

Orbits in a stochastic Schwarzschild geometry

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We study geodesics in the Schwarzschild space-time affected by an uncertainty in the mass parameter described by a Gaussian distribution. This study could serve as a first attempt at investigating possible quantum effects of black hole space-times on the motion of matter in their surroundings as well as the role of uncertainties in the measurement of the black hole parameters.

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I. INTRODUCTION

Black holes were always one of the characterizing predictions of general relativity (GR) [1,2], and the recent detection of gravitational waves from the merging of black hole binaries [3] has further boosted the interest in such astrophysical objects. The mathematical properties of these vacuum solutions of the Einstein equations are already problematic at the classical level, where it is well known that no sensible energy-momentum tensor can be associated with them [4]. It becomes even more problematic at the quantum level, since our very limited understanding from semiclassical approaches yields the famous Hawking radiation [5] and a bunch of paradoxes (see, e.g., Refs. [6–11], and references therein).

On the other hand, our observational capacities remain relatively weak in determining with precision what astrophysical black holes are in nature, and a host of alternative, somewhat more exotic, compact objects are actively being investigated in the present literature (see, e.g., Refs. [11–22]). In this respect, it is very important to determine the physical consequences of alternative models of compact objects or alternative descriptions of gravity on observable quantities, regardless of the origin of such deviations from the simple black hole metrics of GR.

For example, horizon quantum mechanics [23–26] offers an alternative perspective to the semiclassical approach to gravity, whose aim is to put under the spotlight the quantum features of a black hole's geometric structure inherited by a purely quantum mechanical description of its

source. In this regard, the location of trapping surfaces becomes fuzzy because of the quantum mechanical nature of the source, thus providing a clear motivation for the study presented here.

Another motivation can be found in light of the theory of stochastic gravity (see, e.g., [27] and references therein), which offers an extension of semiclassical gravity based on the classical Einstein equations sourced by the stress-energy tensor of quantum matter fields by including the contribution of quantum fluctuations to the vacuum expectation values of matter. Specifically, these fluctuations are accounted for by means of a noise kernel bitensor, and the semiclassical Einstein equations are replaced by the so-called Einstein-Langevin equations.

In this work, after reviewing some generalities on the structure of the orbits in both Newtonian gravity and GR, we analyze the geodesics of a Schwarzschild space-time¹ [28,29]

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

affected by an uncertainty in the mass parameter modeled in terms of a Gaussian distribution.

II. ORBITS IN NEWTONIAN GRAVITY AND IN THE SCHWARZSCHILD GEOMETRY

The equation that governs the radial motion of a test particle in the Schwarzschild metric (1) or in Newtonian physics can be written as [1,2]

¹We use units with $c = 1$ and denote the Newton constant with G .

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$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 = \frac{E^2}{2} - V(r) \equiv \mathcal{E} - V(r), \quad (2)$$

where we denoted with E and L , respectively, the conserved energy and angular momentum per unit mass in the massive case ($\varepsilon = 1$) or the conserved energy and angular momentum in the massless case ($\varepsilon = 0$). The potential in the above equation reads

$$V(r) = \frac{\varepsilon}{2} \left(1 - \frac{2GM}{r} \right) + \frac{L^2}{2r^2} - \gamma \frac{GML^2}{r^3}, \quad (3)$$

where the parameter $\gamma = 1$ in GR and $\gamma = 0$ (and $\varepsilon = 1$) in Newtonian gravity. Equation (2) formally resembles the equation for a classical particle of unit mass and energy \mathcal{E} moving in a one-dimensional potential² V (the conserved energy per unit mass is E , but the effective potential corresponds to $\mathcal{E} = E^2/2$).

A. Circular orbits

The stationary points of the potential V represent circular orbits, whose radius r_c is thus given by

$$0 = \left. \frac{dV}{dr} \right|_{r=r_c} \sim \varepsilon GM r_c^2 - L^2 r_c + 3\gamma GML^2. \quad (4)$$

Of course, these circular orbits are stable (unstable) if they correspond to a minimum (maximum) of the potential.

