New method to generate exact scalar-tensor solutions

Bertrand Chauvineau*

Université Côte d'Azur, Observatoire de la Côte d'Azur, CNRS, Laboratoire Lagrange, 06304 Nice Cedex 4, France

(Received 30 July 2018; published 24 July 2019)

A new method is proposed, that establishes a one to one correspondance between the whole set of static axially symmetric vacuum general relativity solutions and a specific class of stationary axially symmetric scalar-Einstein ($R_{ab} = \partial_a\varphi\partial_b\varphi$) solutions for any mass and angular momentum. The method explicitly takes advantage of the Kerr metric Ricci flatness. This also results in a class of stationary axially symmetric vacuum, ie Kerrlike, Brans-Dicke solutions. A particular solution, that is asymptotically flat, is more closely considered. It converges to Kerr for a vanishing scalar charge, but fails to converge to the Fisher-Janis-Newman-Winicour solution for a vanishing "rotation parameter." This solution exhibits a set of singular points, with a naked subset having a ringlike structure.

DOI: 10.1103/PhysRevD.100.024051

I. INTRODUCTION

Obtaining exact solutions of a relativistic gravity theory is far from being an easy task, due to the highly nonlinear character of the involved field equations. The most known solutions are Schwarzschild, Kerr, and Friedmann-Lemaitre-Robertson-Walker (FLRW), that solve the general relativity (GR) field equation in vacuum for the two first ones. These three solutions (or family of solutions for FLRW) are of obvious usefulness in the astronomical framework. Some other exact GR solutions are known, that are also very useful for astronomical purposes [1,2,3]. On the other hand, an incredibly huge number of solutions, the usefulness of which is not obvious *a priori*, have been obtained so far [4]. However, despite the nonimmediate relevance of a solution, its usefulness should not be underestimated. Indeed, any solution may reveal some unexpected property of the considered theory. It may also happen that a solution only later turns out to be of genuine astronomical relevance: the emergence of the black hole (BH) concept from the Schwarzschild solution is probably the most obvious exemple.

Since many attempts to quantify gravity, and/or to unify gravity with other interactions, return a classical gravitational sector having not a GR, but a scalar-tensor (ST) structure [5,6,7], Brans-Dicke (BD) and ST gravity theories are considered as valuable alternatives to GR, despite the fact that the latter successfully passes solar system tests up to now [8]. (Let us also point out that many ST theories are driven to mimic the GR behavior as a consequence of the cosmic expansion [9,10].) Thence the interest in ST theories, and specifically in BD, that just involves a constant parameter ω instead of an arbitrary function $\omega(\Phi)$. In this context, looking for exact BD/ST solutions is particularly appropriate to point out relevant qualitative features with respect to GR. For instance, the spherical Brans class I solution [11,12] generically exhibits a naked singularity (NakS) or wormhole structure [13] (see also [14] for pioneering works on the detailed significance of the Class I, II, III and IV Brans solutions), such features being absent from the Schwarzschild GR solution. Besides, it has been recently shown that a particle orbiting a large ω Brans class I solution results in an observed (by a far observer) unbound orbital frequency, depending on how much the solution is scalarized [15]. One may then suspect striking qualitative differences in extreme mass ratio binaries BD/ST gravitational radiation, with respect to GR, since GR orbital frequencies cannot exceed the innermost circular orbital value. Nevertheless, let us remind that under some conditions (mainly regularity, asymptotical flatness and finite area horizon), vacuum and stationary BH-like solutions are the same in BD/ST as in GR [16,17].

It is known from long that any vacuum BD solution is associated to a massless scalar filled GR solution, by the means of a conformal transformation [18]. Thence seeking vacuum BD solutions can be reformulated as a scalar-GR problem. From the Hawking's theorem [16], it is clear that any stationary axisymmetric (SAS) vacuum BD solution, but differing from Kerr, should exhibit a NakS structure, unless exhibiting some other peculiar feature that allow it to evade the theorem (like being not asymptotically flat).

A method has been proposed by [19], that allows us to generate a BD vacuum SAS solution from a GR vacuum SAS seed one, provided one is able to solve a given nonlinear PDE system. A class of electrovacuum BD solutions has been obtained by [20], that generalizes the

*Bertrand.Chauvineau@oca.eu

Majumdar-Papapetrou GR solution. The same authors later proposed a method that allows us to generate SAS vacuum BD solutions from SAS vacuum GR ones (and also from static axisymmetric vacuum BD ones) [21], but the solutions then obtained are generally not asymptotically flat. The SAS, but only one coordinate dependent, case has been solved by [22]. The case of ST SAS solutions has been considered by [23], and explicit solutions are given in a few case of very specific ST (not BD) theories. Along the same lines as [19], it is claimed in [24] that any SAS vacuum BD solution can be obtained by nonlinearly combining any SAS vacuum GR solution and any vacuum solution of the Weyl class. A Kerr-type BD solution has been obtained in [25], that was built in such a way that it reduces to the Kerr solution in the $\omega \to \infty$ limit. Inspired by [20] and [22] works, a BD BH-like solution is proposed by [26], but this solution is not asymptotically flat. Motivated by particle collisions near a Kerr-like BD BH, another nonasymptotically flat solution is obtained in [27], that is also derived using the [20] method. Starting from an unusual formulation of the Lewis metric, a two parameters extension of the method initiated in [21] allowed [28] to derive a BD version of the Ernst equation. The method is applied to some examples, but here again, the obtained spacetimes are generally not asymptotically flat. The matter filled case has been considered by [29] in self interacting [i.e., with a potential $V(\Phi)$] BD, but only static axisymmetric spacetimes were considered. Very recently, a way to generate new solutions starting from known ones has been proposed by [30]. The technique makes use of a BD symmetry in the traceless case ($g^{ab}T_{ab} = 0$).

It is worth reminding that a scalar-metric was proposed by [31] as an SAS vacuum BD solution. A conformally related version is given in [32]. This metric, or its [32] form, has been used in several papers to characterize physics in a NakS (versus BH) field [32,33,34,35], and also to suggest rotating antiscalar solutions as alternatives to Kerr BHs [36]. However, the [31] scalar-metric is actually not a vacuum BD solution, as it is explicitly shown in [37] (and also mentioned in [27]). The reason is that the authors of [31] applied, without justification, to a BD spherical vacuum solution (the Brans Class I) the Newman-Janis (NJ) algorithm [38]. (It was shown in [38] that this algorithm is an unexpected way to recover Kerr from Schwarzschild). Let us stress that, even in the GR framework, determining the ability of the NJ algorithm (or some equivalent reformulations) to generate new solutions from a seed one is not an easy task [39]. The NJ algorithm, as well as some modified versions, nevertheless received continuous interest, not only as a way to suggest new (generally nonperfect fluid filled) solutions, but also in the context of other theories, like supergravity. See for instance [40] for a recent review. Related to the NJ finding [38], let us also mention that another (but fully justified) way to derive Kerr from Schwarzschild was obtained in [41].

