Spacetimes foliated by nonexpanding null surfaces in the presence of a cosmological constant

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We prove that every solution to vacuum Einstein's equations with possibly nonzero cosmological constant that is foliated by nonexpanding null surfaces transversal to a single nonexpanding null surface belongs to the family of the near (extremal) horizon geometries. Our results are local, and hold in a neighborhood of the single nonexpanding null surface.

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I. INTRODUCTION

A. Motivation

Spacetimes foliated by nonexpanding null surfaces were introduced by Kundt in the context of explicit ("exact") solutions to Einstein's equations describing plane fronted gravitational waves [1]. In that framework the primary structure is a congruence of null geodesic curves that is nondiverging, shear-free and surface forming [2–4]. Nonexpanding null surfaces [called also nonexpanding horizons (NEHs)] appear as a main tool in the local theory of black holes [5–8], are applicable also to cosmological horizons and the Killing horizons in mathematical relativity, or are used as the boundaries of conformally completed asymptotically flat spacetimes and isolated regions of asymptotically de Sitter spacetimes [9,10]. The theory of the geometry of NEHs was developed and used to interpret the constraints following from Einstein's equations satisfied by a surrounding spacetime M. Isolated horizons (IHs) admit a stronger null symmetry and their class was studied individually. Briefly, one could say that every NEH is a Killing horizon to the zeroth order, while each IH is a Killing horizon to the first order. The physical difference the stronger symmetry makes, is that proper NEHs (that is nonisolated) are used as boundaries of conformally completed asymptotically flat spacetimes [11]. In that case, the lack of the stronger symmetry means gravitational radiation. The IHs, on the other hand, are particularly useful in the description of stationary black hole horizons. Every IH is either extreme or nonextreme, depending on its surface gravity. The extreme IH case is particularly interesting for several reasons. To begin with, the Einstein constraints on extreme IH's geometry take a very nontrivial form [5,6]. The resulting equations, also called near horizon geometry (NHG) equation [12], have been studied intensively [12–15]. However, despite many results, the existence issue of generic, nonaxisymmetric extremal IHs of topologically spherical sections is still unsolved. That problem is relevant for the existence of extremal Killing horizons and potential filling gaps in the black hole uniqueness theorems. Indeed, the extremal case is the least known in the context of the black hole uniqueness [16].

The geometric theory of NEHs can also be applied to the null nonexpanding surfaces foliating Kundt's class spacetimes. The result is a surprising relation between the Kundt's class spacetimes and solutions to the extremal IH constraints [12,17,18]. Indeed, the foliation assumption implies a constraint on NEH geometry. The resulting equation turns out to be a necessary condition for each leaf of the foliation to be transversal to an extreme IH that may or may not exist in the given spacetime. In that way, every Kundt's spacetime defines a family of solutions to the extreme IH equation. Conversely, for every solution to, say, vacuum extreme IH equation there is a special case of the vacuum Kundt's class spacetime [17]. Each of those spacetimes can be obtained from a neighborhood of an extremal Killing horizon by the Bardeen-Horowitz-Reall limit [19,20]. Hence, physically, they can be interpreted as spacetimes near extremal black hole horizons. That is why they are called the near horizon geometries [12]. Remarkably, the NHGs are still exact solutions to Einstein's equations.

What emerges from those considerations is a generalization of NHG: spacetime foliated by NEHs emanating from a single IH. Each generalized in that way NHG spacetime still contains an IH, hence it provides a new example of geometry near IH. A natural question is, whether that class of spacetimes contains more solutions to the vacuum Einstein's equations than the original NHG spacetimes. That question was raised in a previous paper [18] where the vanishing cosmological constant case was considered,

$$\Lambda = 0.$$

In the current paper we investigate that question in the case of nonzero cosmological constant

$$\Lambda \neq 0.$$

This time, we approach the problem with a different technique, namely by using the black hole holograph theorem and its applications [21,22]. The black hole holograph theorem concerns spacetimes that contain two intersecting NEHs. Each vacuum spacetime of this class is determined by the NEHs geometry locally, to the future and to the past of their intersection.

B. Definitions and notation

Throughout the paper we will use the following index notation: the lower case Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots$ correspond to the bundles $T(M)$ and $T^*(M)$ tangent and cotangent, respectively, to four-dimensional spacetime M. The lower case Latin indices $a, b, c, \ldots$ correspond to bundles tangent and cotangent to three-dimensional surfaces in M, while the upper case Latin indices $A, B, C, \ldots$ correspond to tangent or cotangent bundles of 2-surfaces.

