad (44)$$

If $q = q(\sigma)$, $\Phi > 0$ and $q' < 0$, then the similar integration yields

$$q' = -\Phi(q),$$

where the same conformal factor (44) stands in the right-hand side. This case can be united with the previous one by rewriting the equation for q in the form,

$$|q'| = \Phi(q), \qquad q = q(\sigma), \qquad \Phi > 0. \qquad (45)$$

The modulus sign in the left-hand side means that if $q(\sigma)$ is a solution, then the function $q(-\sigma)$ is also a solution.

The static case for $\Phi < 0$ is integrated in the same way,

$$|q'| = -\Phi(q), \qquad q = q(\sigma), \qquad \Phi < 0. \qquad (46)$$

If the solution is homogeneous, $q = q(\tau)$ and $\Phi > 0$, $q' > 0$, then the integration of Eq. (43) yields the equality,

$$q' = -\left(K^{(h)} - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3}\right).$$

That is, the conformal factor must be identified with the right-hand side,

$$\hat{\Phi} = -\left(K^{(h)} - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3}\right). \qquad (47)$$

We denote the expression for the conformal factor through q by hat because in the homogeneous case, it differs by the sign. Thus, homogeneous solutions of Einstein's equations can be written in the form,

$$|q'| = \hat{\Phi}(q), \qquad q = q(\tau), \qquad \hat{\Phi} > 0. \qquad (48)$$

$$|q'| = -\hat{\Phi}(q), \qquad q = q(\tau), \qquad \hat{\Phi} < 0. \qquad (49)$$

If the conformal factor is negative, then the signature of the metric is $(- + --)$. In this case, we return to the previous signature $(+ - --)$ after substitution $\tau \leftrightarrow \sigma$. This transformation allows us to unite static and homogeneous solutions by taking the modulus of the conformal factor in the expression for the metric (35). Then, a general solution of vacuum Einstein's equations with an electromagnetic field (7) in the corresponding coordinate system takes the form,

$$ds^2 = |\Phi|(d\tau^2 - d\sigma^2) - q^2 d\Omega^2, \qquad (50)$$

where the conformal factor Φ is given by Eq. (44). Here, the variable q depends on σ (a static local solution) or τ (a homogeneous local solution) through the differential equation,

$$\left|\frac{dq}{d\zeta}\right| = \pm\Phi(q), \qquad (51)$$

where the sign rule holds

$$\begin{aligned} &\Phi > 0: \quad \zeta = \sigma, \quad \text{the sign} + (\text{static local solution}), \\ &\Phi < 0: \quad \zeta = \tau, \quad \text{the sign} - (\text{homogeneous local solution}). \end{aligned} \qquad (52)$$

Thus, the four-dimensional Einstein's equations imply that there is a metric with one Killing vector field on the surface $\mathbb{U}$ which was considered in full detail in [15]. Now we can construct global solutions (maximally extended along geodesics) of vacuum Einstein's equations using the conformal block method. The number of singularities and zeroes of the conformal factor (44) depends on relations between constants K, M, Q, and Λ. Therefore, there are many qualitatively different global solutions, which are considered in next sections.

The conformal factor (44) has one singularity: the second order pole at $q = 0$. Therefore, according to the rules formulated in [15,16], every global solution corresponds to one of the intervals $(-\infty, 0)$ or $(0, \infty)$. The form of the conformal factor (44) implies that these global solutions are obtained one from the other by the transformation $M \mapsto -M$. Hence, without a loss of generality, we describe global solutions corresponding to both intervals but positive values of M.

Because the conformal factor $\Phi(q)$ is a smooth function for $q \neq 0$, all arising Lorentzian surfaces $\mathbb{U}$ and metrics on them are smooth.

To conclude the section, we compute geometrical invariants which show that the obtained solution of Einstein's equations are nontrivial. First, we compute the scalar curvature $R^{(g)}$ of the surface $\mathbb{U}$. Equations (30) and (31) imply

$$R^{(g)} = -\frac{2K^{(h)}}{m} + \frac{(\nabla m)^2}{2m^2} - \frac{4Q^2}{m^2} = \frac{2K^{(h)}}{q^2} + \frac{2(\nabla q)^2}{q^2} - \frac{4Q^2}{q^4}.$$

Since

$$(\nabla q)^2 = \frac{1}{\Phi}\eta^{\alpha\beta}\partial_\alpha q \partial_\beta q = -\frac{q'^2}{\Phi}$$

both for static and homogeneous solutions, the final expression is

$$R^{(g)} = \frac{2\Lambda}{3} + \frac{4M}{q^3} - \frac{6Q^2}{q^4}. \quad (53)$$

It does not depend on the Gaussian curvature $K^{(h)}$ of a Riemannian surface $\mathbb{V}$ and is singular for $q = 0$ if $M \neq 0$ and/or $Q \neq 0$.