In this paper, a new method, that establishes a one to one correspondence between static axially symmetric vacuum GR solutions and SAS massless scalar GR solutions having a metric with some prior form, is established. This prior form is inspired from the metric found in [27], referred to as the SB solution in the following. It is defined as a modification of the Kerr metric, with given (m, a) parameters, by inserting two unknown metric functions in a suitable manner. These functions are then demanded to be such that the scalar-Einstein field equations are satisfied. Thence, it uses Kerr as a seed in some sense, but in a way that differs from the generating techniques previously reviewed. The definition of some well-suited (α, β) coordinates then allows us to establish the correspondence. Let us stress that, unlike the NJ algorithm, the method is completely justified since it involves the resolution of the relevant field equations. The SB solution is recovered as a special case, and other explicit solutions are built, one of them being asymptotically flat. The application of this new method, but using an *a priori* non-Kerr SAS, or SAS like, metric as a seed, is also briefly discussed. This allows us to spot the specific features of Kerr that result in the established correspondence with vacuum static GR solutions.

A. Outline of the paper

The prior form of the metric and the related index notations are defined in Sec. II. The first half of the Einstein equations is considered in II A. The Klein-Gordon (KG) equation is used in II B, that suggests the definition of the $(\alpha(r, \theta), \beta(r, \theta))$ coordinates. In II C, the second half of the Einstein equations is considered. The correspondence with the general static axisymmetric vacuum GR case is made explicit in II D. A particular asymptotically flat solution is then considered in Sec. III. The links with the BD theory is discussed in Sec. IV. In Sec. V, one reconsiders the method but starting from any SAS-like metric as a seed. The Sec. VI is dedicated to a brief conclusion.

II. THE CONSIDERED SET OF AXISYMMETRIC METRICS

We consider in this paper metrics having the form

$$g_{pq} = e^{A} k_{pq} \tag{1a}$$

$$g_{uv} = e^{B} k_{uv} \tag{1b}$$

where k_{ab} is the Kerr metric of mass m and of angular momentum per unit mass a

$$\begin{pmatrix} k_{00} & k_{03} & 0 & 0 \\ k_{03} & k_{33} & 0 & 0 \\ 0 & 0 & k_{11} & 0 \\ 0 & 0 & 0 & k_{22} \end{pmatrix} = \begin{pmatrix} -V & -w(1-V) & 0 & 0 \\ -w(1-V) & 2w^2 - w^2V + \Sigma\sin^2\theta & 0 & 0 \\ 0 & 0 & \frac{\Sigma}{\Delta} & 0 \\ 0 & 0 & 0 & \Sigma \end{pmatrix} \tag{2}$$

in Boyer-Lindquist coordinates, and where A and B are (r, θ) dependent functions. One has introduced the usual quantities

$$\begin{aligned} w(\theta) &= a\sin^2\theta \\ \Sigma(r,\theta) &= r^2 + a^2\cos^2\theta \\ \Delta(r) &= r^2 - 2mr + a^2 \\ V(r,\theta) &= 1 - \frac{2mr}{\Sigma}. \end{aligned} \tag{3}$$

Besides the usual index convention $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, let us also make the convention

$$\begin{aligned} (p, q, r, s) &\text{ indexes} \in \{0, 3\} \\ (u, v, w, x, y, z) &\text{ indexes} \in \{1, 2\} \end{aligned}$$

while the (a, b, c, d, e) indexes take the four spacetime values. The ordering in (2) makes explicit the block diagonal structure of Kerr's metric. The metric (1a)–(1b) is also block diagonal, each block being "conformally" related to the Kerr corresponding one, but with different "conformal" factors. The requirement (1a)–(1b) implicitly means imposing two prior relations between the four metric functions describing the general form of an SAS metric (see for instance Eq. (1) of [42]).

The metric (1a)–(1b) is required to solve the scalar-Einstein equation

$$R_{ab} = \partial_a\varphi\partial_b\varphi \tag{4}$$

where φ is an (r, θ) dependent scalar field. The special case $A = B = 0$ solves (4) for $\varphi = 0$, since the Kerr metric (2) is Ricci flat.

The SB solution [27] corresponds to $A = 0$ and $e^B = (\Delta \sin^2\theta)^\sigma$, where σ is an integration constant. Of course it should be recovered as a special solution of (4) with the corresponding scalar, that reads $\varphi_{SB} = \sqrt{\frac{\sigma}{2}} \ln(\Delta \sin^2\theta)$. This will be checked later.

From the (r, θ) dependence of the considered scalar-metric, one has $\partial_p(A, B, \varphi) = 0$. The nonzero connexion components then read

$$\Gamma^p_{qu} = K^p_{qu} + \frac{1}{2}\delta^p_q\partial_u A \tag{5a}$$

$$\Gamma^u_{pq} = e^{A-B}\left(K^u_{pq} - \frac{1}{2}k^{ux}k_{pq}\partial_x A\right) \tag{5b}$$

$$\Gamma^u_{vw} = K^u_{vw} + \gamma^u_{vw} \tag{5c}$$

where the K^c_{ab} quantities are the Kerr connexion components

$$K^p_{qu} = \frac{1}{2}k^{pr}\partial_u k_{qr} \tag{6a}$$

$$K^u_{pq} = -\frac{1}{2}k^{ux}\partial_x k_{pq} \tag{6b}$$

$$K^u_{vw} = \frac{1}{2}k^{ux}(\partial_v k_{xw} + \partial_w k_{xv} - \partial_x k_{vw}) \tag{6c}$$

with

$$\gamma^u_{vw} = \frac{1}{2}(\delta^u_v\partial_w B + \delta^u_w\partial_v B - k_{vw}k^{ux}\partial_x B). \tag{7}$$

For convenience, let us introduce the following notation

$$(c, s) = (\cos\theta, \sin\theta). \tag{8}$$

A. The (pq) Einstein equation components

From (4), one has $R_{pq} = 0$, that writes

$$\frac{1}{\sqrt{-g}}\partial_w(\sqrt{-g}\Gamma^w_{pq}) - \Gamma^d_{pc}\Gamma^c_{qd} = 0. \tag{9}$$

Using (5a), (5b), but also $R_{pq}(k_{ab}) = 0$ (Kerr's metric being Ricci flat) and $\partial_x k_{pq} = K^r_{xq}k_{pr} + K^r_{xp}k_{qr}$ (from the Ricci identity on Kerr's metric), (9) yields an equation that involves A only

$$2K^w_{pq}\partial_w A - k_{pq}\left[k^{wx}\partial_w A\partial_x A + \frac{1}{\sqrt{-k}}\partial_w(\sqrt{-k}k^{wx}\partial_x A)\right] = 0. \tag{10}$$

Eliminating the bracket term thanks to the contraction by k^{pq}, one obtains (since $k^{pq}k_{pq} = 2$)

$$(2K^w_{pq} - k_{pq}k^{rs}K^w_{rs})\partial_w A = 0. \tag{11}$$

Using the Kerr metric (2) and the connexion components (5b), (11) yields

$$k^{wx}\partial_x\left(\frac{k_{pq}}{\sqrt{\Delta}|s|}\right)\partial_w A = 0. \tag{12}$$

This equation has to be satisfied by both k_{00}, k_{03} and k_{33}. Writing out the two equations for k_{00} and k_{03}, one obtains two homogeneous equations on $\partial_1 A$ and $\partial_2 A$. One then shows that having a nontrivial solution requires $a = 0$. Thence, A is necessarily constant for $a \neq 0$. One can then set $A = 0$ by redefining B and the ds^2 units.