Nonexpanding horizon (NEH) is a three-dimensional null surface, say, $\mathcal{H}$ contained in spacetime M, such that a null vector field ℓ^a tangent to and orthogonal to $\mathcal{H}$ is nonexpanding. In the theory of NEHs we assume that the spacetime metric tensor $g_{\mu\nu}$ has the signature $-+++$, satisfies the Einstein equations

$$R^{(4)}{}_{\mu\nu} - \frac{1}{2} R^{(4)} g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad (1)$$

and the stress energy tensor satisfies at $\mathcal{H}$ the energy inequalities:

$$T^{\mu}{}_{a} \ell^a T_{\mu b} \ell^b \leq 0 \qquad (2)$$

and

$$\ell^\mu \text{ is future directed} \Rightarrow T^{\mu\nu} \ell_\nu \text{ is future directed.} \qquad (3)$$

Then, the vanishing of the expansion of ℓ^a implies vanishing of the shear, hence the intrinsic degenerate metric tensor g_{ab} induced in $\mathcal{H}$ is Lie dragged by the flow of ℓ,

$$\mathcal{L}_\ell g_{bc} = 0. \qquad (4)$$

It follows from (4) that the spacetime covariant derivative ∇_μ preserves the tangent bundle $T(\mathcal{H})$ and therefore induces therein an intrinsic derivative ∇_a. In that way, every NEH $\mathcal{H}$ is endowed with an intrinsic geometry (g_{ab}, ∇_a). A rotation 1-form potential $\omega^{(\ell)}$ and surface gravity $\kappa^{(\ell)}$ depend on a choice of the null vector ℓ, and are defined as follows:

$$\nabla_a \ell =: \omega_a^{(\ell)} \ell, \qquad \kappa^{(\ell)} := \ell^a \omega_a^{(\ell)}. \qquad (5)$$

They satisfy the zeroth law of NEH mechanics,

$$\nabla_a \kappa^{(\ell)} = \mathcal{L}_\ell \omega_a^{(\ell)}. \qquad (6)$$

While (4) is true for every null vector field ℓ tangent to $\mathcal{H}$, in general

$$[\mathcal{L}_\ell, \nabla_a] \neq 0. \qquad (7)$$

We call $\mathcal{H}$ an isolated horizon (IH), if it admits a stronger symmetry, namely if there is on $\mathcal{H}$ a (nowhere vanishing) null vector field ℓ such that

$$[\mathcal{L}_\ell, \nabla_a] = 0. \qquad (8)$$

For every IH

$$\kappa^{(\ell)} = \text{const}. \qquad (9)$$

IH is called extreme whenever

$$\kappa^{(\ell)} = 0. \qquad (10)$$

In the paper we also consider two transversal NEHs, say $\mathcal{H}$ and $\tilde{\mathcal{H}}$, that is such that

$$S := \mathcal{H} \cap \tilde{\mathcal{H}} \qquad (11)$$

is a two-dimensional spacelike surface. In that case, we chose a pair of null vector fields ℓ and $\tilde{\ell}$ tangent to $\mathcal{H}$ and, respectively, $\tilde{\mathcal{H}}$, such that

$$\ell^\mu \tilde{\ell}_\mu|_S = -1. \qquad (12)$$

Then, the pull back to S of $\omega^{(\ell)}$ and $\tilde{\omega}^{(\tilde{\ell})}$, respectively are related to each other by

$$\omega_A^{(\ell)} = -\tilde{\omega}_A^{(\tilde{\ell})}. \qquad (13)$$

II. EINSTEIN SPACETIMES FOLIATED BY NEHs TRANSVERSAL TO A SINGLE NEH: THE GENERAL CASE

It will be useful for the reader if we briefly announce the plan and content of the remainder of the article. Below we introduce a class of spacetimes that generalizes the idea of NHG. Each of the spacetimes is formed by NEHs emanating from a single NEH. The spacetime metric tensor satisfies Einstein's equations with an energy-momentum tensor subject to inequalities assumed in the NEH theory. Clearly, the class contains spacetimes that are not NHGs. In Sec. III we impose the vacuum Einstein equations with a possibly nonzero cosmological constant. Then, we prove that the only solutions are the NHG spacetimes. We demonstrate details of the proof that were missing in the previous paper in which the $\Lambda = 0$ case was concerned [18]. Also, we introduce now a new method of identification of the considered spacetimes by an application of the black hole holograph (BHH) theorem [21,22].

