

Superradiant scattering in fluids of light

Angus Prain,^{1,2,*} Calum Maitland,^{1,2,†} Daniele Faccio,^{2,‡} and Francesco Marino³

¹*Institute of Photonics and Quantum Sciences, Heriot-Watt University,
Edinburgh EH14 4AS, United Kingdom*

²*School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, United Kingdom*

³*CNR-Istituto Nazionale di Ottica and INFN, Sez. di Firenze, I-50019 Sesto Fiorentino (FI), Italy*



(Received 11 April 2019; published 18 July 2019)

We investigate the scattering process of Bogoliubov excitations on a rotating photon fluid. Using the language of Noether currents we demonstrate the occurrence of a resonant amplification phenomenon, which reduces to the standard superradiance in the hydrodynamic limit. We make use of a time-domain formulation where superradiance emerges as a transient effect encoded in the amplitudes and phases of propagating localized wave packets. Our findings generalize previous studies in quantum fluids to the case of a non-negligible quantum pressure and can be readily applied also to other physical systems, in particular atomic Bose-Einstein condensates. Finally we discuss ongoing experiments to observe superradiance in photon fluids and how our time domain analysis can be used to characterize superradiant scattering in nonideal experimental conditions.

DOI: [10.1103/PhysRevD.100.024037](https://doi.org/10.1103/PhysRevD.100.024037)

I. INTRODUCTION

Superradiant scattering is an effect whereby waves reflected from a moving medium are amplified, extracting energy and momentum in the process. For an axially symmetric and rotating medium such amplification occurs whenever the following superradiant condition is met by the incident wave:

$$\omega < m\Omega, \quad (1)$$

where ω is the wave's angular frequency, m its angular momentum, and Ω is the magnitude of the angular velocity of the rotating object. That is, $m\Omega$ acts as a cutoff frequency above which no amplification occurs. This phenomenon was famously introduced by Zel'dovich [1] in 1971 who showed that low frequency electromagnetic waves reflected by a rotating conducting cylinder are amplified in this manner, hence the terminology ‘‘Zel'dovich effect’’ for this kind of scattering. Following the original Zel'dovich proposal, it was suggested by Misner [2], motivated by previous works by Penrose [3], that this anomalous reflection also occurs near rotating black holes, coining the terminology ‘‘superradiance’’ for this effect. This led to significant interest in the phenomenon and its deep relation to black hole mechanics [4,5].

Superradiant scattering is not specific to the Zel'dovich effect for conductors or its cousin for black holes. In fluid

mechanics this anomalous reflection effect is known as ‘‘over-reflection’’ and has been the subject of a long line of inquiry dating back to the 1950's [6–9], appearing even before the publication of Zel'dovich's work¹ and culminating recently in a direct experimental observation in a rotating water experiment [10,11].

The parallel between these two communities working on essentially the same physics has remained unknown and is only now coming to light, through developments at a modern interface of gravitation and fluid mechanics known as analogue gravity. Analogue gravity is the observation that gravitational effects may be simulated in fluid mechanical systems,² including quantum fluids, in the lab. In particular, attention has focused on the simulation of wave scattering effects in the strong gravitational fields that exist near astrophysical black holes [13,14]. This possibility has recently been realized experimentally in a draining fluid vortex [15].

In completely unrelated developments, a new approach to quantum fluids has emerged in the form of quantum fluids of light, where effective photon-photon interactions of a monochromatic laser beam propagating in a nonlinear medium lead to a collective behavior of the many photon system, leading to a superfluid behavior [16–19]. Such

¹For example, Eq. (4.1) in Ref. [7] and the condition $Z < 0$ in Eq. (14) in [6], both derived in 1957 and describing the conditions under which over-reflection occurs are both precisely equivalent to the condition (1) as can be easily checked.

²Amongst other physical systems: See [12] and references therein for a modern overview of this interdisciplinary research field.

*angusprain@gmail.com

†cm350@hw.ac.uk

‡Daniele.Faccio@glasgow.ac.uk

photon fluids have been shown to be suitable candidates for the simulation of analogue curved spacetimes [20,21], having received recent experimental attention [22]. In particular it has been established that the Kerr metric of a rotating black hole can be realized in a photon fluid, which exhibits a superradiant scattering spectrum [23]. However, this spectrum applies in the acoustic limit in which the quantum pressure (optical diffraction) is completely neglected and the reflection coefficient describes the amplitude of continuous plane waves as measured by an observer infinitely far away from the black hole. Neither of these conditions can be truly realized in an experiment.

In the present work we show that superradiant scattering persists in nonlinear photon fluids when the quantum pressure is not negligible; that this effect occurs across the full nonlinear dispersion relation and is not specific to the acoustic part of the spectrum where the analogy with gravity is present. We do so through a transient formalism making use of time-dependent wave packets possessing finite spatial extent, essential for practical analysis of an analogue experiment. This work can be viewed as a natural continuation of work on analogue superradiant scattering in fluids [20,23–32]. However, it can also be viewed from the perspective of nonlinear optics alone, as a novel phenomenon not previously considered, or as a natural extension of the previous work on over-reflection in ordinary fluids to the case of superfluids such as Bose Einstein condensates (BEC) or photon fluids. We anticipate a close relationship between this effect and other amplification phenomena associated with rotation and angular momentum optics (see for example Refs. [33–35]).

In what follows we will explicitly work with the full optical equations, demonstrating the condition (1) for optical excitations across the full spectrum, without recourse to an analogy with curved spacetime and show how, in the acoustic, i.e., the superfluid limit, we recover the traditional language and results of superradiance in an analogue spacetime. Due to a very close relationship between the governing equations for photon fluids and Bose-Einstein condensates (BEC), the present work also serves as a roadmap for future experiments in BEC seeking to observe superradiant scattering.

In Sec. II we review how the Bogoliubov-de-Gennes equations governing the evolution of coupled pairs of acoustic perturbations in the photon fluid are derived from the nonlinear Schrödinger equation using the Madelung transformation, making the connection to optical quantities explicit. In Sec. III we show how these perturbations may superradiantly scatter from a simple rotating flow analogous to the cylindrical conductor in the Zel'dovich effect. A formula is derived for the amplitude of superradiant reflection from this flow in the presence of quantum pressure, based on the conservation of a Noether current. In Sec. IV conservation relation is then proved and explored in greater detail, indicating how it might be used

to measure superradiance under very general conditions, including some particularities of real optical experiments. Finally in Sec. V we show our results' connection to models which neglect quantum pressure and analogues for rotating black holes within them.