2. Metric signature $(-+++)$

If $m > 0$, then the signature of the metric is opposite $(-+++)$, and we introduce the parametrization,

$$m := q^2, \qquad q > 0,$$

instead of Eq. (36). Performing the same calculation as in the previous section, we obtain the first order equation for q,

$$\left|\frac{dq}{d\zeta}\right| = \pm\Phi(q), \quad (54)$$

where M is an integration constant and

$$\Phi(q) := \left(K^{(h)} - \frac{2M}{q} - \frac{Q^2}{q^2} + \frac{\Lambda q^2}{3}\right). \quad (55)$$

Here, we must take into account that for getting the signature $(-+++)$, we have to make the interchange $\tau \leftrightarrow \sigma$. We see that for drawing the Carter–Penrose diagram one has to simply make the replacement $Q^2 \mapsto -Q^2$ and $\Lambda \mapsto -\Lambda$ as compared to the signature $(+---)$.

Now, we describe all the spatially symmetric global solutions of Einstein's equations with an electromagnetic field which are defined by zeroes and their types of the conformal factor $\Phi(q)$.

B. Spherically symmetric solutions $K^{(h)} = 1$

In the considered case, global spherically symmetric solutions, that is pairs $(\mathbb{M}, \hat{g})$, have the form $\mathbb{M} = \mathbb{U} \times \mathbb{S}^2$, where $\mathbb{U}$ is the maximally extended Lorentzian surface, which is depicted by the Carter–Penrose diagram. The four-dimensional metric on $\mathbb{M}$ has the form (50), where $d\Omega^2$ is the metric on the sphere $\mathbb{S}^2$ for the signature $(+---)$. If the signature is opposite $(-+++)$, then we have to replace $Q^2 \mapsto -Q^2$ and $\Lambda \to -\Lambda$ in the conformal factor and change the sign of $d\Omega^2$ in metric (50). Due to the existence of one Killing vector on a Lorentzian surface $\mathbb{U}$, we are able to classify all global solutions. To construct Carter–Penrose diagrams, we use the conformal block method described in [15] (see also, [16]). First, we consider solutions of the signature $(+---)$, and then with the signature $(-+++)$.

1. Metric signature $(+---)$

If the metric signature is $(+---)$, then the conformal factor is

$$\Phi(q) = 1 - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3} =: \frac{\varphi(q) + 3Q^2}{3q^2}, \quad (56)$$

where we introduced the auxiliary function,

$$\varphi(q) := -\Lambda q^4 + 3q^2 - 6Mq, \quad (57)$$

which is needed for further analysis. The case $Q = 0$ was analyzed in [14]. Therefore, we classify solutions for $Q \neq 0$. Without a loss of generality, we consider the case $Q > 0$, because only Q^2 enters the conformal factor.

A conformal factor (56) has the second order pole Q^2/q^2 at zero and the following asymptotic at infinity:

$$\Phi \approx 1 - \frac{\Lambda q^2}{3}, \qquad q \to \infty.$$

If the cosmological constant is equal to zero, then the metric is asymptotically flat. For $\Lambda > 0$ and $\Lambda < 0$, we have asymptotically de Sitter and anti–de Sitter spacetime, respectively.

A global solution corresponds to one of the intervals $q \in (0, \infty)$ or $q \in (-\infty, 0)$ and $M > 0$, because the curvature has a singularity (53) at zero, and space-time is not extendable through this point. Roots of the conformal factor (56) correspond to horizons of space-time, and Carter–Penrose diagrams are defined by the number and type of zeroes of the conformal factor [15]. Thus, we have to analyze the number and type of zeroes of the conformal factor (56) for all possible values of constants Λ, $M \geq 0$, and $Q > 0$.

Note that the conformal factor (56) is invariant with respect to the transformation,

$$M \to -M, \qquad q \to -q.$$

Therefore, instead of constructing global solutions on the interval $q \in (0, \infty)$ for all values of M, we restrict ourselves not only for non-negative $M \geq 0$, but on two intervals $q \in (-\infty, 0)$ and $(0, \infty)$. This simplifies the analysis of the conformal factor.

We start with the simplest and well-known case $\Lambda = 0$.

2. Metric signature $(+---)$. The case $\Lambda = 0$

If the cosmological constant vanishes, then zeroes of conformal factor (56) are defined by the quadratic equation,

$$q^2 - 2Mq + Q^2 = 0, \quad (58)$$

which has two roots,

$$q_\pm = M \pm \sqrt{M^2 - Q^2}. \quad (59)$$

The Reissner–Nordström solution.—For $Q < M$, there are two positive simple roots. This solution is called the Reissner–Nordström solution [4,5] and depicted by the Carter–Penrose diagram S1 shown in Fig. 1. It was also found by H. Weyl [19]. The solution has two horizons at q_- and q_+ and a naked timelike singularity at $q = 0$. The conformal factor tends to unity at infinity, and, consequently, the Reissner–Nordström solution is asymptotically flat. Arrows on the diagram show directions in which the solution can be periodically extended in time. Instead of periodic extension, there is the possibility to identify the opposite horizons. The singularity at $q = 0$ is timelike, and an observer can approach it as close as he likes in conformal blocks I or III, and then enter universe III or I by going through conformal block IV. Therefore, the Reissner–Nordström solution does not describe a black hole.