Let us start with a general case of spacetime $(M, g_{\mu\nu})$ foliated by NEHs transversal to a single NEH. We admit an arbitrary energy-momentum tensor $T_{\mu\nu}$ in (1) and use only the energy inequalities (2) and (3) to write a general formula for the metric tensor in suitably adapted coordinates.

Let

$$\mathbb{R} \ni u \mapsto \mathcal{H}_u \subset M \tag{14}$$

be the family of nonexpanding null surfaces, and $\tilde{\mathcal{H}}$ be the transversal nonexpanding null surface. Locally, in a suitable neighborhood of a point of $\tilde{\mathcal{H}}$, we choose the function u to be one of spacetime coordinates. A vector field ℓ^μ defined by

$$\ell_\mu = -\nabla_\mu u \tag{15}$$

is tangent to each of the leaves $\mathcal{H}_u$, null, and satisfies

$$\ell^\mu \nabla_\mu \ell = 0. \tag{16}$$

Define the second coordinate, a function v, such that

$$\ell^\mu \nabla_\mu v = 1, \quad \text{and} \quad v_{|\tilde{\mathcal{H}}} = 0. \tag{17}$$

Locally, for every point of $\tilde{\mathcal{H}}$, we may restrict our considerations to a neighborhood in M, such that every $\mathcal{H}_u$ and $\tilde{\mathcal{H}}$ has the topology

$$\mathcal{H}_u, \tilde{\mathcal{H}} \equiv S \times \mathbb{R}, \tag{18}$$

where S is a two-dimensional surface, and the intersection topologically is equivalent to S,

$$\mathcal{H}_u \cap \tilde{\mathcal{H}} \equiv S. \tag{19}$$

The remaining two spacetime coordinates x^A, $A = 1, 2$ are introduced first on the intersection

$$\mathcal{H}_{u_0} \cap \mathcal{H}, \tag{20}$$

for arbitrarily chosen value u_0 of the parameter u, next extended along $\tilde{\mathcal{H}}$ such that they are constant along the null generators, and finally along each $\mathcal{H}_u$ by

$$\ell^\mu \nabla_\mu x^A = 0. \tag{21}$$

In the resulting coordinates

$$(x^\mu) = (x^A, v, u) \tag{22}$$

the metric tensor g takes the following form:

$$g_{\mu\nu} dx^\mu dx^\nu = g_{AB} dx^A dx^B - 2du[dv + W_A dx^A + H du]. \tag{23}$$

The coordinates are defined up to transformations

$$u' = U'(u), \qquad v' = \left(\frac{dU'}{du}\right)^{-1} v, \qquad x'^A = X'^A(x^B). \tag{24}$$

The components g_{AB}, W_A and H of the metric tensor are constrained by our assumption on the NEHs $\tilde{\mathcal{H}}$ and $\mathcal{H}_u$. We analyze the constraints now.

To begin with, the assumption that $\tilde{\mathcal{H}}$ is null, the second condition in (17) and the assumption that x^A are constant along the null generators of $\tilde{\mathcal{H}}$ imply that the vector field ∂_u is orthogonal to $\tilde{\mathcal{H}}$, hence

$$W_A(x, 0, u) = 0 = H(x, 0, u), \tag{25}$$

where we denoted $x \coloneqq (x^A)$.

Second, the energy inequalities (2) and (3) imply constrains on the dependence on u along $\tilde{\mathcal{H}}$, and on the dependence on v along every $\mathcal{H}_u$ of some functions, even if we do not specify the energy-momentum tensor $T_{\mu\nu}$ in (1). Indeed, to begin with

$$g_{AB}(x, v, u)_{,v} = 0, \quad \text{and} \quad g_{AB}(x, 0, u)_{,u} = 0. \tag{26}$$

The first equality follows from the nonexpanding of each $\mathcal{H}_u$ and every metric tensor (23) with the first Eq. (26) is called the Kundt's class metric [1–3]. The second equality follows from the nonexpanding of $\tilde{\mathcal{H}}$. Consequently,

$$g_{AB}(x, v, u) = g_{AB}(x). \tag{27}$$

Next, for every $\mathcal{H}_u$, the rotation 1-form potential is

$$\omega^{(\ell)} = \omega^{(\ell)}_A dx^A = \frac{1}{2} W_{A,v} dx^A. \tag{28}$$