II. PHOTON FLUIDS

In this work our photon fluid involves the propagation of a monochromatic optical beam of wave number k in a self-defocusing nonlinear medium. Writing the electric field from a monochromatic source as an amplitude and phase, $\epsilon = E(\mathbf{x}, z) \exp(ikz - i\omega_0 \tilde{t})$, where $\mathbf{x} = (x, y)$ refers to the transverse spatial coordinates, the so-called paraxial approximation is accurate whenever $|\nabla^2 \epsilon|/k^2 \sim |\partial_z \epsilon|/k \ll 1$, where ∇^2 is the transverse Laplacian. In this approximation, the slowly varying envelope E satisfies by the well-known nonlinear Schrödinger equation (NLSE) [36],

$$i\partial_z E + \frac{1}{2k} \nabla^2 E - k \frac{n_2}{n_0} |E|^2 E = 0. \quad (2)$$

Here n_0 is the refractive index and n_2 the nonlinearity both evaluated at the carrier frequency ω_0 [23]. The photon fluid is the (two-dimensional) transverse part of the electric field E while the z -direction plays the role of a time coordinate. The laboratory time variable \tilde{t} plays no role in this description as it has dropped out as a result of the paraxial approximation. To this end we rewrite Eq. (2) as

$$i\partial_t E + \alpha \nabla^2 E - \beta |E|^2 E = 0, \quad (3)$$

where we have introduced the timelike coordinate $t = n_0 z / c$ with c the speed of light in vacuum, and we have defined the constants

$$\alpha = \frac{c}{2n_0 k} \quad \text{and} \quad \beta = \frac{ckn_2}{n_0^2}, \quad (4)$$

which parametrize the strength of optical diffraction and nonlinearity, respectively. In the defocusing case we have case $\beta > 0$, while $\alpha > 0$ always. The connection with fluid dynamics is drawn when we express the field E in terms of its amplitude and phase $E = \sqrt{\rho} e^{i\phi}$. Then Eq. (3) is equivalent to the coupled set:

$$0 = \dot{\rho} + \nabla \cdot (\rho \mathbf{v}) \quad (5)$$

$$0 = \dot{\psi} + \frac{1}{2} \mathbf{v}^2 + c_s^2 - \alpha^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}. \quad (6)$$

Apart from the optical coefficients, Eqs. (5) and (6) are identical to those describing the density and the phase dynamics of a two-dimensional BEC in the presence of

repulsive atomic interactions [37]. In this mapping onto superfluids the optical intensity ρ is equivalent to a fluid density, $\mathbf{v} := 2\alpha\nabla\phi$ to a fluid velocity, and $c_s^2 := 2\alpha\beta\rho = c^2 n_2 \rho / n_0^3$ is the squared sound speed. The optical non-linearity, corresponding to the atomic interaction, provides the bulk pressure $P = c^2 n_2 \rho^2 / (2n_0^3)$.³ The last term in (6) $\propto \alpha^2$, the quantum pressure, has no equivalent in classical fluid mechanics and arises from the wave nature of light and is due to diffraction. This term is significant in rapidly varying and/or low-intensity regions such as dark-soliton cores and close to boundaries. This set of equations is equivalent to a Klein-Gordon equation in curved spacetime when one neglects this term, as is common in the analogue gravity literature. In this work we retain it for two reasons. Firstly, we will show that superradiant scattering persists beyond the analogy with gravity and curved spacetime deep into the dispersive regime. Secondly, it allows closer accuracy with realistic experiments, in which diffraction can never be entirely ignored.

Small perturbations ψ on top of the background beam E_0 of the form

$$E_{\text{total}} = E_0(1 + \psi), \quad (7)$$

with $|\psi| \ll |E_0|$, satisfy the Bogoliubov de-Gennes (BdG) equation [38–40]:

$$\left(\partial_t + \mathbf{v} \cdot \nabla - i \frac{\alpha}{\rho} \nabla \cdot \rho \nabla \right) \psi + i\beta\rho(\psi + \bar{\psi}) = 0, \quad (8)$$

when both E_0 and E_{total} obey the NLSE (3) (the overline notation $\bar{\psi}$ denotes the complex conjugate of ψ). Note that this choice of perturbation variable ψ differs from the alternative decomposition common in optics $E_{\text{total}} = E_0 + \tilde{\psi}$ by a multiplicative factor of the background field E_0 . That is, ψ is a dimensionless perturbation while $\tilde{\psi} \equiv E_0\psi$ is an electric field.

For stationary⁴ beams, precisely as in a BEC, the solutions of (8) always come in pairs of modes with opposite frequencies,

$$\psi = \psi_S + \psi_I = U(\mathbf{x})e^{-i\omega t} + \bar{V}(\mathbf{x})e^{i\omega t}, \quad (9)$$

which we label as positive $\propto e^{-i\omega t}$ and negative $\propto e^{i\omega t}$ modes. In regions of constant flow and intensity ($\mathbf{v} = \text{const}$, $\rho = \text{const}$) or equivalently in a Wentzel-Kramers-Brillouin approximation,⁵ local plane wave solutions with $U = e^{i\mathbf{k}\cdot\mathbf{x}}$ to

the BdG equation must satisfy the familiar Bogoliubov dispersion relation:

$$(\omega - \mathbf{k} \cdot \mathbf{v})^2 - \alpha^2 \mathbf{k}^4 - c_s^2 \mathbf{k}^2 = 0. \quad (10)$$

Note that this is a position dependent dispersion relation, depending on the location in the flow through $\mathbf{v}(\mathbf{x})$ and $c_s(\mathbf{x})$. Including the frequencies, the complete positive mode,

$$\psi_S = e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}, \quad (11)$$

labeled by the pair (ω, \mathbf{k}) can be shown to locally induce a negative mode labeled by $(-\omega, -\mathbf{k})$,

$$\psi_I = \zeta e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\omega t}. \quad (12)$$

The constant of proportionality ζ can be explicitly calculated to be

$$\zeta = \frac{1}{\beta\rho} (\omega - \mathbf{k} \cdot \mathbf{v} - \alpha \mathbf{k}^2 - \beta\rho), \quad (13)$$

as can be easily verified by inserting this pair into (9), then eventually (8), neglecting derivatives of the background quantities.