Extremal black hole.—For $Q = M$, the conformal factor is

$$\Phi = \frac{(q - M)^2}{q^2}.$$

It has one positive root of second order at $q = M$. The corresponding Carter–Penrose diagram is shown in Fig. 1, S4. It is called an extremal black hole, though there is no black hole since the singularity is timelike and the horizon surrounding the singularity is absent. There is also a space-reflected diagram.

Naked singularity.—For $Q > M$, horizons are absent, and we have a naked singularity shown in Fig. 1, S5. There is also a space-reflected diagram.

3. Metric signature $(+---)$. The case $\Lambda > 0$

For positive cosmological constant, zeroes of the conformal factor are defined by the fourth order equation,

$$\varphi(q) + 3Q^2 = 0, \quad (60)$$

where a function $\varphi(q)$ is given by the fourth order polynomial (57). To draw Carter–Penrose diagrams, we do not need to know exact position of zeroes. We have to know only their existence and type. Therefore, we analyze the function $\varphi(q)$ qualitatively and then move its graphic up, which corresponds to increasing the value of Q^2.

First, we differentiate the function (60),

$$\varphi'(q) = -4\Lambda q^3 + 6q - 6M,$$
$$\varphi''(q) = -12\Lambda q^2 + 6 = -6(2\Lambda q^2 - 1). \quad (61)$$

The asymptotics of the function $\varphi(q)$ $(\Lambda > 0)$ and its derivatives for $q = 0$ and $q \to \infty$ are easily found

$$\varphi(0) = 0, \qquad \varphi(q \to \infty) \approx -\Lambda q^4,$$
$$\varphi'(0) = -6M, \qquad \varphi'(q \to \infty) \approx -4\Lambda q^3, \quad (62)$$
$$\varphi''(0) = 6, \qquad \varphi''(q \to \infty) \approx -12\Lambda q^2.$$

Zeroes of function $\varphi(q) + 3Q^2$ require more work. As we see later, their number does not exceed 3. To find the types of zeroes, we have to know the local extrema of the function $\varphi(q)$, which become zeroes of order two or three after shifting on $3Q^2$.

Local extrema of the function φ are defined by a cubic equation (the solution is given, e.g., in [20]),

$$q^3 - \frac{3}{2\Lambda} q + \frac{3M}{2\Lambda} = 0. \quad (63)$$

There are three qualitatively distinct cases depending on the value of constant,

$$\Upsilon := -\frac{1}{8\Lambda^3} + \frac{9M^2}{16\Lambda^2}. \quad (64)$$

Namely,

$$\Upsilon > 0 \quad \Leftrightarrow |M| > \tfrac{1}{3}\sqrt{\tfrac{2}{\Lambda}} \quad - \quad \text{one real and two complex conjugate roots},$$
$$\Upsilon = 0 \quad \Leftrightarrow |M| = \tfrac{1}{3}\sqrt{\tfrac{2}{\Lambda}} \quad - \quad \text{three real roots (at least two roots coincide)},$$
$$\Upsilon < 0 \quad \Leftrightarrow |M| < \tfrac{1}{3}\sqrt{\tfrac{2}{\Lambda}} \quad - \quad \text{three different real roots.}$$

We start with the simplest case, $\Upsilon = 0$. This equality implies a restriction on “mass”,

$$\Upsilon = 0 \Leftrightarrow M = \frac{1}{3}\sqrt{\frac{2}{\Lambda}}. \quad (65)$$

Moreover, roots of Eq. (63) take the simple form,

$$M = \frac{1}{3}\sqrt{\frac{2}{\Lambda}}: \quad q_1 = -\sqrt{\frac{2}{\Lambda}}, \quad q_{2,3} = \frac{1}{2}\sqrt{\frac{2}{\Lambda}}, \quad (66)$$

As we see, there are one simple negative root and one positive root of second order for a positive “mass” (65).