The fact that $g_{pq} = k_{pq}$ has two straightforward consequences: (1) the equatorial ($\theta = \pi/2$) circular orbits and their linear planar stability (both do not involve g_{11}) are obtained solving the same equations as the Kerr's case, and (2) the horizon and ergosphere are the "same" as Kerr's

$$r_h = m + \sqrt{m^2 - a^2} \tag{13}$$

$$r_e(\theta) = m + \sqrt{m^2 - a^2 c^2}. \tag{14}$$

To be precise, let us point out that these claims just concern the functions that describe these orbits and surfaces in terms of θ and of the (m, a) parameters. Indeed, the metric components g_{uv} enter their geometric and relative properties. For instance, the geometric radial distance between two circular orbits having circumferences C and C' depends on g_{11}.

Since $A = 0$, the metric (1a)–(1b) achieves the form

$$ds^2 = k_{00}dt^2 + 2k_{03}dtd\phi + k_{33}d\phi^2 + e^B(k_{11}dr^2 + k_{22}d\theta^2). \tag{15}$$

B. The Klein-Gordon equation

To pursue the integration, it would be sufficient to solve the (uv) components of (4), since the KG equation is a direct consequence of (4). It is nevertheless useful to write out the KG equation

$$\partial_a(\sqrt{-g}g^{ab}\partial_b\varphi) = 0. \tag{16}$$

Using (15), it reads

$$\partial_1(s\Delta\partial_1\varphi) + \partial_2(s\partial_2\varphi) = 0. \tag{17}$$

This form suggests defining the alternative radial coordinate

$$\rho \equiv \ln\left(\frac{r - m + \sqrt{\Delta}}{\sqrt{m^2 - a^2}}\right) \tag{18}$$

that yields

$$\partial_\rho = \sqrt{\Delta}\partial_1 \tag{19a}$$

$$r - m = \sqrt{m^2 - a^2}\cosh\rho \tag{19b}$$

$$\Delta = (m^2 - a^2)\sinh^2\rho. \tag{19c}$$

This allows rewriting (17) in the form

$$\partial_\rho(sS\partial_\rho\varphi) + \partial_2(sS\partial_2\varphi) = 0. \tag{20}$$

For convenience, one has introduced the following notation

$$(C, S) = (\cosh\rho, \sinh\rho). \tag{21}$$

This form returns two obvious solutions

$$\varphi_{SB} = \Lambda \ln(Ss) \tag{22a}$$

$$\varphi_2 = \Lambda Cc \tag{22b}$$

where Λ is any constant, and φ_{SB} the scalar entering the SB solution [27]. The form of these two solutions suggests defining new (α, β) coordinates by

$$(\alpha, \beta) = (Ss, Cc). \tag{23}$$

In terms of the initial Boyer-Lindquist coordinates, these coordinates read

$$\alpha = \frac{\sqrt{r^2 - 2mr + a^2}}{\sqrt{m^2 - a^2}}\sin\theta \tag{24a}$$

$$\beta = \frac{r - m}{\sqrt{m^2 - a^2}}\cos\theta \tag{24b}$$

and behave as $(r\sin\theta, r\cos\theta)$ for $r \to \infty$, up to the $(m^2 - a^2)^{-1/2}$ factor. From (23) and the trigonometric identities $c^2 + s^2 = 1$ and $C^2 - S^2 = 1$, one obtains the following relations

$$\begin{aligned}(S^2 + s^2)\partial_\alpha(C, S) &= Cs(S, C)\\ (S^2 + s^2)\partial_\alpha(c, s) &= Sc(-s, c)\\ (S^2 + s^2)\partial_\beta(C, S) &= Sc(S, C)\\ (S^2 + s^2)\partial_\beta(c, s) &= Cs(s, -c)\end{aligned} \tag{25}$$

that turn out to be useful in the calculations to do. The derivation operators transform as

$$\begin{aligned}\partial_\rho &= Cs\partial_\alpha + Sc\partial_\beta\\ \partial_2 &= Sc\partial_\alpha - Cs\partial_\beta.\end{aligned} \tag{26}$$

Reinserting in (20) returns, after a lengthy calculation, and using (25), the KG equation in the nice form

$$\partial_\alpha(\alpha\partial_\alpha\varphi) + \alpha\partial_\beta\partial_\beta\varphi = 0. \tag{27}$$

This form points out two other obvious solutions, besides (22a) and (22b)

$$\varphi_3 = \Lambda\beta\ln\alpha \tag{28a}$$

$$\varphi_N = \frac{\Lambda}{\sqrt{\alpha^2+\beta^2}}. \tag{28b}$$

The solution (28a) is nothing but the product of (22a) and (22b), that turns out to be a solution too. The solution (28b) results from the fact that (27) is the classical Laplacian equation written in cylindrical coordinates, in the case of an axisymmetric potential. For this reason, we will refer to (28b) as being the Newtonian scalar.

It may be worth spotting that if φ solves (27), so do its successive derivatives with respect to β. (Remark that $\varphi_{SB} = \partial_\beta\varphi_3$.) Along these lines, new solutions can also be obtained by integration with respect to β. For instance

$$\varphi_5 = \Lambda\ln\left(\beta+\sqrt{\alpha^2+\beta^2}\right) \tag{29a}$$

$$\varphi_6 = \Lambda\left[\beta\ln\left(\beta+\sqrt{\alpha^2+\beta^2}\right)-\sqrt{\alpha^2+\beta^2}\right] \tag{29b}$$

also solve (27), and are related to φ_N by $\varphi_N = \partial_\beta\varphi_5 = \partial_\beta\partial_\beta\varphi_6$.

See the Appendix A for a quicker demonstration of (27), using the (α,β) coordinates.