The relation (10) is symmetric under the simultaneous replacement $\mathbf{k} \rightarrow -\mathbf{k}$ and $\omega \rightarrow -\omega$, and, therefore, negative modes also satisfy the Bogoliubov dispersion relation, while sitting in a different location on the dispersion curves as we show in Fig. 1. Note that both positive and negative modes always have the same group and phase velocity and, hence, always travel in the same direction with the same speed.

The coupled mode-pair nature of the solutions (9) will be crucial for deriving the superradiant reflection in Sec. III

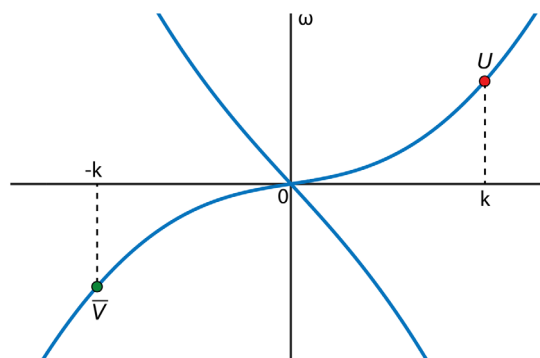


FIG. 1. Bogoliubov dispersion curve for an acoustic probe of the photon fluid, including a component of the fluid flow along the probe’s propagation direction [$\mathbf{v} = (-v, 0)$]. The positive U and negative \bar{V} modes are indicated with red and green circles, respectively, which together form coupled pairs of solutions to the BdG equation. The coupling between these modes is provided by the optical nonlinearity.

³Note that the sound speed is $c_s^2 = \partial P / \partial \rho$, as required.

⁴Note that in the optical context stationary means invariant along the propagation direction z , which acts as effective time $t = n_0 z / c$ for the photon fluid.

⁵This approximation is valid whenever the wavelength of a mode is much shorter than the typical length scale on which the background flow and intensity is varying and provides the intuitive picture of a wave which slowly changes its amplitude and phase on an inhomogeneous background.

and is worth reemphasizing. Normal mode solutions to the BdG equation always come in positive-negative pairs. These pairs are locally “almost” complex conjugates of one another: they are of opposite phase, but there is an amplitude difference given locally by ζ in Eq. (13).

We finish this section with a short note to clarify the connection with real optical experiments. Only measurements of the total field E_{total} can be made in real experiments, which is the sum of two components E_0 and $E_0\psi$. By conducting an unperturbed $\psi = 0$ experiment, E_0 is known and can be subtracted from E_{total} . What is left in the measured electric field is the product $E_0\psi$, which from Eqs. (7) and (9) we have seen will consist of the pair

$$E_S = E_0\psi_S \quad (14)$$

$$E_I = E_0\psi_I. \quad (15)$$

What typically occurs in optical experiments is that the background field E_0 (the “pump” beam) is provided by one laser with a second independent laser providing the electric field perturbations E_S, E_I . In this way the variable ψ , which satisfies the Bogoliubov dispersion, is a mixture of the pump and probe beams with a phase, which is the sum of the pump and probe phase and intensity, which is the product. For this reason the measured fields E_S and E_I do not satisfy the same Bogoliubov dispersion relation.

III. SUPERRADIANCE IN PHOTON FLUIDS

We will now demonstrate that the BdG Eq. (8) possesses superradiant scattering solutions in general (for all values of α) and not just in the acoustic limit. We will do this by monitoring the radial propagation of a localized annular wave packet, comprised of the individual positive and negative wave packets, as it approaches and scatters from an optical Zel’dovich cylinder. We will make use of a conserved quantity (the conservation relation will be proved in Sec. IV) to relate the reflected and transmitted wave packets.

Consider the idealized background (pump) beam profile $\rho = \text{const}$ and

$$v_\theta = \begin{cases} r\Omega, & r < r_0 \\ 0, & r > r_0 \end{cases}, \quad v_r = 0, \quad (16)$$

with $\Omega = \text{constant}$. Such a beam is the analogue of the rigidly rotating circular body on which the traditional Zel’dovich effect occurs. We choose this background as a simple example for which the analysis is straightforward. It is also an opportunity to introduce methods which later we shall use in more general backgrounds after having established the existence of the basic superradiant mechanism here.

Consider now, on top of this background, an annular positive mode wave packet ψ_S of fixed positive frequency

$\omega > 0$ and positive angular momentum $m > 0$, initially in the region $r > r_0$, propagating inwards towards the rotating “body”. Associated with this positive mode is of course a negative mode ψ_I , which completes the exact solution to (8):

$$\psi = \psi_S + \psi_I. \quad (17)$$

Due to the angular symmetry and stationarity of the background, the frequencies and angular momenta of the positive and negative modes are conserved and of opposite sign:

$$\begin{aligned} \psi_S &= U(r)e^{-i\omega t + im\theta} \\ \psi_I &= \bar{V}(r)e^{+i\omega t - im\theta}. \end{aligned} \quad (18)$$

The functions U and \bar{V} implicitly carry the envelope and radial momentum of the packets.

We can, without loss of generality, assume that locally these initial wave packets, being in the region of zero flow, form parts of propagating Bessel modes, i.e., Hankel functions,⁶ and that within the initial wave packet envelope we may approximate

$$U(r) \simeq \frac{e^{-ikr}}{\sqrt{r}}. \quad (20)$$

The wave numbers ω, k, m are not independent, being determined by the dispersion relation (10) in the $r > r_0$ region which, for our Zel’dovich background, is shown in Fig. 2 where we note the “mass gap” arising from the angular momentum $m > 0$. Using the same arguments that led to Eq. (13), initially we have that

$$\bar{V}(r) \simeq \frac{1}{\beta\rho} (\omega - \alpha\tilde{k}^2 - \beta\rho) \frac{e^{+ikr}}{\sqrt{r}}, \quad (21)$$

where $\tilde{k}^2 = k^2 + (m^2 - 1/4)/r^2$ is the local wave number.⁷ Again, the functions $U(r)$ and $\bar{V}(r)$ implicitly encode the wave packets’ Gaussian envelopes, which are chosen such that initially $\int r dr |\psi|^2 = 1$. Recall that the positive and negative modes have the same phase and group velocities, and, hence, both move towards the rotating flow.

⁶It is straightforward to show that the inward propagating Hankel function pair,

$$f = H_m^+(kr)e^{-i\omega t + im\theta} + \zeta H_m^-(kr)e^{+i\omega t - im\theta}, \quad (19)$$

exactly solves the BdG equation in the exterior zero flow region with \mathbf{k} satisfying (10) and ζ given by (13).