1. Newtonian gravity

In Newtonian gravity ($\gamma = 0$), we have the following well-known results:

- (a) For massless particles ($\varepsilon = 0$), no circular orbits exist. In fact, bound orbits do not exist, in general (massless particles move on a straight line).
- (b) For massive particles ($\varepsilon = 1$), there are stable circular orbits at the radius

$$r_c = \frac{L^2}{GM} \quad (5)$$

as well as bound orbits that oscillate around the radius r_c . In general, if the energy is greater than the asymptotic value³ $E = 1$, the orbits are unbound (parabolas or hyperbolas); otherwise, they are bound (circles or ellipses).

2. General relativity

In GR ($\gamma = 1$), the term GML^2/r^3 becomes important when r is small and the potential V vanishes at the

²The potential V is actually the *potential energy* (per unit mass).

³This follows from $V(r \rightarrow \infty) = 1/2$ in Eq. (2).

Schwarzschild radius $r = 2GM$. The following general results are also well known:

- (a) For massless particles ($\varepsilon = 0$), we find

$$r_c = 3GM \equiv r_{\text{ph}}, \quad (6)$$

which represents the innermost (unstable) circular orbit of a photon. Note that the radius of this orbit does not depend on L .

- (b) For massive particles ($\varepsilon = 1$), the zeros of Eq. (4) are given by

$$r_{\pm} = \frac{L^2}{2GM} \left(1 \pm \sqrt{1 - \frac{12G^2M^2}{L^2}} \right) \equiv \frac{L^2(1 \pm \chi)}{2GM}. \quad (7)$$

Hence, when $L > \sqrt{12}GM \approx 3.46GM$ (that is, χ is real), there is an inner unstable circular orbit ($r_{\text{in}} = r_-$) and an outer stable circular orbit ($r_{\text{out}} = r_+$). For large L , we have

$$\lim_{L \rightarrow \infty} r_{\text{in}} = 3GM, \quad (8)$$

so that the unstable orbit approaches the massless orbit (6), whereas

$$r_{\text{out}} \sim \frac{L^2}{GM} \quad (9)$$

and the stable circular orbit moves farther and farther away, approaching the Newtonian expression (5). Conversely, by decreasing L the two orbits come closer together and coincide for $L = \sqrt{12}GM$. The common radius of this (stable) circular orbit is⁴

$$r_c = 2r_{\text{ph}} = 6GM. \quad (10)$$

Finally, no circular orbits are possible when $L < \sqrt{12}GM$ (since χ becomes imaginary).

B. Noncircular orbits

Noncircular orbits in GR are not perfectly closed ellipses. Nonetheless, they can be viewed, to a good level of approximation, as ellipses that precess. Thus, it is rather convenient to describe the evolution of the radial coordinate r as a function of the angular coordinate φ , i.e., $r = r(\varphi)$.

Recalling that $L = r^2 d\varphi/d\tau$ [1], from Eq. (2) we obtain

$$\left(\frac{dr}{d\varphi} \right)^2 = \frac{E^2 - \varepsilon}{L^2} r^4 + \frac{2\varepsilon GM}{L^2} r^3 - r^2 + 2\gamma GM r. \quad (11)$$

⁴This is the innermost stable circular orbit.

Letting $x \equiv L^2/GMr$, Eq. (11) becomes

$$\left(\frac{dx}{d\varphi}\right)^2 = \frac{L^2(E^2 - \varepsilon)}{G^2M^2} + 2\varepsilon x - x^2 + \gamma \frac{2G^2M^2}{L^2} x^3, \quad (12)$$

which, upon differentiating with respect to φ , can also be written as

$$\frac{d^2x}{d\varphi^2} = \varepsilon - x + \gamma \frac{3G^2M^2}{L^2} x^2. \quad (13)$$

Note that $x = 1$ corresponds to the Newtonian circular orbit (5).