The 0th law (6) and (16) imply

$$W_A(x, v, u)_{,vv} = 0. \tag{29}$$

This is one of the well-known vacuum Kundt equations (see [2]). Hence,

$$W_A(x, v, u) = 2v\omega^{(\ell)}_A(x, u). \tag{30}$$

One can also notice that at the transversal nonexpanding null surface $\tilde{\mathcal{H}}$, as a tangent null vector field we can use the vector

$$\tilde{\ell} := \partial_u. \tag{31}$$

Since by the construction

$$\tilde{\ell}^\mu \ell_\mu = -\partial_u u = -1 \quad \text{at } \tilde{\mathcal{H}} \cap \mathcal{H}_u \tag{32}$$

for every value of the parameter u, the family of 1-form potentials $\omega^{(\ell)}_A(x,u)$ (labeled by u and each defined on the corresponding $\mathcal{H}_u$) is related to a single rotation 1-form potential $\tilde{\omega}^{(\tilde{\ell})}(x,u)$ on $\tilde{\mathcal{H}}$,

$$\tilde{\omega}^{(\tilde{\ell})}(x,u) = \tilde{\kappa}^{(\tilde{\ell})}(x,u)du - \omega^{(\ell)}_A(x,u)dx^A, \tag{33}$$

where the surface gravity $\tilde{\kappa}^{(\tilde{\ell})}$ can be calculated from the function H, namely

$$\tilde{\kappa}^{(\tilde{\ell})}(x,u) = -\partial_v H(x,0,u). \tag{34}$$

In the vacuum case investigated in the next subsection, the surface gravity $\tilde{\kappa}^{(\tilde{\ell})}$ will be zero eventually. Proving the vanishing of $\tilde{\kappa}^{(\tilde{\ell})}$ will be the main technical task. The geometric meaning of that property is that the function u provides an affine parametrization of the null geodesics in $\tilde{\mathcal{H}}$.

Of course the 0th law (6) still applies to $\tilde{\kappa}^{(\tilde{\ell})}$ and $\tilde{\omega}^{(\tilde{\ell})}$, and it reduces to

$$\partial_A \tilde{\kappa}^{(\tilde{\ell})}(x,u) = -\partial_u \omega^{(\ell)}_A(x,u). \tag{35}$$

Hence, ruling out the surface gravity $\tilde{\kappa}^{(\tilde{\ell})}(x,u)$ we obtain the constraint on H and $\omega^{(\ell)}_A$,

$$\partial_A \partial_v H(x,0,u) = \partial_u \omega^{(\ell)}_A(x,u). \tag{36}$$

In summary, the functions $g_{AB}(x,v,u), W_A(x,v,u)$ and $H(x,v,u)$ in (23) satisfy (27), (30), and (25) where the functions $g_{AB}(x)$ and $\partial_v H(x,v,u)$ are arbitrary, and the functions $\omega^{(\ell)}_A(x,u)$ are constrained by (36). Now, the surfaces

$$\mathcal{H}_{u'}:\ u = u' \tag{37}$$

are NEHs for every u', and so is the surface

$$\tilde{\mathcal{H}}:\ v = 0. \tag{38}$$

The momentum energy tensor, determined by (1) and the resulting metric tensor (23), satisfies the energy inequalities (2) and (3).

The pull back $R^{(4)}{}_{AB}$ of spacetime Ricci tensor to any 2-surface such that

$$u = \text{const}, \qquad v = \text{const} \tag{39}$$

can be expressed by the metric tensor g_{AB} and the 1-form $\omega^{(\ell)}_A$ as follows:

$$\frac{1}{2}R^{(4)}{}_{AB} = \nabla_{(A}\omega^{(\ell)}_{B)} - \omega^{(\ell)}_A \omega^{(\ell)}_B + \frac{1}{2}R_{AB}, \tag{40}$$

where ∇_A and R_{AB} are the covariant derivative and the Ricci tensor, respectively, of the metric tensor $g_{AB}(x)$. This equation written in an equivalent way in terms of the spacetime metric components is very well known and can be found in [2,3]. For us, it is a relation between the spacetime Ricci tensor and the data $(g_{AB}, \omega^{(\ell)}_B)$. This is not a constraint, though, unless additional assumptions are made about $T_{\mu\nu}$.

On the similar basis, the $R^{(4)}{}_{vu}(x,v,u)$ component of the spacetime Ricci tensor is determined by $g_{AB}(x)$, $\omega^{(\ell)}_A(x,u)$ and the function $H(x,v,u)$ [2], namely

$$R^{(4)}{}_{uv}(x,v,u) = g^{AB}(x)[-\nabla_A \omega^{(\ell)}_B(x,u) + 2\omega^{(\ell)}_A(x,u)\omega^{(\ell)}_B(x,u)] + H_{,vv}(x,v,u) \tag{41}$$

and at this point is an identity, rather than a constraint.