⁷This locally varying wave number arises from the approximation of a Bessel function by an exponential. The full Bessel function has the constant Laplacian eigenvalue $-k^2$ for the radial wave number.

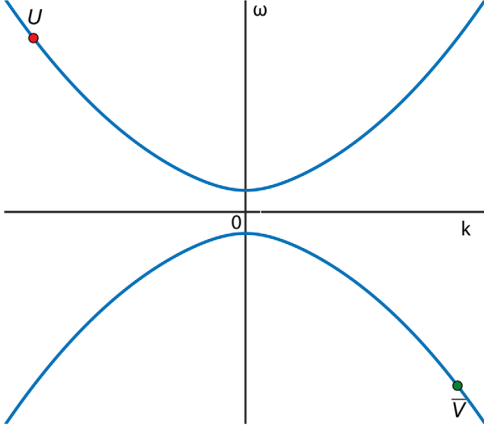


FIG. 2. The initial positive U (red circle) and negative \bar{V} (green circle) modes' locations on the BdG dispersion in the exterior region, with U and \bar{V} as specified by Eqs. (20) and (21), respectively. Both modes have negative group velocity and, hence, move radially inwards towards the cylinder.

With this description of the initial condition we now let the wave packets propagate and scatter from the rotating cylinder. At late times, the incident wave packet has split into reflected and transmitted components which are separated in space. Due to the stationarity and axisymmetry, the quantum numbers m and ω are conserved throughout the scattering so that at late times the solution is again of the form (18) but now with the two oppositely propagating wave packet components in both the positive and negative channels:

$$U(r) = \mathbf{R}_U(r) + \mathbf{T}_U(r) \quad (22)$$

$$\bar{V}(r) = \bar{\mathbf{R}}_V(r) + \bar{\mathbf{T}}_V(r). \quad (23)$$

The exterior reflected components again take the known Bessel form, which we again approximate with exponentials

$$U(r) \simeq R_U \frac{e^{+ikr}}{\sqrt{r}} + T_U(r) \quad (24)$$

$$\bar{V}(r) \simeq \bar{R}_V \frac{e^{-ikr}}{\sqrt{r}} + \bar{T}_V(r). \quad (25)$$

Using the local proportionality result (13) (i.e., that positive and negative modes are locally conjugates of each other, up to a real constant) we know that the reflected components of the positive and negative modes satisfy

$$\bar{R}_V = \bar{R}_U \times \frac{1}{\beta\rho} (\omega - \alpha\tilde{k}^2 - \beta\rho), \quad (26)$$

so that we have

$$\bar{V}(r) \simeq \bar{R}_U \frac{1}{\beta\rho} (\omega - \alpha\tilde{k}^2 - \beta\rho) \frac{e^{-ikr}}{\sqrt{r}} + \bar{T}_V(r). \quad (27)$$

We have not explicitly written down the radial phase distributions of the transmitted wave packets T_U and \bar{T}_V as we were not able to find simple expressions—they are the normal modes of the rotating disc background which are of Bessel-like form. Their exact forms are unimportant, however, since in order for the pair of transmitted wave packets to locally be a solution to Eq. (8) the relationship between the transmitted modes must be

$$\bar{T}_V = \frac{1}{\beta\rho} (\omega - m\Omega + \alpha\nabla_m^2 - \beta\rho) \bar{T}_U, \quad (28)$$

where $\nabla_m^2 = r^{-1} \partial_r r \partial_r - m^2/r^2$. Assuming that our transmitted wave packet is sufficiently localized we can, without loss of generality, assume that \bar{T}_U is locally an eigenfunction of ∇_m^2 with a (possibly m dependent) eigenvalue⁸ $-\lambda_m^2$. We therefore have

$$\bar{T}_V = \frac{1}{\beta\rho} (\omega - m\Omega - \alpha\lambda_m^2 - \beta\rho) \bar{T}_U. \quad (29)$$

Hence the full solution at late times is given by $\psi_S + \psi_I$ decomposed as in Eq. (18) with $U(r)$ as in Eq. (24), and

$$\begin{aligned} \bar{V}(r) = \bar{R}_U \times \frac{1}{\beta\rho} (\omega - \alpha\tilde{k}^2 - \beta\rho) \frac{e^{-ikr}}{\sqrt{r}} \\ + \frac{1}{\beta\rho} (\omega - m\Omega - \alpha\lambda_m^2 - \beta\rho) \bar{T}_U(r). \end{aligned} \quad (30)$$

We wish to relate the power of the reflected positive mode R_U to the power of the incident positive mode, which we have chosen here to be 1. In particular, we want to find conditions under which the reflected power can be greater than the incident power, so that superradiance can occur. To this end we need to relate the solutions at early times to the solution at late times. This will be achieved by making use of a conservation relation, which provides a quantity which is conserved over the duration of the evolution. By evaluating this quantity at early and late times and equating the two we can explicitly relate the reflected and incident amplitudes.

We will prove the following conservation relation in Sec. IV:

$$\int r dr (|U|^2 - |V|^2) = \text{const.} \quad (31)$$

Assuming that this relation holds we proceed by evaluating the left hand side at early and late times by inserting the

⁸There is the possibility of complex eigenvalues here which will occur if the transmitted modes are too close to the origin which can be avoided by choosing r_0 sufficiently large.

asymptotic forms [early (20), (21) and late (24), (30)]. We find after some algebra the relation:

$$1 = |\mathcal{R}|^2 + \sigma^2 \frac{(\omega - m\Omega)}{\omega} |\mathcal{T}|^2, \quad (32)$$

where we have introduced the total integrated powers $|\mathcal{R}|^2 = \int r dr |R_V|^2$ (and similarly for \mathcal{T}), and

$$\sigma^2 = \frac{\alpha \lambda_m^2 + \beta \rho}{\alpha \tilde{k}^2 + \beta \rho} \quad (33)$$

is a positive constant.

This final relation (32) is our main result and shows that $|\mathcal{R}| > 1$ whenever the superradiance condition (1) holds, as claimed at the beginning of this section.

We note that this result is completely general, holds even if we are far from the ‘‘acoustic’’ or linear part of the dispersion relation (10), and does not require any radial flow structures. The result generalizes previous work in this aspect, see for instance [23,29,41,42]. Figure 3 plots σ^2 , which characterizes the deviation from classical superradiant reflection, for varying the strength of β (proportional to the optical nonlinearity supporting the photon fluid). As $\beta \rightarrow \infty$, $\sigma^2 \rightarrow 1$ for all phonon frequencies, recovering the acoustic limit.