C. The (uv) Einstein equation components

From (4), one has $R_{uv} = \partial_u\varphi\partial_v\varphi$, that writes

$$\frac{1}{\sqrt{-g}}\partial_w(\sqrt{-g}\Gamma^w_{uv}) - \partial_u\partial_v\ln\sqrt{-g} - \Gamma^q_{up}\Gamma^p_{vq} - \Gamma^y_{ux}\Gamma^x_{vy} = \partial_u\varphi\partial_v\varphi. \tag{30}$$

Using (5a), (5c) and (7), with $A=0$, but also $R_{uv}(k_{ab}) = 0$ (Kerr's metric is Ricci flat) and $\partial_y k_{uv} = K^x_{yu}k_{vx} + K^x_{yv}k_{ux}$ (from the Ricci identity on Kerr's metric), (30) yields

$$K^p_{vp}\partial_u B + K^p_{up}\partial_v B - \frac{1}{\sqrt{-k}}k_{uv}\partial_w(\sqrt{-k}k^{wz}\partial_z B) = 2\partial_u\varphi\partial_v\varphi. \tag{31}$$

Making explicit the (11), (12) and (22) components yields, using the ρ radial coordinate and (19a)

$$Cs\partial_\rho B - Ss\partial_\rho\partial_\rho B - S\partial_2(s\partial_2 B) = 2Ss(\partial_\rho\varphi)^2 \tag{32a}$$

$$Sc\partial_\rho B + Cs\partial_2 B = 2Ss\partial_\rho\varphi\partial_2\varphi \tag{32b}$$

$$2Sc\partial_2 B - S\partial_2(s\partial_2 B) - Ss\partial_\rho\partial_\rho B - Cs\partial_\rho B = 2Ss(\partial_2\varphi)^2. \tag{32c}$$

Let us now rewrite these equations in (α,β) coordinates. Combining the equation (32b), and the difference of (32a) and (32c), returns

$$\partial_\alpha B = \alpha[(\partial_\alpha\varphi)^2 - (\partial_\beta\varphi)^2] \tag{33a}$$

$$\partial_\beta B = 2\alpha\partial_\alpha\varphi\partial_\beta\varphi. \tag{33b}$$

It appears that the integrability condition of (33a) and (33b) is ensured by the KG equation (27). An (uv) equation remains to be written, that can be built from the sum of (32a) and (32c). It turns out that this equation is solved thanks to (27).

See the Appendix A for a quicker demonstration of these results, using the (α,β) coordinates from the start.

D. Generating solutions

From the previous sections, a (φ, B) solution can be obtained by (1) solving first the KG equation (27), and then (2) integrating the system (33a)–(33b). There is then a direct correspondence with the issue of seeking a static axisymmetric solution of vacuum GR. Indeed, such a metric can be written [2]

$$ds^2 = -e^{2U}dt^2 + e^{-2U}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2 d\phi^2] \tag{34}$$

where the metric functions $U(\rho,z)$ and $\gamma(\rho,z)$ have to solve

$$\partial_\rho(\rho\partial_\rho U) + \rho\partial_z\partial_z U = 0 \tag{35a}$$

$$\partial_\rho\gamma = \rho[(\partial_\rho U)^2 - (\partial_z U)^2] \tag{35b}$$

$$\partial_z\gamma = 2\rho\partial_\rho U\partial_z U. \tag{35c}$$

Thence, any $(U(\rho,z),\gamma(\rho,z))$ solution of (35a)–(35c) immediately results in a $(\varphi(\alpha,\beta), B(\alpha,\beta))$ solution of the problem considered in this paper, by just making the changes $(\rho,z)\to(\alpha,\beta)$ and $(U,\gamma)\to(\varphi,B)$. There is then a one to one correspondence between these two problems. Let us stress that the (α,β) definition depends on the Kerr's mass and angular momentum, in such a way that the correspondence works for any (m,a) parameters. Note that while both problems are GR issues, the former corresponds to a static field in a vacuum spacetime, while the later to a rotating field in a massless scalar filled spacetime.

Considering the four first solutions of the KG equation obtained in II B, one obtains

$$\varphi_{SB} = \Lambda\ln\alpha \Rightarrow B_{SB} = \Lambda^2\ln\alpha \tag{36a}$$

$$\varphi_2 = \Lambda\beta \Rightarrow B_2 = -\Lambda^2\frac{\alpha^2}{2} \tag{36b}$$

$$\varphi_3 = \Lambda\beta\ln\alpha$$
$$\Rightarrow B_3 = \Lambda^2\left(\beta^2\ln\alpha - \frac{1}{4}\alpha^2[1-2\ln\alpha+2(\ln\alpha)^2]\right) \tag{36c}$$

$$\varphi_N = \frac{\Lambda}{\sqrt{\alpha^2+\beta^2}} \Rightarrow B_N = -\frac{\Lambda^2\alpha^2}{2(\alpha^2+\beta^2)^2} \tag{36d}$$

The (φ_{SB}, B_{SB}) solution is the SB solution [27]. It is clear from (24a)–(24b) and (15) that the SB, the (φ_2, B_2) and the (φ_3, B_3) solutions are not asymptotically flat, a point that makes their astrophysical usefulness debatable. On the other hand, the (so named in the following) scalar-Newtonian solution (φ_N, B_N) is asymptotically flat.

From the previously pointed out correspondence, the SB solution is associated to the Minkowski spacetime (in some non-Cartesian coordinates), while the scalar-Newtonian one is associated to the Curzon-Chazy spacetime [2].

It is worth also having a look on the fifth solution of the KG equation obtained in II B. Integrating for B_5 yields

$$B_5 = 2\Lambda^2 \ln\left(1 + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) \tag{37}$$

i.e., in terms of (r, θ) coordinates, using (23)

$$e^{B_5} = \left(1 + \frac{(r-m)c}{\sqrt{r^2 - 2mr + a^2 + (m^2-a^2)c^2}}\right)^{2\Lambda^2} \tag{38}$$

This, with (15), suggests that the spacetime is not asymptotically flat. From the previous correspondence, the (φ_5, B_5) solution is associated to the Gautreau-Hoffman spacetime [2].

The (φ_6, B_6) solution is obviously not asymptotically flat. Indeed, (4) shows that it is even not Ricci flat at infinity, since $\partial_\alpha\varphi_6 = -\Lambda\alpha(\beta + \sqrt{\alpha^2+\beta^2})^{-1}$ and $\partial_\beta\varphi_6 = \varphi_5 = \Lambda\ln(\beta + \sqrt{\alpha^2+\beta^2})$ do not vanish at infinity.

III. THE SCALAR-NEWTONIAN SOLUTION

Since it is asymptotically flat, let us have a closer look on the scalar-Newtonian solution. Its metric (36d) explicitly reads

$$\begin{aligned} ds_N^2 &= k_{00}dt^2 + 2k_{03}dtd\phi + k_{33}d\phi^2 \\ &\quad + \exp\left(-\frac{\Lambda^2(m^2-a^2)(r^2-2mr+a^2)s^2}{2[r^2-2mr+a^2+(m^2-a^2)c^2]^2}\right) \\ &\quad \times (k_{11}dr^2 + k_{22}d\theta^2). \end{aligned} \tag{39}$$

The g_{00} and g_{03} metric tensor components being identical to Kerr, the m and a constants have the same ADM mass and angular momentum meanings. As already mentioned, the Kerr horizon (13) and ergosphere (14) surfaces are recovered, since they just depend on the g_{pq} metric components, which are the same as Kerr.

The scalar associated to (39) reads, in terms of (r, θ) coordinates

$$\varphi_N = \frac{\Lambda\sqrt{m^2-a^2}}{\sqrt{r^2-2mr+a^2+(m^2-a^2)c^2}}. \tag{40}$$

The explicit θ dependence of φ_N shows that the (39)–(40) solution is not the BD-Kerr solution obtained in [19] (equations (24)–(26) of [19]).