III. EINSTEIN SPACETIMES FOLIATED BY NEHs TRANSVERSAL TO A SINGLE NEH: THE VACUUM CASE

Suppose now that the metric tensor (23) defined in the previous subsection satisfies the vacuum Einstein equations,

$$R^{(4)}{}_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{42}$$

In order to find the set of (local) solutions, one could impose the equations (42) on the spacetime metric tensor introduced in the previous subsection.

A. The vacuum case with an additional symmetry assumption

It is easy to solve the resulting equations completely and explicitly for the subset of spacetimes that satisfy the following additional symmetry:

$$\partial_u W(x,v,u) = 0. \tag{43}$$

We will focus in this subsection on that case.

From the point of view of the NEH $\tilde{\mathcal{H}}$, in this case

$$\partial_u \tilde{\omega}^{(\tilde{\ell})}_A(x,u) = 0, \tag{44}$$

hence

$$\partial_A \tilde{\kappa}^{(\tilde{\ell})}(x,u) = 0, \tag{45}$$

that is, the surface gravity of the vector field $\tilde{\ell}$ is a function of u only. That implies that there is a coordinate transformation

$$u = U(u'), \qquad v = \frac{V(v',u')}{dU/du'} \tag{46}$$

such that

$$\tilde{\kappa}^{(\tilde{\ell}')} = 0, \tag{47}$$

hence

$$\partial_{v'} H'(x,0,u') = 0. \tag{48}$$

Therefore, without lack of generality, we suppose from now on that

$$\partial_v H(x,0,u) = 0. \tag{49}$$

From the point of view of the family of NEHs $\mathcal{H}_u$, the equality (36) implies that the rotation 1-form potentials (labeled by u) are actually all the same, that is there exists a 1-form $\omega_A(x)$ such that

$$\omega_A^{(\ell)}(x,u) = \omega_A(x). \tag{50}$$

Next, for every $g_{AB}(x)$ and $\omega_A(x)$, the identity (41) becomes an equation on H. Its integration implies

$$H(x,v) = \frac{1}{2}(\nabla_A \omega^A(x) - 2\omega_A(x)\omega^A(x) - \Lambda)v^2. \tag{51}$$

The identity (40) now becomes a constraint on $g_{AB}(x)$, $\omega_A(x)$, and Λ, namely

$$\nabla_{(A}\omega_{B)} - \omega_A\omega_B + \frac{1}{2}R_{AB} - \frac{1}{2}\Lambda g_{AB} = 0. \tag{52}$$

In that way all the components of the metric tensor $g_{\mu\nu}(x,v,u)$ have been determined by a metric tensor $g_{AB}(x)$ and a 1-form $\omega_A(x)$ defined on the two-dimensional manifold S, provided they satisfy the new constraint (52). Indeed, we have established a formula:

$$g = g_{AB}(x)dx^A dx^B - 2du\bigg(dv + 2v\omega_A(x)dx^A + \frac{1}{2}v^2(\nabla_A\omega^A(x) - 2\omega_A(x)\omega^A(x) - \Lambda)du\bigg) =: g^{(NHG)}. \tag{53}$$

Finally, it can be checked by inspection that the remaining components of the spacetime Riemann tensor do satisfy the vacuum equations (42). The spacetimes (53) are known as the NHG solutions.

B. The vacuum case without the additional assumption

In this subsection we relax the assumption (43) and turn to a general case of the metric tensor $g_{\mu\nu}$ (23)–(42). We will show now by using the results on extremal IHs [5] [see Eqs. (B.53) and (B. 54) therein] that $g_{\mu\nu}$ necessarily admits the symmetry (43), unless the Weyl tensor vanishes on $\tilde{\mathcal{H}}$. In the latter case, the spacetime has a triple (at least) principal null direction. In a conformally nonflat case, that makes the foliation $u \mapsto \mathcal{H}_u$ unique. Application of the black hole holograph theorem [21,22] completes our solution to the problem.