In Fig. 4 we show the results of numerical simulations of the BdG Eq. (8) by plotting the spectrum of $|\mathcal{R}|^2$ as a function of the initial ω for a range of values of the quantum pressure parameter α . We obtain these spectra by repeatedly simulating the scattering of a wave packet from the flow in Eq. (16) for different frequencies ω and calculating $|\mathcal{R}|^2$ by integrating the current J^0 (to be defined in Sec. IV) over the region $r > r_0$ after reflected and transmitted components

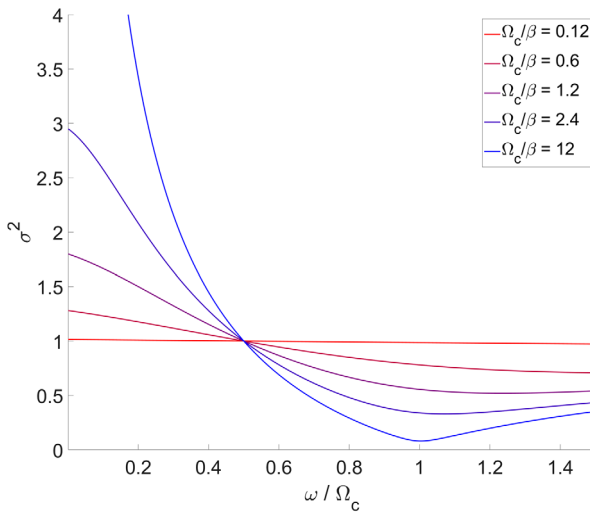


FIG. 3. Modification σ^2 of the over-reflection coefficient vs frequency, scaled to the over-reflection cutoff frequency $\Omega_c = m\Omega$, for different values of $\Omega_c/\beta\rho$.

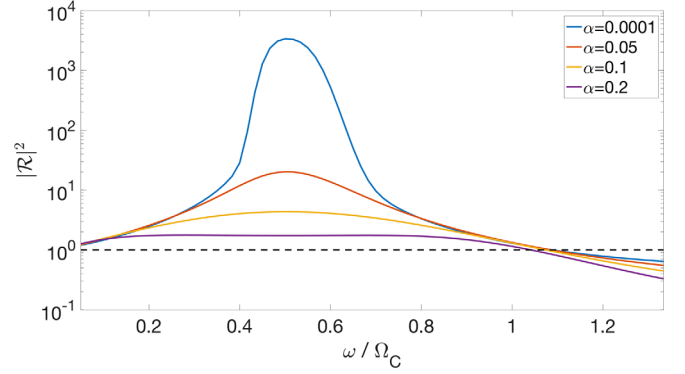


FIG. 4. Superradiant scattering of Gaussian wave packets from an optical Zel’dovich cylinder. The reflected amplitude is measured using our integrated current method developed in Sec. IV.

have become well separated. Note that $\alpha \rightarrow 0$ with $\alpha\beta = \text{const}$ is the acoustic limit. We see the presence of the critical cutoff frequency $\Omega_c = m\Omega$ [see Eq. (1)] above which superradiance ceases, independently of the quantum pressure parameter α , and also that superradiance appears to be more dramatic as one approaches the acoustic limit $\alpha \ll 1$. There also appears to be a resonant effect which amplifies the reflection at half the cutoff frequency, for which $\sigma^2 = 1$ as the wave numbers λ_m and \tilde{k} of modes on either side of the cylinder boundary are of equal magnitude but opposite sign. This resonance becomes more pronounced as α decreases. We point out that this resonance is well known and corresponds to the classical transmission resonance from quantum mechanics (see for example [43]) for this particular sharp background configuration. It can be shown that in the absence of dispersion, the term due to the rotational flow in Eq. (16) that appears in the wave equation for either U or V maps to a smooth Rosen-Morse type potential with a corresponding transmission resonance as discussed in [43]. In fact the reflection and transmission coefficients of the nondispersive wave equation are normalized by $\lambda_m + \tilde{k}$ and, hence, diverge at the resonance frequency $\omega = \Omega_c/2$, for which $\lambda_m = -\tilde{k}$. This divergence is regularized by finite dispersion or any smooth interpolation of the flow in Eq. (16), resulting in a bounded reflection resonance as seen in Fig. 4.

What we have shown is that superradiance is possible in photon fluids, obeys the cutoff condition in Eq. (1) for any strength of the quantum pressure, and it does not require an acoustic approximation—the superradiance condition for amplification depends only on the initial frequency and momentum of the mode as well as the rotational velocity of the reflecting medium. The result intrinsically involved the coupled positive-negative mode structure of the solutions and a conservation relation, which required knowledge and monitoring of both parts of the solution. This is in contrast to many results in the acoustic approximation in BEC and earlier work in photon fluids whereby only the positive

component was considered. Interestingly, the generalization to the dispersive case requires one to measure both components.

We have carried out this demonstration in a particular simple background flow as a proof of principle for superradiance in photon fluids. We believe the mechanisms and structures involved in this version are not specific to this background; the exploration of dispersive superradiance on more general backgrounds is left for future work, but we describe below a generalization of superradiance which is naturally applied to modern optical experiments.

IV. CONSERVED CURRENTS AND A GENERAL DEFINITION OF SUPERRADIANCE

In this section we will prove the crucial conservation relation (31) using a symmetry argument from an action principle for the BdG equation. In the previous section we used this relation to demonstrate superradiance in a single monochromatic scattering event on a particular background. We will show that the conservation relation (31) arises as the conserved charge associated with a symmetry in an action, giving rise to the BdG Eq. (8). By partially integrating the resulting Noether conservation relation down to a finite radius one obtains a measure of the total Noether charge in the integrated region. This “exterior charge” is not positive definite nor conserved and increases when superradiance occurs. We show that this coincides with a transmitted mode with negative Noether charge escaping the integrated region. Based on this new perspective, we propose a more general definition of superradiance, which is applicable to modern nonlinear optical experiments which reduces to the standard one in the traditional setting.