A. Singularities

One knows that the Kerr metric is singularity free outside its (external) horizon (13), and that the horizon itself is a regular surface. However, from the Hawking theorem [16], this should not be true for the spacetime (39) since $\partial\varphi_N \neq 0$ while (1) it is asymptotically flat and, (2) its horizon surface is finite. Indeed, the scalar curvature reads, from (4)

$$R = g^{uv}\partial_u\varphi_N\partial_v\varphi_N \tag{41}$$

which, from (36d), yields

$$R = \frac{16\Lambda^2}{\Sigma}\frac{C^2S^2+c^2s^2}{(S^2+c^2)^2}\exp\left(\frac{\Lambda^2S^2s^2}{2(S^2+c^2)^2}\right). \tag{42}$$

One sees that (42) diverges whatever the way $S^2 + c^2 \to 0$, i.e., the way $(c, S) \to (0, 0)$ (and only in this case for $r \geq r_h$). The points having $\theta = \pi/2$ and $\rho = 0$ are then NakS points. Thence, the horizon is not regular everywhere, since its equator $\theta = \pi/2$, and the equator only (considering points having $r \geq r_h$), is scalar curvature singular.

However, the fact that the scalar curvature does not diverge on the points having $(r \geq r_h, \theta) \neq (r_h, \pi/2)$ is not sufficient to prove that none of these points is singular. A quantity that is often regarded as a reliable singularity indicator is the Kretschmann scalar

$$\hat{K} \equiv R_{abcd}R^{abcd} \tag{43}$$

where R_{abcd} is the Riemann-Christoffel curvature tensor. From (5a)–(5c), one finds that

$$\begin{aligned} R_{pqrs} &= e^{-B}Q_{pqrs} \\ R_{pquv} &= Q_{pquv} \\ R_{puqv} &= Q_{puqv} + q_{puqv} \\ R_{uvwx} &= e^{B}(Q_{uvwx} + q_{uvwx}) \\ R_{pqru} &= R_{puvw} = 0 \end{aligned} \tag{44}$$

where Q_{abcd} is the Kerr's metric Riemann-Christoffel curvature tensor, and

$$\begin{aligned} q_{puqv} &= k_{pr}K^r_{qw}\gamma^w_{uv} \\ q_{uvwx} &= k_{uy}(\partial_w\gamma^y_{vx} - \partial_x\gamma^y_{vw} + K^z_{vx}\gamma^y_{wz} + K^y_{wz}\gamma^z_{vx} - K^z_{vw}\gamma^y_{xz} \\ &\quad - K^y_{xz}\gamma^z_{vw} + \gamma^y_{wz}\gamma^z_{vx} - \gamma^y_{xz}\gamma^z_{vw}). \end{aligned} \tag{45}$$

If $B = 0$, which implies $\gamma^u_{vw} = 0$ from (7), $\hat{K}$ is finite since the Kerr's metric is regular in the considered region.

Any divergence of $\hat{K}$ can then only appear from the γ^z_{vw} and $\partial\gamma^z_{vw}$, i.e., from the ∂B_N and $\partial\partial B_N$, quantities entering the q_{abcd} terms in (44). It is then easy to see from (36d) that a divergence of $\hat{K}$ can only occur at points where $c = S = 0$. The set of Kretschmann singularity points is then included in the singular scalar curvature points set. This strongly suggests that the horizon is regular at points not belonging to the equatorial NakS ring $(r, \theta) = (r_h, \pi/2)$.[1]

Let us stress that it was recently found by [43] that the (not asymptotically flat) SB solution, i.e., (36), also exhibits such a ringlike NakS structure.

1. A bit more on the scalar curvature singularities

From (18), the calculations performed so far are *a priori* only relevant in the regions $r \geq m + \sqrt{m^2 - a^2}$ and $r \leq m - \sqrt{m^2 - a^2}$, in which the quantity $\sqrt{\Delta}$ entering (18) is well defined. However, sense can be given to these calculations, and to the related findings, by analytic continuation in the spacetime region located inside the external and internal horizons $r = m \pm \sqrt{m^2 - a^2}$, where Δ is negative.[2] It then results that the points satisfying the condition $S^2 + c^2 = 0$, i.e., $C^2 = s^2$, or, from (19b)

$$r_{\text{sing}}(\theta) = m \pm \sqrt{m^2 - a^2}\sin\theta \tag{46}$$

in terms of (r, θ) coordinates, are also scalar curvature singular points. They define a close 2-surface in the constant t subspaces, that fully belongs to the region between the horizons, apart from the $\theta = \pi/2$ points that are located on the equators of the two horizons. Thence, apart from the $(r, \theta) = (m + \sqrt{m^2 - a^2}, \pi/2)$ events previously discussed, all these singular points are hidden to any external observers, and then do not affect their behaviors.

Accordingly, we will only consider the (external) ringlike NakS part of the (46) singular set in the following.

B. The $a = 0$ subcase (static case)

Let us now specify the scalar-Newtonian solution (36d) to the $a = 0$ case. The vacuum Kerr's solution returns the (vacuum) spherical Schwarzschild solution for $a = 0$. In the nonvacuum, but massless scalar filled, case, a spherical solution is known, often named the JNW metric, referring to the Janis-Newman-Winicour 1968 paper [44]. It is worthwhile to point out that this solution was in fact discovered earlier by Fisher in his 1948 paper [45], but using an areal radial coordinate. It seems then fair to use the FJNW acronym when referring to this solution, and so will I do in this paper. The FJNW solution includes Schwarzschild as a limit (non scalarized) case. It is then natural to suspect the existence of a massless scalar filled GR solution, that would depend on a "rotation parameter" a, and that would return (1) the Kerr spacetime for a vanishing scalar, and (2) the FJNW solution for $a = 0$.

The solution (36d), or (39)–(40), indeed returns the Kerr spacetime for a vanishing scalar, i.e., for $\Lambda = 0$. On the other hand, making $a = 0$ in (39) does not return the FJNW solution, but

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \exp\left(-\frac{\Lambda^2 m^2 (r^2 - 2mr)s^2}{2(r^2 - 2mr + m^2c^2)^2}\right) \times \left[\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2\right] + r^2 s^2 d\phi^2. \tag{47}$$

This static but non spherical solution is known from long, see Eqs. (17)–(18) of Penney's paper [46]. From (40), the scalar field reads

$$\varphi = \frac{m\Lambda}{\sqrt{r^2 - 2mr + m^2c^2}}. \tag{48}$$

The metric (47) has a horizon, that reads

$$r = 2m \tag{49}$$

and whose equator's points $\theta = \pi/2$, and only these $(r = 2m)$ points, are singular, as it can be directly checked from the $(S^2 + c^2)$ expression entering (42), that reads for $a = 0$

$$S^2 + c^2 = \left(\frac{r}{m} - 1\right)^2 - s^2. \tag{50}$$

(Other, but hidden to external observers, singular points can also be defined from $S^2 + c^2 = 0$, along the lines of the subsection III A 1.) This agrees with the fact that the metric, while static since $g_{03} = 0$, is not spherical, because of the presence of the (θ dependent) exponential in front of the bracket in (47). Incidentally, this also shows that (47) cannot be FJNW in disguise. The ring character of the NakS, stuck on the horizon, is obviously a property inherited from the $a \neq 0$ general case.