The proof proceeds as follows. For an arbitrary value of the parameter u, consider the corresponding intersection

$$\tilde{\mathcal{H}} \cap \mathcal{H}_u, \tag{54}$$

thereon, the induced metric tensor $g_{AB}(x)$ (27) and the rotation 1-form potential $\omega_A^{(\ell)}(x,u)$ (28). It follows from (40) and (42) that the data $[g_{AB}(x), \omega_A^{(\ell)}(x,u)]$ satisfies a constraint

$$\nabla_{(A}\omega_{B)}^{(\ell)}(x,u) - \omega_A^{(\ell)}(x,u)\omega_B^{(\ell)}(x,u) + \frac{1}{2}R_{AB}(x) - \frac{1}{2}\Lambda g_{AB}(x) = 0. \tag{55}$$

Hence, the given $g_{AB}(x)$, we have a family of solutions $\omega_A^{(\ell)}(x,u)$ (labeled by u) to the Eq. (55). Moreover, the integration of (36) with respect to u from u_0 to u_1 shows that there exists a function $b(x,u_0,u_1)$ such that

$$\omega_A^{(\ell)}(x,u_1) = \omega_A^{(\ell)}(x,u_0) + \nabla_A b(x,u_0,u_1). \tag{56}$$

The issue of the existence of two solutions $\omega_A^{(\ell)}(x,u_1)$ and $\omega_A^{(\ell)}(x,u_0)$ to the Eq. (55) related to each other by the gradient of a function was investigated in [5]. According to Proposition 1 in [13] 1-form $\omega_A^{(\ell)}(x,u)$ satisfies the following equation:

$$\bar{m}^A(\nabla_A + 3\omega_A^{(\ell)}(x,u))\Psi_2 = 0, \tag{57}$$

where m^A is a null frame defined locally on S and $\Psi_2(x)$ is a complex valued invariant of the tensor g_{AB} and the 1-form ω_A, namely

$$\Psi_2 = -R + \frac{\Lambda}{6} - i\eta^{AB}\nabla_A\omega_B \tag{58}$$

(η_{AB} is the area 2-form defined by g_{AB}). It is easily seen now that (56) and (57) imply

$$\Psi_2(x)\nabla_A b(x,u_0,u_1) = 0. \tag{59}$$

In the consequence, if

$$\Psi_2 \neq 0, \tag{60}$$

at a point of $\tilde{\mathcal{H}}\cap\mathcal{H}_u$ of the coordinates $(x,0,u)$, then for all the values of the variables $x' = (x'^1, x'^2)$ sufficiently close to x,

$$\omega^{(\ell)}_A(x',u_1) = \omega^{(\ell)}_A(x',u_0), \quad \text{for all } u_0, u_1 \tag{61}$$

and we are back in the case solved in the previous section. The same conclusion is valid for all the spacetime M, if (60) holds for an open and dense subset of $\tilde{\mathcal{H}}\cap\mathcal{H}_u$.

From the spacetime point of view, for every point $p\in M$, the number $\Psi_2(x(p))$ is one of the Newman-Penrose coefficients of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ of the spacetime metric tensor $g_{\mu\nu}$, namely

$$\Psi_2 = \frac{1}{2}C_{\alpha\beta\gamma\delta}\ell^\alpha\ell'^\beta(\ell^\gamma\ell'^\delta - m^\gamma\bar{m}^\delta), \tag{62}$$

where $(m^\alpha, \bar{m}^\alpha, \ell^\alpha, \ell'^\alpha)$ is a null frame (adapted to the foliation $\mathcal{H}_u$).

That component of the Weyl tensor defined by a frame adapted to a NEH is constant along it. In our case it is constant along the horizons $\mathcal{H}_u$ and also $\tilde{\mathcal{H}}$. That fact is manifested by the independence of Ψ_2 of the variables v and u. Suppose that

$$\Psi_2(x) = 0 \tag{63}$$

for an open set of the values of coordinates x^A, that is an open subset of the intersection $\tilde{\mathcal{H}}\cap\mathcal{H}_u$, for any value of u. Then, the Weyl tensor component Ψ_2 vanishes on an open subspace $M'\subset M$ generated by the planes labeled by x such that (63), and arbitrary pairs of values of (v,u). When Ψ_2 vanishes at a point of a NEH, then the given null direction orthogonal to the horizon becomes (at least) a triple principal null direction of the Weyl tensor. In our case, that applies to the NEHs $\mathcal{H}_u$ and $\tilde{\mathcal{H}}$. In the consequence, the foliation $u\mapsto\mathcal{H}_u$ of M' is orthogonal to the at least triple null direction of the Weyl tensor. At each point of the subset of $\tilde{\mathcal{H}}\cap\mathcal{H}_u$, there two distinct principal null directions, one orthogonal to $\tilde{\mathcal{H}}$ and one orthogonal to $\mathcal{H}_u$ to each at least triple, which makes the Weyl tensor vanish at those points. Now, if the Weyl tensor is still not 0 on an open and dense subset of M', then the foliation $u\mapsto\mathcal{H}_u$ is a unique NEH foliation admitted in M' by this metric tensor, up to reparametrizations

$$u = U(u'). \tag{64}$$

That follows from the uniqueness of a triple/fourfold null direction of a nonvanishing Weyl tensor. That observation will be useful below, when we characterize $g_{\mu\nu}$ by applying the black hole holograph (BHH) theorem [21,22].