For simplicity we again work in the special Zel’dovich case described in Eq. (16) to outline the idea. Since the background is axially symmetric, the positive-negative mode decomposition of the perturbation takes the general form

$$\psi = U(t, r)e^{im\theta} + \bar{V}(t, r)e^{-im\theta}, \quad (34)$$

where we have absorbed the frequencies into the U and \bar{V} amplitudes. Even if the flows given in Eq. (16) are idealized, such a decomposition will hold in more general backgrounds including under nonstationary and finite-size experimental conditions.

By some simple manipulations Eq. (8) is equivalent to the following decoupled pair of higher order partial differential equations for the amplitudes U and \bar{V} :

$$\begin{aligned} (\partial_t + im\Omega + i\alpha\nabla_m^2)(\partial_t + im\Omega - i\alpha\nabla_m^2)U - 2\alpha\beta\nabla_m^2 U &= 0 \\ (\partial_t - im\Omega + i\alpha\nabla_m^2)(\partial_t - im\Omega - i\alpha\nabla_m^2)\bar{V} - 2\alpha\beta\nabla_m^2 \bar{V} &= 0. \end{aligned} \quad (35)$$

Here, we introduced the shorthand $\tilde{\Omega}(r) = \Omega\text{Heaviside}(r_0 - r)$ and then for simplicity dropped the tildes and the functional dependence on r , leaving these implicit. We shall refer to these as the BdG-wave equations as they are second order in time partial differential equations with fourth order spatial derivatives. These equations follow as the Euler-Lagrange equations for the following action:

$$\begin{aligned} S := \int r dr dt [&(\partial_t - im\Omega - i\alpha\nabla_m^2)\bar{V}] \\ &\times [(\partial_t + im\Omega - i\alpha\nabla_m^2)U] \\ &- 2\alpha\beta[(\partial_r U)(\partial_r \bar{V}) + m^2 U \bar{V}]. \end{aligned} \quad (36)$$

Moreover, this action (36) is invariant under the global $U(1)$ phase shift:

$$\begin{pmatrix} U \\ V \end{pmatrix} \rightarrow \begin{pmatrix} U \\ V \end{pmatrix} e^{i\lambda} \simeq \begin{pmatrix} U \\ V \end{pmatrix} + i\lambda \begin{pmatrix} U \\ V \end{pmatrix}. \quad (37)$$

The Noether current associated with this symmetry has components which can be simplified, using the BdG equations of motion, to

$$\begin{aligned} J^0 &= -i\beta(|U|^2 - |V|^2) \\ J^1 &= \alpha\beta(V\partial_r \bar{V} - \bar{V}\partial_r V - \bar{U}\partial_r U + U\partial_r \bar{U}). \end{aligned} \quad (38)$$

The current J^μ satisfies the standard conservation relation $\partial_\mu J^\mu = 0$, and therefore the integrated total charge $Q := \int r dr J^0$ is constant in time. This is precisely the result we set out to prove.

This establishes Eq. (31) from an interesting new perspective to the traditional one: *The Bogoliubov normalization condition (31) can be understood as a conserved Noether charge Q associated with a $U(1)$ symmetry (37) in the action for the decoupled “BdG-wave” equations.*

Given the conserved current, one can integrate it over an annular subregion with boundary instead of all space. The resulting relation

$$\partial_t \left(\int_{\text{region}} r dr J^0 \right) + J^1|_{\text{boundary}}(t) = 0 \quad (39)$$

tells us that as long as J^1 is zero at the boundary, the total charge in the integrated region is conserved. For example, when a wave packet moves across the boundary the total charge in the region will change, and that change is governed by Eq. (39).

Crucially, under special circumstances the flux $J^1|_{\text{boundary}}$ can be negative. This occurs when a mode with negative Noether charge exits the region. These special circumstances reduce to Eq. (1) when such a condition makes sense but is more general than (1), applying also to

nonstationary multimode situations without the need to decompose into angular m -eigenmodes.

As an example, let us consider the idealized wave packets of Sec. III. Such modes are characterized by triples (ω, m, k) for the positive modes and $(-\omega, -m, -k)$ for the negative ones. Let us call the total charge of the initial condition Q_i and let us normalize the solution by dividing by Q_i so that the total initial charge is 1. Then the charge of the final solution is of course the sum of the charges of the reflected and transmitted components, since the integral splits into disjoint sub integrals

$$Q_f = Q_f^T + Q_f^R. \quad (40)$$

Of course, since the charge is conserved, we have $Q_f = Q_i$. By direct calculation one can show that

$$Q_f^T = \sigma^2 \frac{(\omega - m\Omega)}{\omega} |T|^2 \quad (41)$$

in the notation of Eq. (32). Therefore we see the component structure of the relation (32) emerging. The left-hand side is the (normalized) initial charge, while the right-hand side is the (normalized) final charge, which is the sum of the reflected and transmitted charges. The sum of the reflected and transmitted charges is always equal to the initial charge, and whenever $\omega < m\Omega$ the transmitted charge is negative while the reflected charge is greater than 1.

In Fig. 5 we plot the real part of the positive mode and the local charge of the full solution⁹ (scaled by the initial charge). We see that the transmitted charge is negative while the reflected charge is greater than the incident charge.

Integrating the charge density up to the radius r_0 and plotting this over time one observes an increasing function in time. In this idealized example the total exterior charge as a function of time is a very simple step function as shown in Fig. 6. In that figure we have also shown the integration of the charge density over the complement region $r < r_0$. In that region there is initially zero charge but this decreases when the negatively charged transmitted modes enter the integration region.

These observations allow us to make an operational definition of superradiance: superradiance occurs whenever the total charge in an exterior region increases in time. The total charge is measured by Eq. (31) which can be obtained by measuring only the amplitudes of the positive and negative modes at a single instant. The condition defined by Eq. (1) applies to monochromatic waves in stationary situations. Under those conditions our new definition is precisely equivalent to the standard one (1). Our new

⁹The charge of the positive mode alone is not a meaningful quantity and only its combination with the negative mode has a well-defined charge; see Eq. (31).