The fact that the FJNW metric does not appear as a subcase of (47) means that (39) cannot be interpreted as a "rotating version of FJNW." It also strongly suggests the existence of asymptotically flat Kerr like solutions of (4), that do not fulfill (1a)–(1b). Indeed, recovering FJNW as a rotationless limit case, with its properties ([15]), is incompatible with the $A = 0$ conclusion obtained in II A.

[1]Extending to the $r < r_h$ region, one sees on (42) that $\Sigma = 0$ is a scalar curvature singularity of the considered spacetime (it is a singularity, but not a scalar curvature one, in Kerr's spacetime). The spacetime has then two ring singularities: one is naked, the second being hidden (the latter being the counterpart of the usual Kerr's, in some sense).

[2]Let us keep in mind that r is a timelike coordinate in this region, since $g_{rr} = \Sigma/\Delta$ is negative then.

C. Orbits in the equatorial plane of the static solution

As already mentioned, the presence of the scalar field affects neither the existence condition of equatorial circular orbits nor their linear stability (using the radial coordinate defined by the form of the metric (15)). The situation is very similar to the case of a scalarized version of the γ-metric reported in [47]. However, this does not mean that the physics is unaffected by the scalar, even in the equatorial plane. The aim of this subsection is to illustrate this point. For convenience, we specify to the static case, that makes all the calculations easily tractable.

In the $\theta = \pi/2$ plane, the metric (47) simplifies into

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \exp\left(-\frac{\Lambda^2 m^2}{2(r^2 - 2mr)}\right)\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{51}$$

One has already pointed out that the exponential term in g_{11} impacts the distance between close circular orbits. The effect is mostly important near the $r = 2m$ NakS, where the distance between two close orbits of radii r and $r + dr$ goes to zero, because of the vanishing of the exponential factor. Related to this, the (coordinate) time needed for a photon to radially propagate from $r = 2m$ to an external observer at coordinate $r_{\rm obs}$, that reads

$$\Delta t(\text{from } r = 2m \text{ to } r = r_{\rm obs}) = \int_{2m}^{r_{\rm obs}} \frac{r dr}{r - 2m} \exp\left(-\frac{\Lambda^2 m^2}{4(r^2 - 2mr)}\right), \tag{52}$$

converges for $\Lambda \neq 0$, unlike what happens in the Schwarzschild case. On the other hand, the convergence does not occur for a radial polar photon ($\theta = 0$ or π), since the argument of the exponential term in (47) cancels then. More generally, the convergence does not occur for any photon crossing the Penney's $r = 2m$ sphere at any non-equatorial point, since then $c \neq 0$, so that the exponential goes to 1 when $r \to 2m$. However, let us also spot that despite the convergence of (52), the far observed behavior of a clock at rest close to the Penney's sphere is frozen, even in the equatorial plane, since $g_{00}(r = 2m) = 0$ for any θ, the $\pi/2$ case included. The consequence is that in the equatorial plane, while circular close to the NakS (non-geodesic) motions are frozen (the areal NakS circumference being $4\pi m$), radial infallings are not.

Let us point out that the same observations, i.e., (1) (coordinate) time convergence for a radial photon propagating from any NakS point, but (2) frozen rest clock behavior near any NakS point, also occurs in the FJNW spherical metric [44,45]. This is obvious from the isotropic form of this metric [15]

$$ds^2 = -\left(\frac{r - k}{r + k}\right)^{2\lambda} dt^2 + \left(\frac{r + k}{r}\right)^4 \left(\frac{r - k}{r + k}\right)^{2-2\lambda} (dr^2 + r^2 d\Omega^2) \tag{53}$$

where $\lambda \in]0, 1[$ ($\lambda = 1$ corresponding to the GR Schwarzschild metric).[3] At least for these two metrics, the time convergence for a photon reaching the horizon only concerns orbits approaching points belonging to the NakS location.

IV. BRANS-DICKE VACUUM SOLUTIONS

It is well-known [48] that in a four-dimensional spacetime, the conformal transformation

$$g_{ab} = \Phi \bar{g}_{ab} \tag{54}$$

yields, for any constant ω (supposed to be $> -3/2$)

$$\int \left[\Phi \bar{R} - \frac{\omega}{\Phi}(\bar{\partial}\Phi)^2\right]\sqrt{-\bar{g}} d^4x = \int \left(R - \frac{1}{2}(\partial\varphi)^2\right)\sqrt{-g} d^4x \tag{55}$$

where Φ is any positive scalar function and

$$\varphi = \sqrt{2\omega + 3} \ln \Phi. \tag{56}$$

This means that the vacuum BD action of the BD gravitational field $(\Phi, \bar{g}_{ab})$ identifies with the GR action, with gravitational field g_{ab}, but filled by the (matter source) massless scalar φ. Thence, any solution of (4) is conformally associated to a vacuum BD solution [18]. This (ω) BD solution reads

$$\Phi = \exp\left(\frac{\varphi}{\sqrt{2\omega + 3}}\right) \tag{57a}$$

$$\bar{g}_{ab} = \exp\left(-\frac{\varphi}{\sqrt{2\omega + 3}}\right) g_{ab}. \tag{57b}$$

It is then possible to built a vacuum SAS BD solution from any vacuum static axisymmetric GR solution, using first the correspondance reported in II D. Let us mention that the Kerrlike BD solution built this way from the scalar-Newtonian solution (36d), or (39)–(40), is also asymptotically flat, since φ_N vanishes in far regions.

[3] Let us point out that in the FJNW case, the fact that close to the NakS rest clocks are frozen does not mean that circular motions are frozen, since the NakS circumference is zero, in such a way that an infinite local orbital frequency comes into play. Indeed, close to NakS circular geodesics (that exist for $\lambda < 1/2$) return a divergent far observed frequency [15].

Experiments constrain BD/ST theories to satisfy $\omega_0 > 4.10^5$, where ω_0 is the BD parameter, or the present value of $\omega(\Phi)$ in the ST case [8]. In such circumstances, vacuum BD gravity, and also to some extent ST gravity, is asymptotically equivalent to massless scalar filled GR [15,28,49]. In other words, the solutions obtained solving (27) and (33a)–(33b) can directly serve as vacuum BD/ST solutions in the large ω case, without having to explicitly consider the conformal correspondence (57a)–(57b). Linked to this, let us remind the reader that for any scalar function φ chosen "independently on ω," (57a) yields

$$\Phi = 1 + \frac{\varphi}{\sqrt{2\omega}} + O\left(\frac{1}{\omega}\right). \tag{58}$$

Despite that Φ goes to a constant value, the $\frac{\omega}{\Phi^2}\partial_a\Phi\partial_b\Phi$ term entering the full BD equation does then not vanish in the large ω limit, but results in a $\partial_a\varphi\partial_b\varphi$ contribution. This is coherent with the fact that a vacuum BD solution does not reduce to a GR vacuum solution (that would have been Kerr in the SAS case) in the "$\omega \to \infty$ limit" [49].