Given a metric tensor $g_{\mu\nu}$ such that (23)–(42), consider an arbitrary value u_0 of the parameter u, the corresponding intersection

$$\tilde{\mathcal{H}}\cap\mathcal{H}_{u_0}, \tag{65}$$

the induced thereon metric tensor $g_{AB}(x)$ (27) and the rotation 1-form potential $\omega^{(\ell)}_A(x,u_0)$ (28). According to the BHH theorem, the data $[g_{AB}(x), \omega^{(\ell)}_A(x,u_0)]$ determines $g_{\mu\nu}$ in (a part of) some spacetime neighborhood of $\tilde{\mathcal{H}}\cup\mathcal{H}_{u_0}$ to the future and to the past of $\tilde{\mathcal{H}}\cap\mathcal{H}_{u_0}$ modulo spacetime diffeomorphisms preserving $\tilde{\mathcal{H}}$ and $\mathcal{H}_{u_0}$. On the other hand, as pointed out above, the data $[g_{AB}(x), \omega^{(\ell)}_A(x,u_0)]$ satisfies the constraint

$$\nabla_{(A}\omega^{(\ell)}_{B)}(x,u_0) - \omega^{(\ell)}_A(x,u_0)\omega^{(\ell)}_B(x,u_0) + \frac{1}{2}R_{AB}(x) - \frac{1}{2}\Lambda g_{AB}(x) = 0. \tag{66}$$

The comparison with (52) and the BHH theorem shows that there exists a NHG metric tensor $g^{(NHG)}{}_{\mu\nu}$ (53) that also matches the same holographic data $[g_{AB}(x), \omega^{(\ell)}_A(x,u_0)]$ and coincides with $g_{\mu\nu}$ in some spacetime neighborhood of $\tilde{\mathcal{H}}\cup\mathcal{H}_{u_0}$ intersected with the future and the past of $\tilde{\mathcal{H}}\cap\mathcal{H}_{u_0}$. What strengthens the application of the BHH theorem in that case is the arbitrariness of u_0. Given a point

$$\tilde{p}\in\tilde{\mathcal{H}} \tag{67}$$

we can first apply the BHH theorem choosing the holographic data at

$$u_0 > u(\tilde{p}) \tag{68}$$

and next another data at

$$u_1 < u(\tilde{p}). \tag{69}$$

Then the data at u_0 provides coordinates (x', v', u') such that $g_{\mu\nu}$ takes the form (53) for

$$u' \leq u_0, \qquad v' \geq 0 \tag{70}$$

while the data at u_1 provides coordinates (x'', v'', u'') such that $g_{\mu\nu}$ takes the form (53) for

$$u'' \geq u_1, \qquad v'' \leq 0. \tag{71}$$

In that way we can cover both sides of the part of the horizon $\tilde{\mathcal{H}}$ between the sections defined by u_0 and u_1, respectively. What the BHH theorem does not provide is the differentiability of the glueing of the coordinates (x', v', u') with the coordinates (x'', v'', u'') along $\tilde{\mathcal{H}}$, which corresponds to $v' = 0 = v''$. The coordinates (x', v', u') define on one side of $\tilde{\mathcal{H}}$ a NEH foliation

$$u' \mapsto \mathcal{H}'_{u'}, \tag{72}$$

where $\mathcal{H}'_{u'}$ are defined to be the constancy surfaces. The coordinates (x'', v'', u'') provide on another side of $\tilde{\mathcal{H}}$ a NEH foliation

$$u'' \mapsto \mathcal{H}''_{u''}, \tag{73}$$

where $H''_{u''}$ are the constancy surfaces of u''. The problem is that *a priori*, those NEH foliations might not coincide with the original NEH foliation $u \mapsto \mathcal{H}_u$. However, if the spacetime admits a unique NEH foliation, then the foliations $u' \mapsto \mathcal{H}'_{u'}$ and $u'' \mapsto \mathcal{H}''_{u''}$ coincide with the foliation $u \mapsto \mathcal{H}_u$. In the consequence, the coordinates (x', v', u') and (x'', v'', u''), respectively, are related to the coordinates (x, v, u) by a transformation (24). It follows that

$$u'' = U''(u'). \tag{74}$$

Using the property

$$\tilde{\kappa}^{(\tilde{\ell}'')} = 0 = \tilde{\kappa}^{(\tilde{\ell}')} \tag{75}$$

the relation becomes

$$u'' = a''u' + b'', \qquad a'', b'' = \text{const}. \tag{76}$$

Using those transformations, without the lack of generality, we can assume that

$$u''|_{\tilde{\mathcal{H}}} = u', \qquad x''^A|_{\tilde{\mathcal{H}}} = x'^A. \tag{77}$$

Those conditions define unique coordinates in M via (17)–(21), hence

$$(x', v', u') = (x'', v'', u'') \tag{78}$$

everywhere in M.