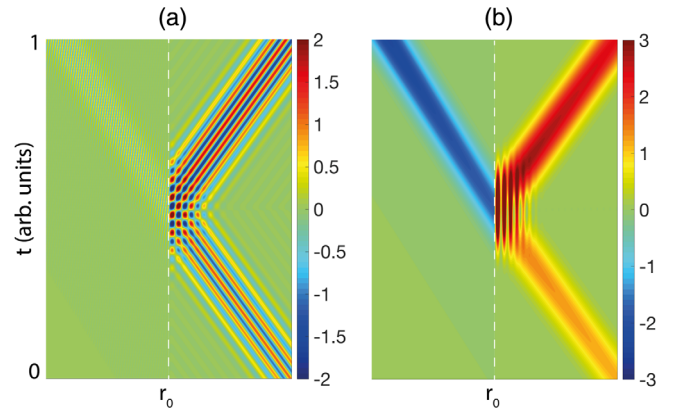


FIG. 5. Real part of the positive mode $\Re(U(r, t))$ undergoing superradiance (a) and associated current/Noether charge density $J_0(r, t)$ as defined in Eq. (38) (b). The white dashed line indicates the edge of the Zel'dovich cylinder $r = r_0$. Note that the transmitted modes have the peculiar property that their phase velocity points in the opposite direction to the group velocity as can be seen in (a).

definition is much more general, allowing for arbitrary initial conditions, incomplete scattering (in which the reflected and transmitted components have not had time to propagate away from scattering boundary and separate spatially), and nonstationary backgrounds. This is perfectly suited to real optical experiments where the initial perturbations might be spatially extended over either side of the scattering region and have complicated functional forms, being composed of many frequencies. Looking at Fig. 6 we also note that since the exterior current is monotonically increasing with time, superradiance may be verified without having to observe the complete scattering process by simply detecting a small growth of the Noether charge beyond $r > r_0$. Experimental systems ordinarily have losses which constrain the time scales in which modes

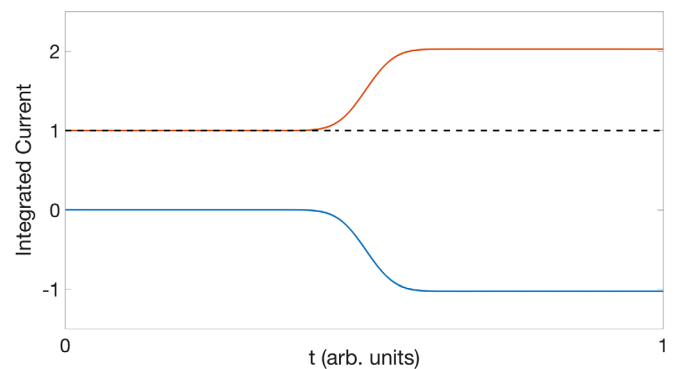


FIG. 6. Integrals of the Noether charge density shown in Fig. 5(b) inside (blue, lower curve) and outside (red, upper curve) the Zel'dovich cylinder over time, relative to the total charge density at $t = 0$. The straight dotted black line shows the global current density integral, showing that total Noether charge is conserved throughout the scattering.

can propagate without significant distortion. Freeing ourselves from the confines of the asymptotic analysis “at infinity” by moving to a transient formulation, we bring the theory in line with realistic experiments and the possibility of experimental detection of superradiance.

V. RELATIONSHIP WITH ANALOGUE MODELS AND ROTATING BLACK HOLES

As we mentioned in the introduction, there is a well-known analogy between wave propagation in strong gravitational fields and the propagation of excitations in (possibly effectively) moving media. In practice, and under certain approximations, excitations obey the d’Alembertian equation associated with an effective spacetime metric,

$$0 = \square_g \psi = \frac{1}{\sqrt{\mathbf{g}}} \partial_\mu (\sqrt{\mathbf{g}} g^{\mu\nu} \partial_\nu \psi), \quad (42)$$

where the effective metric $g_{\mu\nu}$ is a $D + 1$ dimensional matrix with inverse $g^{\mu\nu}$ and determinant \mathbf{g} . The metric coefficients are functions of the background upon which the excitations propagate (see [12] for a review). Normally the analogy is valid in an approximation of the perturbation dynamics, one in which the dispersion relation takes a linear “relativistic” form with a universal propagation speed. Indeed the dispersion relation associated with Eq. (42) is the quadratic $g^{\mu\nu} k_\mu k_\nu = 0$, and does not contain the quartic terms of Eq. (10), for example.

The strength of our results in the previous sections lies precisely in their validity beyond the acoustic approximation, which we no longer need to monitor throughout a scattering in order to trust the result. Our superradiance result for wave packet propagation (32) holds over the entire Bogoliubov spectrum; the price we pay for the generality is that both positive and negative modes need to be measured and included in the analysis. As a by-product of this formulation we were able to talk about characterization of generalized superradiance for arbitrary perturbations. The purpose of this section is to draw a parallel between our formulation and a similar one which exists in the purely relativistic case, or equivalently in the acoustic limit. This is intended on the one hand to show how our results smoothly reduce to traditional relativistic results as well as building a bridge between the optical and gravitational dynamics beyond the acoustic approximation.

The BdG Eq. (8) is a set of first order in time partial differential equations (PDEs), one for ψ and another for $\bar{\psi}$ which are coupled. They can be rearranged without approximation into the decoupled 2nd order in time PDE:

$$\begin{aligned} & \left(\partial_t + v \cdot \nabla + i \frac{\alpha}{\rho} \nabla \cdot \rho \nabla \right) \frac{1}{\rho} \left(\partial_t + v \cdot \nabla - i \frac{\alpha}{\rho} \nabla \cdot \rho \nabla \right) \psi \\ & - 2\alpha\beta \frac{1}{\rho} \nabla \cdot \rho \nabla \psi = 0, \end{aligned} \quad (43)$$

and its complex conjugate. Under the appropriate Wentzel-Kramers-Brillouin approximation, both these PDEs possess the Bogoliubov dispersion (10). This is very similar to what we did in Eq. (35) but here we have made no assumptions about the form of the solution ψ .

Equation (43), however, is not a wave equation *per se* as it stands due to the third term inside each of the brackets. This nonhyperbolic PDE can be approximated by a hyperbolic equation when α is small while keeping the product $\alpha\beta$ fixed as we mentioned at the end of Sec. III. In this limit Eq. (43) can be reexpressed, using Eq. (5), as

$$(\partial_t + \nabla \cdot \mathbf{v})(\partial_t + \mathbf{v} \cdot \nabla) \psi - \nabla \cdot c_s^2 \nabla \psi = 0, \quad (44)$$

where $c_s = 2\alpha\beta$ is the universal (but position dependent) propagation speed, which is precisely the d’Alembertian describing the dynamics of a massless minimally coupled scalar field propagating in a geometry with metric

$$ds^2 = \left(\frac{\rho}{c_s} \right)^2 [-c_s^2 dt^2 + (d\mathbf{x} - \mathbf{v} dt)^2]. \quad (45)$$

Metrics of the form (45) are capable of describing a wide range of astrophysically relevant axisymmetric geometrical structures, such as black hole event horizons (locations where $v_r^2 - c_s^2$ changes sign), ergo-regions (regions where $\mathbf{v}^2 > c_s^2$), and time dependent cosmological horizons. These structures are the necessary and sufficient geometric ingredients for Hawking radiation, superradiance, and cosmological particle production, respectively, and as such are attractive analogues to simulate in the lab due to the absence of experimental verification of these astrophysical phenomena.