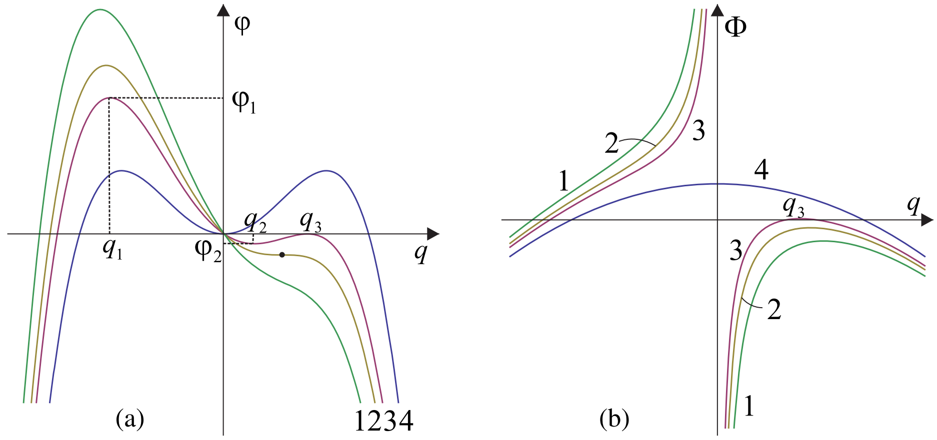

FIG. 2. Auxiliary function $\varphi(q)$ for $\Lambda > 0$ (a) and conformal factor $\Phi(q)$ for $Q = 0$ (b). The curves correspond to the following values of the constant: (1) $\Upsilon > 0$, (2) $\Upsilon = 0$, (3) $\Upsilon < 0$, and (4) $\Upsilon = -\frac{1}{8\Lambda^3}$. Local extrema for curve 3 on the left picture are located at points q_1, q_2, and q_3. For curve 2, local maximum and minimum coincide, that is $q_2 = q_3$, and are denoted by the fat point. For curve 1, there is only one maximum for negative q. Curve 4 on the left is symmetric with respect to substitution $q \mapsto -q$, has local minimum at $q = 0$, and two maxima at points $q = \pm\sqrt{\frac{3}{2\Lambda}}$.

If the inequality $\Upsilon < 0$ holds, then the real roots of the cubic equation (63) are (see, e.g., [20]),

$$q_3 = \sqrt{\frac{2}{\Lambda}}\cos\frac{\alpha}{3}, \qquad q_{2,1} = -\sqrt{\frac{2}{\Lambda}}\cos\left(\frac{\alpha}{3} \pm \frac{\pi}{3}\right), \tag{67}$$

where

$$\cos\alpha \coloneqq -3M\sqrt{\frac{\Lambda}{2}}.$$

Since we consider only nonnegative M, then $\alpha \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. It implies the existence of one negative root q_1 and two positive: q_2 and q_3. We enumerate the zeroes in Eq. (67) in such a way, that, in the limit,

$$M \to \frac{1}{3}\sqrt{\frac{2}{\Lambda}},$$

they take values (66).

If $\Upsilon > 0$, then we have only one negative root q_1. Its exact position can be written, but it is not needed.

Figure 2(a) shows qualitative behavior of the function $\varphi(q)$ for $\Lambda > 0$ and different values of $M \geq 0$. Now, to construct all global solutions which exist in the theory for a signature $(+---)$, we have to analyze the zeroes of the conformal factor $\Phi(q)$; qualitative behavior of which for $Q = 0$ is shown in Fig. 2(b). Zeroes of the conformal factor and their type coincide with that of function $\varphi(q) + 3Q^2$. Therefore, we have to shift up curves 1–4 in Fig. 2(a) on $3Q^2$ to analyze its qualitative behavior. The number and type of zeroes depend on curves 1–4 and on the value of the shift $3Q^2$. All possible Carter–Penrose diagrams are drawn in Fig. 1.

The conformal factor depicted by curve 4 in Fig. 2(b) does not have a zero at $q = 0$. It corresponds to de Sitter space and is degenerate at this presentation of the problem ($M = 0$, $Q = 0$), which is not considered here because of the assumption $Q > 0$.

For a qualitative description of the behavior of the conformal factor, we introduce the notation,

$$\varphi_1 \coloneqq \varphi(q_1), \qquad \varphi_2 \coloneqq \varphi(q_2), \qquad \varphi_3 \coloneqq \varphi(q_3), \tag{68}$$

where φ_1 is the maximum, φ_2 is local minimum, and φ_3 is local maximum of the auxiliary function $\varphi(q)$. One can easily verify, that, for $\Lambda > 0$ and $q < 0$, the maximum is positive, $\varphi_1 > 0$. On a positive half line $q > 0$, the local minimum is always negative, $\varphi_2 < 0$, and local maximum φ_3 can take negative as well as positive values,

$$0 < M < \frac{1}{\sqrt{\Lambda}}, \qquad \varphi_3 > 0,$$

$$M = \frac{1}{\sqrt{\Lambda}}, \qquad \varphi_3 = 0,$$

$$M > \frac{1}{\sqrt{\Lambda}}, \qquad \varphi_3 < 0.$$

When Eq. (65) holds, local minimum and maximum coincide: $q_2 = q_3$. Now we list all possibilities in the considered case.

Three horizons.—Under the condition,

$$-\varphi_3 < 3Q^2 < -\varphi_2, \tag{69}$$

the conformal factor has three simple zeroes on a positive half line. The corresponding Carter–Penrose diagram of a surface $\mathbb{U}$ is given by S2 in Fig. 1. Here, we have two timelike naked singularities. Arrows show that this diagram can be either periodically continued in space- and timelike directions, or opposite horizons can be identified. If we identify horizons in one direction, then topologically the surface $\mathbb{U}$ is a cylinder. If identification is performed in both directions, then it is a torus.