For the purpose of finding analytical solutions to Eq. (12), we define the dimensionless parameters

$$\alpha = \frac{GM}{L}, \quad \beta = \frac{GM}{EL} = \frac{\alpha}{E}, \quad \rho = \frac{2G^2M^2}{L^2} = 2\alpha^2, \quad (14)$$

so that Eq. (12) will read

$$\left(\frac{dx}{d\varphi}\right)^2 = \left(\frac{1}{\beta^2} - \frac{\varepsilon}{\alpha^2}\right) + 2\varepsilon x - x^2 + \gamma\rho x^3. \quad (15)$$

Analogously, Eq. (13) can be rewritten as

$$\frac{d^2x}{d\varphi^2} = \varepsilon - x + \frac{3}{2}\gamma\rho x^2. \quad (16)$$

1. Newtonian gravity

For Newtonian gravity ($\gamma = 0$), Eq. (15) reduces to

$$\begin{aligned} \left(\frac{dx}{d\varphi}\right)^2 &= \left(\frac{1}{\beta^2} - \frac{\varepsilon}{\alpha^2}\right) + 2\varepsilon x - x^2 \\ &\equiv -(x - x_1)(x - x_2), \end{aligned} \quad (17)$$

where the roots $x_1 = \varepsilon - \kappa$ and $x_2 = \varepsilon + \kappa$, with

$$\kappa = \sqrt{\frac{1}{\beta^2} - \frac{\varepsilon}{\alpha^2} + \varepsilon^2}. \quad (18)$$

The general solution to Eq. (17) is then given by

$$\begin{aligned} x(\varphi) &= x_1 + (x_2 - x_1) \sin^2\left(\frac{\varphi}{2} + \delta\right) \\ &= \varepsilon + (x_1 - \varepsilon) \cos(\varphi + 2\delta), \end{aligned} \quad (19)$$

where δ is an integration constant determined by the initial condition $x(\varphi_0) = x_0$. Equation (19) describes a conic of eccentricity

$$e = \frac{x_2 - x_1}{x_2 + x_1} = \frac{\kappa}{\varepsilon}. \quad (20)$$

For massive particles ($\varepsilon = 1$), we see that

- (i) if $x_1 > 0$ and $x_1 \neq x_2$ (i.e., $0 < \kappa < 1$), the orbit is an ellipse where x_1 and x_2 represent the distances of furthest and closest approach, respectively;
- (ii) if $x_1 = x_2$ (i.e., $\kappa = 0$), the orbit is circular, with the radius given by the Newtonian value (5) (this happens when $E^2 = 1 - G^2M^2/L^2$);
- (iii) if $x_1 = 0$ (i.e., $\kappa = 1$, namely, $\alpha = \beta$), the orbit is a parabola and x_2 represents the distance of closest approach;
- (iv) if $x_1 < 0$ (i.e., $\kappa > 1$), the orbit is a hyperbola and x_2 represents again the distance of closest approach.

For massless particles ($\varepsilon = 0$), $-\kappa \leq x \leq \kappa$ and the orbit is always an unbound straight line.

2. General relativity

In the GR case ($\gamma = 1$), the cubic polynomial on the right-hand side of Eq. (15) admits three roots, which we denote by x_1 , x_2 , and x_3 , so that

$$\left(\frac{dx}{d\varphi}\right)^2 = \rho(x - x_1)(x - x_2)(x - x_3), \quad (21)$$

with $x_1 + x_2 + x_3 = 1/\rho$. The general solution of Eq. (21) is

$$x(\varphi) = x_1 + (x_2 - x_1) \operatorname{sn}^2\left(\frac{\varphi}{2} \sqrt{\rho(x_3 - x_1)} + \delta, k\right), \quad (22)$$

where $\operatorname{sn}(z, k)$ is the Jacobi elliptic function with argument z and elliptic modulus

$$k = \sqrt{\frac{x_2 - x_1}{x_3 - x_1}}. \quad (23)$$

The integration constant δ is again found from the initial condition $x(\varphi_0) = x_0$.