For example, a rotating draining fluid vortex flow $v_r \propto 1/r$, $v_\theta \propto 1/r$ has been the subject of much attention in the literature from this perspective [15] and is an analogue for a rotating black hole spacetime, possessing both an event horizon and an ergo-region. In optics the rotating spacetime with $v_\theta \propto 1/r$ naturally arises for an optical beam with orbital angular momentum (OAM) (see [23] for example). Expanding cosmologies naturally arise in this fashion by switching off the trapping potential of a confined BEC [44].

The action for which (44) is the extremal condition is given by

$$S = \frac{1}{2} \int dt d^n x \sqrt{\mathbf{g}} g^{\mu\nu} (\partial_\mu \psi) (\partial_\nu \bar{\psi}), \quad (46)$$

which possesses conserved current

$$J^\mu = \sqrt{g} g^{\mu\nu} (\bar{\psi} \partial_\nu \psi - (\partial_\nu \bar{\psi}) \psi), \quad (47)$$

and Noether charge

$$\begin{aligned}
Q(t) &= \int d^n x \sqrt{\mathbf{g}} g^{\mu 0} (\bar{\psi} \partial_\mu \psi - \text{c.c.}) \\
&= \int d^n x (\bar{\psi} (\partial_t + \mathbf{v} \cdot \nabla) \psi - \text{c.c.}), \quad (48)
\end{aligned}$$

which is conserved $dQ/dt = 0$ (see [45,46] for example).

As we did in Sec. IV, integrating the current density J^0 only over a subregion \mathcal{M} with boundary $\partial\mathcal{M}$ results in the “partial charge” $Q_{\mathcal{M}}$ which will increase when negative norm component modes cross $\partial\mathcal{M}$. This is precisely what happens during superradiance: a positive charge wave packet propagates towards the inhomogeneous rotational flow, scatters from it into a positive charge reflected mode, and a negative charge transmitted packet (see Fig. 5). Of course the sum of these two components is always equal to the initial charge (since total charge is conserved); the charge of the reflected mode will be greater than the initial incident mode.

Looking closely at the currents in Eq. (38), it is not clear what becomes of them in the acoustic limit $\alpha \rightarrow 0$ with $\alpha\beta = \text{const.}$ However, their raw forms

$$\begin{aligned}
J^0 &= U(\partial_t - imv - i\alpha\nabla_m^2)\bar{V} \\
&\quad - \bar{V}(\partial_t + imv - i\alpha\nabla_m^2)U \quad (49)
\end{aligned}$$

and

$$J^1 = -2\alpha\beta[U(\partial_x\bar{V}) - \bar{V}(\partial_xU)] + \mathcal{O}(\alpha) \quad (50)$$

provide more insight. We see that the current simply converges to the d’Alembertian current (47) in the acoustic limit.

Note also by Eq. (12) we see that in the acoustic limit the positive and negative modes become degenerate being merely conjugates of one another, and from Eq. (35) we see that both are governed by the same curved spacetime d’Alembertian.

For completeness, the acoustic spacetime associated with our idealized Zel’dovich beam profile in Eq. (16) is given by

$$ds^2 \propto -c_s^2 dt^2 + dr^2 + r^2(d\theta - \Omega dt)^2. \quad (51)$$

Such a spacetime can obviously not arise as a solution to dynamical gravitational equations such as the Einstein equations due to the nondifferentiability at r_0 . However, the spacetime possesses an ergo-region if the rotating region is large enough; $r_0 > c_s/\Omega$ since $g_{tt} = -c_s^2 + r^2\Omega^2$. While

the condition $\omega < m\Omega$ can always be fulfilled for incident wave packets, even for small Zel’dovich cylinders, cylinders with $r_0 < c_s/\Omega$ sit inside the turning points of the radial Bessel modes when $\omega < m\Omega$, where they transition to polynomial decay and therefore do not interact strongly with the modes.¹⁰

VI. CONCLUSION

In summary, we have explored the problem of super-radiant scattering in the time domain using conserved Noether currents. This has allowed us to generalize the conservation condition predicting the amplitude of super-radiant reflections to account for quantum pressure, which is always present in BEC and photon fluid analogues while making the limit to the superfluid regime explicit where the analogy with curved spacetime is available. It has also made a general transient description of superradiance possible, which permits arbitrary initial conditions and a wide class of backgrounds from which it is apparent that an event horizon is not necessary to observe over-reflection from a rotating spacetime. This transient formalism will be crucial for future experiments on superradiance in photon fluids, in which practical constraints limit potential super-radiance measurements to very short time scales. We plan on conducting a complete numerical study of generalized superradiance from this perspective in future work.

ACKNOWLEDGMENTS

D. F. and A. P. acknowledge financial support from the EPSRC (UK, Grant No. EP/M009122/1) and EU Horizons 2020 (Marie-Sklodowska Curie Actions). This work has also received funding from the European Union’s Horizon 2020 research and innovation programme under Grant Agreement No. 820392. C. M. acknowledges studentship funding from EPSRC under CM-CDT Grant No. EP/L015110/1.

¹⁰This can be estimated by the following argument. Locally, outside the rotating region, a radial plane wave $\exp(-i\omega t + im\theta - ik(r)r)/\sqrt{r}$ has position dependent dispersion $\omega^2 \simeq c^2 k(r)^2 + c^2 m^2/r^2$ so that $c^2 k(r)^2 = \omega^2 - c^2 m^2/r^2$. If $\omega < m\Omega$ then $c^2 k(r)^2 < m^2\Omega^2 - c^2 m^2/r^2$. As this mode approaches the critical radius $r_{\text{crit}} = c_s/\Omega$ the right-hand side of this inequality goes to zero, implying the wavelength goes to infinity and the mode ceases to be oscillatory. Hence, rotating regions smaller than r_{crit} cannot superradiate despite satisfying the superradiance condition, in consistency with the spacetime picture which tells us there is no ergo-region for such small cylinders.

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