One simple horizon and timelike singularity.—The conformal factor has one simple zero on the positive half line under the following conditions:

$$\begin{array}{lllll} \Lambda>0, & \Upsilon<0, & M=0, & & Q\neq 0,\\ \Lambda>0, & \Upsilon<0, & M>0, & \varphi_3<0, & 3Q^2<-\varphi_3,\\ \Lambda>0, & \Upsilon<0, & M>0, & \forall\ \varphi_3, & 3Q^2>-\varphi_2,\\ \Lambda>0, & \Upsilon=0, & M>0, & & 3Q^2\neq-\varphi_2,\\ \Lambda>0, & \Upsilon>0, & M>0, & & Q\neq 0,\\ \Lambda>0, & \forall\ \Upsilon, & M<0, & \forall\ \varphi_3, & Q\neq 0. \end{array} \qquad (70)$$

This global solution is depicted by the Carter–Penrose diagram S7. It has timelike singularity.

Triple horizon.—Under the conditions,

$$\Lambda>0, \qquad \Upsilon=0, \qquad M>0, \qquad 3Q^2=-\varphi_2. \qquad (71)$$

local maximum and minimum of an auxiliary function $\varphi(q)$ coincide: $q_2=q_3$, and the conformal factor has a zero of third order at the point q_2 (triple horizon). This case is depicted by diagram S6. It coincides with diagram S7, but there is one important difference: the saddle point q_2 in the center of the diagram is geodesically complete.

This diagram is interesting from physical standpoint. Consider a spacelike section of this diagram. If the section does not go through the saddle point, which is located in the center of the diagram, then it is an interval of finite length with singular ends where two-dimensional curvature becomes infinite. If the space section goes through the saddle point then it is the union of two half-infinite intervals, because the central point in the center of the diagram is the space infinity. If we introduce now the global evolution parameter T, for instance, the vertical line on the diagram, then the topology of space sections change during evolution: for some value of T, there are two half-infinite intervals instead of one finite interval. This example shows that changing the topology of space in time can occur already at the classical level. This type of diagram appeared first in two-dimensional gravity with torsion [8].

Two horizons with double local minima.—Under the conditions,

$$\Lambda>0, \qquad \Upsilon<0, \qquad M>0, \qquad 3Q^2=-\varphi_2, \qquad (72)$$

the conformal factor has one zero of second order at the point q_2 and one simple zero at some point lying to the right from q_2. This solution is depicted by Carter–Penrose diagram S8 with two timelike singularities, which can be periodically extended in a timelike direction.

Two horizons with double local maximum.—Under the conditions,

$$\Lambda>0, \qquad \Upsilon<0, \qquad M>0, \qquad \varphi_3<0, \qquad 3Q^2=-\varphi_3, \qquad (73)$$

the conformal factor has one double zero at q_3 and one simple zero at some point lying to the left from q_2. This solution corresponds to a Carter–Penrose diagram S3 with two timelike singularities, which can be periodically extended in a spacelike direction.

4. Metric signature $(+---)$. The case $\Lambda<0$

For a negative cosmological constant, the conformal factor has the same form and asymptotics remain the same (62). Equation (63) and constant (64), defining the roots, do not change. We see that values of a constant Υ are positive for all Λ and M. Consequently, Eq. (63) has only one non-negative real root. Moreover, now branches of auxiliary function $\varphi(q)$ are directed upwards as shown in Fig. 3, and three new Carter–Penrose diagrams appear in the spherically symmetric case.

The conformal factor depicted by curve 2 in Fig. 3(b) has a zero at point $q=0$. It corresponds to anti–de Sitter space and is the degenerate case in the problem under consideration ($M=0$, $Q=0$).

Now we list all possibilities for negative cosmological constant.

Timelike singularity.—Under the conditions,

$$\begin{array}{llll} \Lambda<0, & M>0, & & 3Q^2>-\varphi_4,\\ \Lambda<0, & M\leq 0, & \varphi_3<0, & 3Q\neq 0, \end{array} \qquad (74)$$

the conformal factor does not have zeroes, and, consequently, horizons are absent. In this case, the Carter–Penrose diagram has the lens form S9 in Fig. 1. There is also a space-reflected diagram.

Naked singularity.—Under the conditions,

$$\Lambda<0, \qquad M>0, \qquad 3Q^2=-\varphi_4, \qquad (75)$$

the conformal factor has one positive root of second order at the minimum of the auxiliary function at q_4. In this case, the Carter–Penrose diagram is S10 in Fig. 1. In contrast to the naked singularity S4, the right complete infinity $q=\infty$ is timelike. It is due to the asymptotic behavior of the