The roots x_1 , x_2 , and x_3 can be either all real or one real and two complex conjugates. In the first case, we name the roots such that $x_1 \leq x_2 \leq x_3$, while in the second case we denote the real root as x_1 . The following situations are therefore possible:

- (i) $x_1 < x_2 < x_3$.—If all three roots are distinct real numbers, the second derivative

$$\begin{aligned} \frac{d^2x}{d\varphi^2} &= \frac{\rho}{2} [(x - x_2)(x - x_3) + (x - x_1)(x - x_3) \\ &\quad + (x - x_1)(x - x_2)] \end{aligned} \quad (24)$$

is positive, negative, and positive at $x = x_1$, $x = x_2$, and $x = x_3$, respectively. It follows that a graph of x versus φ can either oscillate between x_1 and x_2

or move away from x_3 towards infinity (which corresponds to $r \rightarrow 0$). If $x_1 < 0$, only part of an *oscillation* will actually occur. This corresponds to the particle coming from infinity, getting near the central mass, and then moving away again toward infinity, like the hyperbolic trajectory in the Newtonian case.

- (ii) $x_1 < x_2 = x_3$.—If the particle has just the right amount of energy (for a given angular momentum), x_2 and x_3 will merge. The radius $r_{\text{in}} \sim 1/x_2 = 1/x_3$ is called the *inner radius*, and we have $3/2\rho \leq r_{\text{in}} \leq 3\rho$. There are three solutions in this case:
- (1) the orbit spirals into r_{in} , approaching the inner radius (asymptotically) as a decreasing exponential in φ , τ , or t ;
 - (2) one can have a circular orbit at the inner radius r_{in} ; or
 - (3) one can have an orbit that spirals down from the inner radius r_{in} towards the central singularity.
- Since $\text{sn}(z, 1) = \tanh(z)$, in this case Eq. (22) simplifies to

$$x(\varphi) = x_1 + (x_2 - x_1) \tanh^2 \left(\frac{\varphi}{2} \sqrt{\rho(x_2 - x_1)} + \delta \right). \quad (25)$$

- (iii) $x_1 = x_2 < x_3$.—A circular orbit also results when $x_1 = x_2$. In this case, the radius $r_{\text{out}} \sim 1/x_1 = 1/x_2$ is called the *outer radius*.
- (iv) $x_2, x_3 \in \mathbb{C}$.—If the particle approaches the center with enough energy and sufficiently low angular momentum, then only x_1 will be real. This corresponds to the particle spiraling and falling into a black hole with a finite change in φ .

Equivalent forms of the above solutions are briefly reviewed in the Appendix.

III. THE CENTRAL MASS AS A RANDOM VARIABLE

We shall now consider the mass M as a random variable, assumed to be normally distributed with mean value M_0 and standard deviation σ_M (the variance is σ_M^2) [30]. Equivalently, we may denote the mass M as $M_0 + \delta M$, where δM is a random variable with zero mean value, $\mu_{\delta M} = 0$, and variance σ_M^2 , that is,

$$\delta M \sim \mathcal{N}(0, \sigma_M^2), \quad (26)$$

where $\mathcal{N}(\mu_z, \sigma_z^2)$ is the normalized Gaussian distribution of mean value μ_z .

Since δM is a random variable, any trajectory parametrized as $r = r(\varphi)$ will be a functional of this random variable, which, for a given value of the angle φ , will therefore become a random variable itself. In order to stress

that the angle φ is now seen as a parameter and δM is treated as an independent (random) variable, we shall employ the notation $r = r_\varphi(\delta M)$.

Since Eq. (12) has the known analytical solution (22), it is conceivable to carry out the analysis of the probability distribution of r_φ , representing the position of the particle at a given angle φ , without relying on any perturbation methods.

In order to show explicitly the dependence on the mass M , we first rewrite Eq. (21) [equivalent to Eq. (12)] as

$$\left(\frac{du}{d\varphi} \right)^2 = \frac{\rho}{M} (u - u_1)(u - u_2)(u - u_3), \quad (27)$$

where we introduced the (dimensionful) variable

$$u = Mx = \frac{L^2}{Gr}, \quad (28)$$

and $u_i = Mx_i$, for $i = 1, 2, 3$. The solution of Eq. (27) [equivalent to the solution (22)] is now given by

$$u(\varphi) = u_1 + (u_2 - u_1) \text{sn}^2 \left(\frac{\varphi}{2} \sqrt{\frac{\rho}{M} (u_3 - u_1)} + \delta, k \right), \quad (29)$$

where

$$k = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}} \quad (30)$$

and $u(\varphi_0) = u_0$. It is important to remark that $u_1, u_2, u_3, \delta, \rho$, and k are now all functions of M . Since we are assuming M to be a random variable with a given probability density function f_M , the variable $u_\varphi \equiv u(\varphi)$ will turn into a random variable with probability distribution f_{U_φ} .