Let us remark that while the massive geodesics are not the same in the scalar-Einstein and in the corresponding BD solutions, they are asymptotically identical in the $\omega \to \infty$ limit, since the conformal factor is constant in this limit.

V. THE METHOD BEYOND KERR AS A SEED

In this section, one discusses to which extent the method displayed in Sec. II could be generalized to other contexts, using another SAS, or SAS like, metric than Kerr as a seed. Incidently, this spots why the Kerr metric's properties result in the efficiency of the method in looking for scalar-Einstein solutions.

Let us then consider any four-dimensional metric η_{ab} having the form

$$(\eta_{ab}) = \begin{pmatrix} (\eta_{pq}) & 0 \\ 0 & (\eta_{uv}) \end{pmatrix} \quad \text{with} \quad \partial_p\eta_{ab} = 0 \tag{59}$$

where the meaning of the $(a, b, \ldots)$, $(p, q, \ldots)$, and $(u, v, \ldots)$ indexes are the same as in Sec. II. The metric is Lorentzian, in such a way that $\det(\eta_{ab}) = \det(\eta_{pq})\det(\eta_{uv}) < 0$, but it is not decided which part[4] [i.e., $\det(\eta_{pq})$ or $\det(\eta_{uv})$] is <0. Since $\eta_{uv}(x^w)dx^u dx^v$ can be considered as a two-dimensional metric by itself, it is not restrictive to impose that the η_{uv} part of η_{ab} is conformally flat. Thence, let us use coordinates such that

$$\eta_{uv} = e^F m_{uv} \tag{60}$$

where F depends on the (x^u) coordinates, and m_{uv} is diagonal with $|m_{11}| = |m_{22}| = 1$. It could be worth remarking that the conformal form (60) is preserved by some well-suited redefinition of the coordinates (see Appendix B).

Let us consider another metric, related to η_{ab} by

$$g_{pq} = e^A \eta_{pq} \tag{61a}$$

$$g_{uv} = e^B \eta_{uv} = e^{B+F} m_{uv} \tag{61b}$$

where

$$\partial_p A = \partial_p B = 0. \tag{62}$$

A lengthy but straightforward calculation results in the following relations between the Ricci components of the two metrics

$$e^{B-A}R_{pq}(g) = R_{pq}(\eta) - \frac{1}{2}e^{-F}m^{ux}[(\partial_u\eta_{pq})\partial_x A + \eta_{pq}(\partial_u\partial_x A + \partial_u A\partial_x A + (\partial_u \ln\sqrt{|\bar{\eta}|})\partial_x A)] \tag{63}$$

$$\begin{aligned} 2R_{uv}(g) = {} & 2R_{uv}(\eta) + (\partial_v \ln\sqrt{|\bar{\eta}|})\partial_u B + (\partial_u \ln\sqrt{|\bar{\eta}|})\partial_v B - m_{uv}m^{wx}[(\partial_w \ln\sqrt{|\bar{\eta}|})\partial_x B + \partial_w\partial_x B] \\ & + \partial_v(B + F - \ln\sqrt{|\bar{\eta}|})\partial_u A + \partial_u(B + F - \ln\sqrt{|\bar{\eta}|})\partial_v A - m_{uv}m^{wx}\partial_w(B+F)\partial_x A - \partial_u A\partial_v A - 2\partial_u\partial_v A \end{aligned} \tag{64}$$

where $R_{ab}(h)$ denotes the Ricci curvature tensor components of the metric h_{ab}, and where $\bar{\eta} = \det(\eta_{pq})$. Let us remark that from (63), one obtains[5]

$$e^{B-A}\left[R_{pq}(g) - \frac{1}{2}g^{rs}R_{rs}(g)g_{pq}\right] = R_{pq}(\eta) - \frac{1}{2}\eta^{rs}R_{rs}(\eta)\eta_{pq} - \frac{1}{2}e^{-F}\sqrt{|\bar{\eta}|}m^{ux}\partial_u\left(\frac{\eta_{pq}}{\sqrt{|\bar{\eta}|}}\right)\partial_x A. \tag{65}$$

[4]Only in the $\det(\eta_{uv}) > 0$ case can the metric be interpreted as describing an SAS spacetime.

[5]Let us spot that, for any metric h_{ab} having the form (59), $R_{pq}(h) - \frac{1}{2}h^{rs}R_{rs}(h)h_{pq}$ is *not* the (pq) component of the Einstein tensor, since $h^{rs}R_{rs}(h)$ is not the h_{ab} scalar curvature $R(h) = h^{ab}R_{ab}(h) = h^{rs}R_{rs}(h) + h^{uv}R_{uv}(h)$, but just part of it.

The (65) relation enlightens the origin of the $A = 0$ conclusion derived in II A from (12): it results from (1) the Kerr Ricci flatness, and (2) the fact that the (pq) Ricci components of the scalar-Einstein metric are null because of the SAS symmetry and of the very specific source term entering (4).

Specifying to metrics (61a)–(61b) having $A = 0$, (63)–(64) reduce to

$$e^B R_{pq}(g) = R_{pq}(\eta)$$
$$2R_{uv}(g) = 2R_{uv}(\eta) + Q_{uv} \tag{66}$$

where

$$Q_{uv} \equiv (\partial_v \ln \sqrt{|\bar\eta|})\partial_u B + (\partial_u \ln \sqrt{|\bar\eta|})\partial_v B - m_{uv} m^{wx}[(\partial_w \ln \sqrt{|\bar\eta|})\partial_x B + \partial_w \partial_x B]. \tag{67}$$

Here another specificity of the Kerr metric came into play in the case focused in Sec. II: namely, the fact that $\sqrt{|\bar k|}$ is precisely the product $\sinh\rho \sin\theta$ (up to a constant factor), in the (ρ, θ) coordinates defined by (18). Thence, the transform (B1a), which preserves the (60) form (see Appendix B), results in $\sqrt{|\bar\eta|} \propto \alpha$ in the new coordinates. Thanks to this, the quantities (67) achieve the form ($m_{\alpha\alpha} = m_{\beta\beta} = 1$ for the Kerr metric)

$$Q_{\alpha\alpha} = \frac{1}{\alpha}\partial_\alpha B - \partial_\alpha\partial_\alpha B - \partial_\beta\partial_\beta B \tag{68a}$$

$$Q_{\beta\beta} = -\frac{1}{\alpha}\partial_\alpha B - \partial_\alpha\partial_\alpha B - \partial_\beta\partial_\beta B \tag{68b}$$

$$Q_{\alpha\beta} = \frac{1}{\alpha}\partial_\beta B \tag{68c}$$

that enter the equations (33a), (33b), (A6a) and (A6b).