Finally, the only conformally flat solutions to vacuum Einstein's equations (42) are the anti-de Sitter, de Sitter and Minkowski spacetime, respectively, depending on the sign of Λ. Each of those spacetimes does admit locally defined coordinates that make the metric tensor take the form (53).

IV. SUMMARY

In Sec. II we introduced and characterized in detail a family of spacetimes that is a generalization of the NHGs. They are still a subclass of the Kundt's class, generically of the Petrov type II. Each of our generalized NHG spacetimes consists of NEHs emanating from a single, transversal NEH $\tilde{\mathcal{H}}$. The NEH $\tilde{\mathcal{H}}$ becomes an extremal IH if and only if ${R^{(4)}}_{AB}$ in (40) is independent of u, that is if it is constant along the null generators of $\tilde{\mathcal{H}}$. As long as the only conditions assumed about the energy-momentum tensor $T_{\mu\nu}$ are that it satisfies the NEH inequalities (2) and (3) with respect to the NEH $\tilde{\mathcal{H}}$ as well as the NEHs that set the foliation $u \mapsto \mathcal{H}_u$ of spacetime, and otherwise $T_{\mu\nu}$ is arbitrary, a generic member of the family is a proper generalization of an NHG spacetime. The generalized NHG spacetimes may be interesting on their own.

The main result of our work concerns vacuum Einstein's equations with (possibly vanishing) cosmological constant. We have proved the following theorem:

Theorem.—Suppose M is a four-dimensional spacetime equipped with a metric tensor g that satisfies Einstein's equations,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{79}$$

and admits a foliation by nonexpanding null surfaces transversal to a single nonexpanding null surface $\tilde{\mathcal{H}}$; then, locally, in a neighborhood of every point of $\tilde{\mathcal{H}}$, the metric g can be written in the form (53).

The resulting spacetime is known in the literature as NHG [12]. It admits (at least) a two-dimensional vector space of Killing vectors, linear combinations of

$$K \coloneqq \partial_u, \qquad L \coloneqq u\partial_u - v\partial_v. \tag{80}$$

The two NEHs

$$u = u_0, \quad \text{and} \quad v = 0, \tag{81}$$

respectively, form the bifurcated Killing horizon of the Killing vector $L - u_0 K$. The NEH $v = 0$ itself is the extremal Killing horizon of K. This is an example of more general geometries studied recently that admit double Killing horizons [23,24].

For higher dimensional spacetimes, the BHH theorem we have used above has not been proven in the literature. For our purposes, it would be sufficient, if it were true, that induced geometries of two intersecting NEHs set a complete data of the characteristic Cauchy problem for the vacuum Einstein's equations with cosmological constant. Then, the part of our proof that uses the BHH theorem would naturally pass to the higher dimensions and spacetime metric tensor would coincide with the higher dimensional NGH on one-sided neighborhoods of the transversal NEH.

In our proof we used the Newman-Penrose components of the Weyl tensor and the notion of the multiple principal null directions. Therefore a remark on the Petrov type of the resulting spacetimes is in order. If a solution (g_{AB}, ω_A) to the NHG equation (52) admits a symmetry

$$\mathcal{L}_X g_{AB} = 0, \qquad \mathcal{L}_X \omega_B = 0, \tag{82}$$

and (60) holds, then the resulting spacetime is of the Petrov type D. If a spacelike section of $\tilde{\mathcal{H}}$ is compact and connected, then the spacetime is that of extremal Kerr–(anti)-de Sitter throat including the nonrotating case of the extremal Schwarzschild–de Sitter (Kottler) throat spacetime. If a solution (g_{AB}, ω_A) does not admit a symmetry, on the other hand, and (60) still holds, then the corresponding spacetime is of the Petrov type II. The Petrov types III and N are excluded in the conformally nonflat case, because of the properties of nonextremal vacuum Killing horizons [25]. Remarkably, an existence of nonsymmetric solutions to the Eq. (52) on a topological 2-sphere is still an unsolved problem.

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