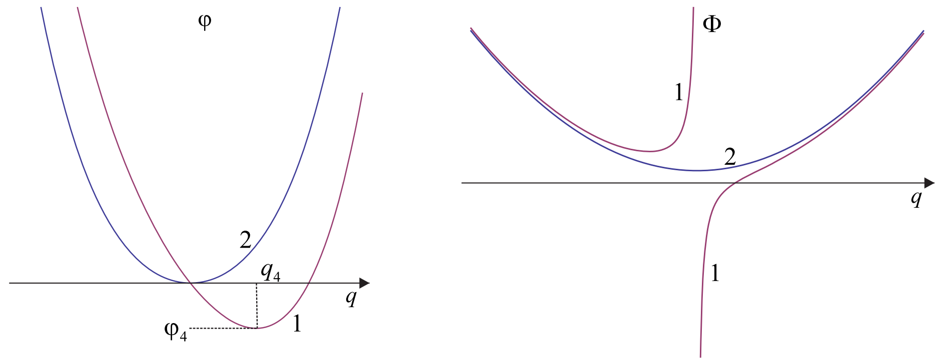

FIG. 3. Auxiliary function $\varphi(q)$ (a) and conformal factor $\Phi(q)$ for $\Lambda < 0$ and $Q = 0$ (b). The curves correspond to the following values of the constant: (1) $\Upsilon > -\frac{1}{8\Lambda^3}$ and (2) $\Upsilon = -\frac{1}{8\Lambda^3}$ ($M = 0$). On the left picture, the only minimum of curve 1 is located at a point q_4. Curve 2 on the left is invariant with respect to the map $q \mapsto -q$ and has a minimum at $q = 0$.

conformal factor at infinity, because space-time is asymptotically anti–de Sitter for $\Lambda < 0$. There is also a space-reflected diagram.

Timelike singularity and two horizons.—Under the conditions,

$$\Lambda < 0, \qquad M > 0, \qquad 3Q^2 < -\varphi_4, \tag{76}$$

the conformal factor has two zeroes. In this case, the Carter–Penrose diagram is given by S11 in Fig. 1. This solution can either be periodically extended in a timelike direction or opposite horizons can be identified. In contrast to diagram S1, space infinities are timelike, which is due to asymptotic at infinity.

Thus, we classified all spherically symmetric global solutions of Einstein's equations with an electromagnetic field for the metric signature $(+---)$. We see, that all solutions of a signature $(+---)$ contain a timelike singularity. Totally, we get 11 topologically inequivalent solutions S1–S11. It is possible to give a more subtle classification taking into account the existence of degenerate and oscillating geodesics. The latter appears if the conformal factor has a local extremum inside one of the conformal blocks. This classification was given for global solutions of two-dimensional gravity with torsion [8].

5. *Metric signature* $(-+++)$

Of course, metric signatures $(+---)$ and $(-+++)$ in general relativity are equivalent with appropriate changes of signs in the action. In our case, the change of signature $(+---) \mapsto (-+++)$ must be followed by the change of the overall sign of action (6). If we do not change the signs in the action and proceed with the signature $(-+++)$, then five new Carter–Penrose diagrams appear S12–S16. The diagram S12 is the same as for the Schwarzschild solution, and diagrams S13–S16 are obtained from S11, S9, S10, and S7 by rotation of 90° degrees, respectively. These solutions are unphysical because the canonical Hamiltonian for physical degrees of freedom of the electromagnetic field becomes a negative definite. Therefore, we skip the detailed analysis.

C. Planar solutions $K^{(h)} = 0$

If the Gaussian curvature of a surface $\mathbb{V}$ equals to zero, then it is either the Euclidean plane $\mathbb{R}^2$, or a cylinder, or a torus (after factorization). Thus, there is a spontaneous $\mathbb{ISO}(2)$ symmetry arising if the surface $\mathbb{V}$ is a Euclidean plane $\mathbb{R}^2$. That is, the space-time metric becomes invariant with respect to the $\mathbb{ISO}(2)$ transformation group on the equations of motion. In Schwarzschild coordinates (ζ, q, y, z), it is written in the form [for $m = -q^2 < 0$, corresponding to a signature $(+---)$],

$$ds^2 = \Phi(q)d\zeta^2 - \frac{dq^2}{\Phi(q)} - q^2 d\Omega_{\mathrm{P}}^2, \tag{77}$$

where

$$\Phi(q) = -\frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3}, \qquad d\Omega_{\mathrm{P}}^2 \coloneqq dy^2 + dz^2. \tag{78}$$

To draw Carter–Penrose diagrams for a Lorentzian surface $\mathbb{U}$, we have to analyze zeroes and asymptotics of a conformal factor $\Phi(q)$. For $Q \neq 0$, we have the second order pole Q^2/q^2 at zero and asymptotic at infinity,

$$\Phi \approx -\frac{\Lambda q^2}{3}, \qquad q \to \infty.$$

On intervals $(0, \infty)$ and $(-\infty, 0)$, the conformal factor is smooth, and, consequently, every global solution corresponding to one of these intervals is smooth. As for spherically symmetric solutions, we consider a positive M on both intervals due to the symmetry transformation $(M, q) \mapsto (-M, -q)$.

We start with the simplest case.

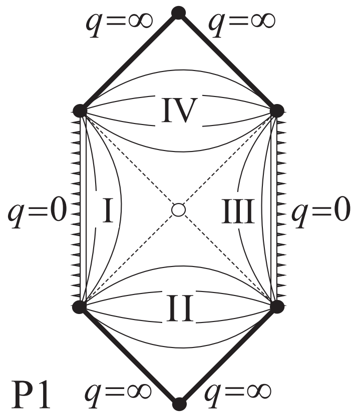

FIG. 4. The Carter–Penrose diagram for a planar solution for $\Lambda = 0$ and $M > 0$.