VI. CONCLUSION AND OUTLOOK

One has established a new nontrivial one to one correspondence that allows us to build new SAS massless scalar-GR solutions, of any mass and angular momentum, from any static axisymmetric vacuum GR solution. While this in turn also results in a way to obtain new SAS vacuum BD solutions, one has pointed out that the so obtained massless scalar-GR solutions can also directly serve as (large ω) vacuum BD/ST solutions for practical purposes.

It was claimed in [25] that a Kerr-like BD solution should: (1) only depend on the three (m, a, ω) parameters, (2) go to Kerr spacetime in the $\omega \to \infty$ limit, and (3) return Schwarzschild's spacetime for $a = 0$. From the results obtained in this paper, it seems that the truth is by far more complex. The solutions explicitly presented in this paper can be interpreted as limit of BD solutions for $\omega \to \infty$, but they differ from Kerr spacetime. The particular case (47) of the scalar-Newtonian solution has $a = 0$ but differs from Schwarzschild. The fact that the (39) solution, for instance, depends on a parameter Λ, besides (m, a), shows that the (ω)BD solution built from it using (57a)–(57b) results in a family of Kerr-like solutions that depends on more than the three (m, a, ω) parameters.

Let us mention that a lot of authors explicitly or implicitly still suppose that (2) is true (see for instance [26] or [27]). Such a presupposition results in a drastic impoverishment of the (large ω) ST potential predictions. Among these the drastically different behavior of far observed orbital frequencies in the spherical case with respect to GR [15]. It would be of interest to know whether such a different from GR behavior could also occur in the rotating case. This requires exploring more general SAS solutions of (4) with $g_{pq} \neq k_{pq}$, which means ruling out the restrictive hypothesis (1a)–(1b).

The possibility to extend the method using seed metrics that differ from Kerr has also been discussed. This incidently points out the specificities of the Kerr metric that make the method particularly efficient when choosing this metric as a seed. Nonetheless, this issue certainly deserves further investigations.

ACKNOWLEDGMENTS

I would like to thank here an anonymous referee for several very interesting historical references, especially concerning the often named JNW (but accordingly here renamed FJNW) scalar-Einstein solution. I would also like to thank Professor Bernard F. Whiting, who suggested discussing the method beyond the Kerr metric as a seed (Sec. V), and who also suggested the discussion entering the Sec. III A 1.

APPENDIX A: CALCULATIONS USING THE (α,β) COORDINATES

Using (18) and (23), one obtains, in the $(\tilde x^a) \equiv (t, \alpha, \beta, \phi)$ coordinates

$$ds^2 = \tilde g_{ab} d\tilde x^a d\tilde x^b$$
$$= k_{00} dt^2 + 2k_{03} dt d\phi + k_{33} d\phi^2 + G(d\alpha^2 + d\beta^2) \tag{A1}$$

where

$$G = \frac{\Sigma e^B}{S^2 + s^2}. \tag{A2}$$

From (23) and the trigonometric identities, one could explicitly obtain (c, s, C, S) in terms of (α, β), but this is not needed here. Indeed, since $\sqrt{-\tilde g} = G s \sqrt{\Delta} = \sqrt{m^2 - a^2}\alpha G$, the d'Alembertian operator reads

$$\Box\varphi = \frac{1}{\sqrt{-\tilde{g}}}[\partial_\alpha(\sqrt{-\tilde{g}}g^{\alpha\alpha}\partial_\alpha\varphi) + \partial_\beta(\sqrt{-\tilde{g}}g^{\beta\beta}\partial_\beta\varphi)]$$
$$= \frac{\sqrt{m^2 - a^2}}{\sqrt{-\tilde{g}}}[\partial_\alpha(\alpha\partial_\alpha\varphi) + \alpha\partial_\beta\partial_\beta\varphi] \quad \text{(A3)}$$

from which one directly obtains (27).

Writing now (A1) in the form

$$ds^2 = k_{00}dt^2 + 2k_{03}dtd\phi + k_{33}d\phi^2 + He^B(d\alpha^2 + d\beta^2) \quad \text{(A4)}$$

(the $B = 0$ version of which being Kerr in $(\tilde{x}^a)$ coordinates), the $\tilde{\Gamma}^*_{**}$ connexion components achieve a form like (5a)–(5c) with $A = 0$, but with (α, β) instead of (r, θ). Thence, (31) is replaced by

$$\tilde{K}^p_{vp}\partial_u B + \tilde{K}^p_{up}\partial_v B - \frac{1}{\alpha He^B}k_{uv}[\partial_\alpha(\alpha\partial_\alpha B) + \alpha\partial_\beta\partial_\beta B]$$
$$= 2\partial_u\varphi\partial_v\varphi \quad \text{(A5)}$$

with $\tilde{K}^p_{up} = \partial_u \ln\alpha$. Writing out the $(uv) = (\alpha\beta)$ component directly returns (33b). The $(uv) = (\alpha\alpha)$ and $(uv) = (\beta\beta)$ components read

$$\partial_\alpha B - \alpha\partial_\alpha\partial_\alpha B - \alpha\partial_\beta\partial_\beta B = 2\alpha(\partial_\alpha\varphi)^2 \quad \text{(A6a)}$$

$$\partial_\alpha B + \alpha\partial_\alpha\partial_\alpha B + \alpha\partial_\beta\partial_\beta B = -2\alpha(\partial_\beta\varphi)^2. \quad \text{(A6b)}$$

Summing (A6a) and (A6b) returns (33a). Inserting ∂B from (33a)–(33b), the remaining equation, for instance (A6b), is satisfied thanks to (27).

APPENDIX B: COORDINATES' CHANGE THAT PRESERVE THE FLATNESS CONFORMAL FORM

The form (60) is preserved through some specific transformations of the coordinates, modulo a redefinition of F. Indeed, the three transformations (x and y being here the two u-like coordinates, not indexes)

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\sin x \sinh y\\ \cos x \cosh y\end{pmatrix} \quad \text{(B1a)}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\sin x \sin y\\ \cos x \cos y\end{pmatrix} \quad \text{(B1b)}$$

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\sinh x \sinh y\\ \cosh x \cosh y\end{pmatrix} \quad \text{(B1c)}$$

result in, respectively

$$d\alpha^2 + d\beta^2 = (\sinh^2 y + \sin^2 x)(dx^2 + dy^2) \quad \text{(B2a)}$$

$$-d\alpha^2 + d\beta^2 = (\sin^2 y - \sin^2 x)(-dx^2 + dy^2) \quad \text{(B2b)}$$

$$-d\alpha^2 + d\beta^2 = (\sinh^2 y - \sinh^2 x)(-dx^2 + dy^2). \quad \text{(B2c)}$$

In the case considered in subsection II B, (B1a) corresponds to the coordinates change (23), where (x, y) correspond to the (ρ, θ) coordinates. The form (A1) of the metric then results from (B2a).

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