1. Metric signature $(+---)$. *The case* $\Lambda = 0$

The conformal factor is

$$\Phi(q) = \frac{Q^2 - 2Mq}{q^2}. \tag{79}$$

It has obviously one simple zero

$$q = \frac{Q^2}{2M}.$$

Moreover, there are only two cases.

Timelike singularity and one horizon.—Under the conditions,

$$\Lambda = 0, \qquad M > 0, \tag{80}$$

the conformal factor has one simple positive zero. The corresponding Carter–Penrose diagram is P1 in Fig. 4. This diagram has the same form as the Schwarzschild black hole S12 but turned over on 90°.

Naked singularity.—Under the conditions,

$$\Lambda = 0, \quad M \leq 0, \tag{81}$$

positive roots of the conformal factor are absent, and we have a naked singularity S5 in Fig. 1. ■

To find zeroes for the nonzero cosmological constant $\Lambda \neq 0$, we introduce an auxiliary function $\phi(q)$ representing the conformal factor for signature $(+---)$ in the form,

$$\Phi(q) =: \frac{\phi(q) + 3Q^2}{3q^2}, \tag{82}$$

where

$$\phi(q) := -6Mq - \Lambda q^4. \tag{83}$$

For the opposite signature, $\mathsf{sign}\hat{g} = (-+++)$, it is needed to make the replacement $Q^2 \mapsto -Q^2$. We see that on intervals $(0, \infty)$ and $(-\infty, 0)$, the number and type of zeroes of the conformal factor coincide with zeroes of the numerator $\phi(q) + 3Q^2$. It means that auxiliary function must be shifted either downwards [signature $(+---)$] or upwards [signature $(-+++)$].

The auxiliary function (83) has two real roots,

$$q = 0, \quad q = \sqrt[3]{-\frac{6M}{\Lambda}},$$

and two complex conjugate roots which do not interest us. The qualitative behavior of the auxiliary function and corresponding conformal factor are shown in Fig. 5. The position of extrema of the auxiliary function is defined by the equality,

$$\phi'(q) = -6M - 4\Lambda q^3 = 0 \quad \Rightarrow \quad q = \sqrt[3]{-\frac{3M}{2\Lambda}}.$$

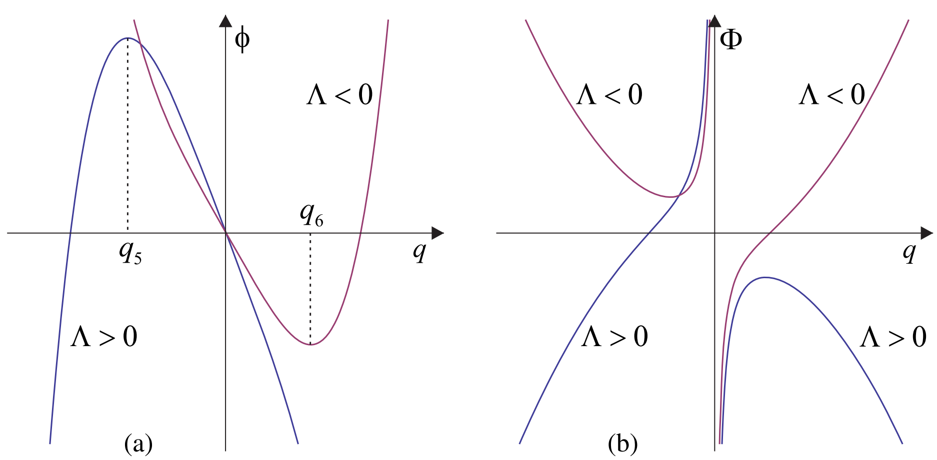

FIG. 5. Auxiliary function $\phi(q)$ (a) and conformal factor $\Phi(q)$ (b) $M > 0$. Maximum and minimum of the auxiliary function are located at points q_5 and q_6 for $\Lambda > 0$ and $\Lambda < 0$, respectively.

TABLE I. Classification of global planar solutions for $\Lambda \neq 0$.

$+---$	$\Lambda > 0$	$\forall M$	$Q \neq 0$	S7
$+---$	$\Lambda < 0$	$M > 0$	$0 < 3Q^2 < -\phi_6$	S11
$+---$	$\Lambda < 0$	$M > 0$	$3Q^2 = -\phi_6$	S10
$+---$	$\Lambda < 0$	$M > 0$	$3Q^2 > -\phi_6$	S9
$+---$	$\Lambda < 0$	$M \leq 0$	$Q \neq 0$	S9
$-+++$	$\Lambda > 0$	$M \geq 0$	$Q \neq 0$	S14
$-+++$	$\Lambda > 0$	$M < 0$	$0 < 3Q^2 < \phi_5$	S13
$-+++$	$\Lambda > 0$	$M < 0$	$3Q^2 = \phi_5$	S15
$-+++$	$\Lambda > 0$	$M < 0$	$3Q^2 > \phi_5$	S14
$-+++$	$\Lambda < 0$	$\forall M$	$Q \neq 0$	S16

We denote them by q_5 and q_6 for $\Lambda > 0$ and $\Lambda < 0$, respectively [see. Fig. 5(a)]. The maximal and minimal values of the auxiliary function are denoted by

$$\phi_{5,6} \coloneqq \phi(q_{5,6}) = \frac{9}{2} M \sqrt[3]{\frac{3M}{2\Lambda}}.$$

It is clear, that $\phi_5 > 0$ for $\Lambda > 0$ and $\phi_6 < 0$ for $\Lambda < 0$.

Detailed analysis show that Carter–Penrose diagrams for all planar solutions for $\Lambda \neq 0$ were already met in the spherically symmetric case. Therefore, to save space, we give the classification of all planar solutions in Table I. Note, that diagrams S7, S9, S10, and S11 differ from diagrams S16, S14, S15, and S13 by a turn of 90° degrees, respectively.

VI. HYPERBOLIC GLOBAL SOLUTIONS

If the Gaussian curvature of a surface $\mathbb{V}$ is negative, $K^{(h)} = -1$, then the surface is a two-sheeted hyperboloid $\mathbb{H}^2$, more precisely, the upper sheet of a two-sheeted hyperboloid (the Lobachevsky plane). It is the universal covering surface for closed Riemannian surfaces of genus two and higher. If $\mathbb{V} = \mathbb{H}^2$, then the isometry group is the Lorentz group $\mathbb{SO}(1,2)$. In this case, the metric in Schwarzschild coordinates $(\zeta, q, \theta, \varphi)$ for the signature $(+---)$ has the form,

$$ds^2 = \Phi(q) d\zeta^2 - \frac{dq^2}{\Phi(q)} - q^2 d\Omega_{\mathrm{H}}^2, \qquad (84)$$

where

$$\Phi(q) = -1 - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3}, \qquad d\Omega_{\mathrm{H}}^2 \coloneqq d\theta^2 + \mathrm{sh}^2\theta d\varphi^2.$$

The conformal factor for this metric differs from that in the spherically symmetric case (56) by the transformation,

$$\Phi \mapsto -\Phi, \qquad M \mapsto -M, \qquad Q^2 \mapsto -Q^2, \qquad \Lambda \mapsto -\Lambda. \qquad (85)$$

In addition, the transformation $Q^2 \mapsto -Q^2$ corresponds to a signature change of the metric, $(+---) \mapsto (-+++)$. Since we have already described global spherically symmetric solutions for all values of M, Q^2, and Λ, all hyperbolic solutions are obtained from spherically symmetric ones by a simple rotation of the Carter–Penrose diagrams by 90°, which corresponds to the transformation $\Phi \mapsto -\Phi$. In this way, we get 16 additional Carter–Penrose diagrams.

VII. CONCLUSION

We assumed that four-dimensional space-time is the warped product of two surfaces, $\mathbb{M} = \mathbb{U} \times \mathbb{V}$, and find a general solution of Einstein's equations with a cosmological constant and an electromagnetic field. These solutions are well-known locally and partly globally. We give a classification of all global solutions in the case when the surface $\mathbb{V}$ is of a constant curvature. Totally, there are 37 topologically different global solutions. These solutions in case B have four Killing vector fields, three of them corresponding to symmetry of the metric on a constant curvature surface $\mathbb{V}$. They are generators of the isometry groups $\mathbb{SO}(3)$, $\mathbb{ISO}(2)$, and $\mathbb{SO}(1,2)$ in cases when the surface $\mathbb{V}$ is a sphere $\mathbb{S}^2$, Euclidean plane $\mathbb{R}^2$, and two-sheeted hyperboloid $\mathbb{H}^2$, respectively. The fourth Killing vector generalizes Birkhoff's theorem. In all cases, there is "spontaneous symmetry emergence" because the existence of Killing vector fields was not assumed at the beginning, and their appearance is the consequence of Einstein's equations. Most probably, part of the constructed solutions are not satisfactory from a physical point of view. For example, for given signs in the Lagrangian and signature of the metric $(-+++)$, the Carter–Penrose diagram for a charged black hole coincides with the Schwarzschild solution. However, the quadratic form of momenta in the canonical Hamiltonian for physical degrees of freedom is not a positive definite (ghosts appearance), and this solution has to be discarded as unphysical. Nevertheless, the given classification of global solutions of Einstein's equations in the form of a warped product of two surfaces is important, because we must know what is to be discarded.

In general relativity, there is another possibility: to consider space-times which are the warped product of a real line with a three-dimensional manifold. These important types of solutions are usually considered in cosmology and require a separate consideration.

An interesting generalization would be the inclusion of a scalar field. If a scalar field depends only on the coordinates on one of the surfaces $\mathbb{U}$ or $\mathbb{V}$, then its energy-momentum tensor is a block diagonal, and we still have three cases as in the present paper. But the subsequent analysis becomes much more complicated. It reduces to two-dimensional gravity with a scalar field which is not integrable in general; see, e.g., [21].

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