As is customary in probability theory, from now on we shall indicate the random variables with capital letters (M and U_φ , respectively, for the mass and the dependent variable u_φ) and with lowercase letters (m and u_φ , respectively) the values taken on by the random variables. We can therefore write Eq. (27) as

$$U_\varphi = u_1 + (u_2 - u_1) \text{sn}^2 \left(\frac{\varphi}{2} \sqrt{\frac{2G^2 M}{L^2} (u_3 - u_1)} + \delta, k \right). \quad (31)$$

In order to determine the probability density function f_{U_φ} of the random variable U_φ , it is useful to work with its cumulative distribution function $\mathcal{F}_{U_\varphi}(u_\varphi)$, defined as the probability \mathcal{P} of U_φ taking on values smaller than or equal to u_φ . Since U_φ is a monotonically increasing function of M [which we shall denote with the symbol g , $U_\varphi = g(M)$], we have

$$\begin{aligned}
 \mathcal{F}_{U_\varphi}(u_\varphi) &= \mathcal{P}(U_\varphi \leq u_\varphi) = \mathcal{P}(0 \leq M \leq m) \\
 &= \int_0^m f_M(\xi) d\xi \\
 &= \int_0^{g^{-1}(u_\varphi)} f_M(\xi) d\xi,
 \end{aligned} \tag{32}$$

where $u_\varphi = g(m)$. Making use of the fundamental theorem of calculus, we obtain

$$\begin{aligned}
 f_{U_\varphi}(u_\varphi) &= \frac{d\mathcal{F}_{U_\varphi}(u_\varphi)}{du_\varphi} = \frac{d\mathcal{F}_{U_\varphi}(u_\varphi)}{dg^{-1}(u_\varphi)} \frac{dg^{-1}(u_\varphi)}{du_\varphi} \\
 &= f_M(m) \frac{dg^{-1}(u_\varphi)}{du_\varphi} \\
 &= \frac{f_M(m)}{g'(m)} = \frac{f_M(m)}{|g'(m)|}.
 \end{aligned} \tag{33}$$

Were the function g monotonically decreasing, the previous derivation could be repeated just changing the sign of the inequality, that is,

$$\begin{aligned}
 \mathcal{F}_{U_\varphi}(u_\varphi) &= \mathcal{P}(U_\varphi \leq u_\varphi) = \mathcal{P}(M > m) \\
 &= \int_m^\infty f_M(\xi) d\xi \\
 &= - \int_\infty^{g^{-1}(u_\varphi)} f_M(\xi) d\xi,
 \end{aligned} \tag{34}$$

which would lead to

$$f_{U_\varphi}(u_\varphi) = - \frac{f_M(m)}{g'(m)} = \frac{f_M(m)}{|g'(m)|}. \tag{35}$$

The case of a nonmonotonic function g can be treated similarly, just splitting the integration range so as to have a monotonic function in each interval. It is then easy to see that, if the function f_{U_φ} has N roots, which we denote as $u_\varphi = g(m_k)$, for $k = 1, 2, \dots, N$, we obtain the general formula

$$f_{U_\varphi}(u_\varphi) = \sum_{k=1}^N \frac{f_M(m_k)}{|g'(m_k)|}. \tag{36}$$

The validity of the previous relation is not limited to U_φ and M . In order to stress this, and for future use, the previous relation can be written as

$$f_Y(y) = \sum_{k=1}^N \frac{f_X(x_k)}{|g'(x_k)|}, \tag{37}$$

for a generic random variable Y function of X (with whatever probability density function f_X), such that $Y = g(X)$.

It is possible to extend the idea described above and to assume other parameters as random variables as well, not just the mass M . For instance, the initial position u_0 of the particle can be assumed as a random variable. In such a case, Eq. (37) can be applied, where the dependent and independent random variables are U_φ and U_0 , respectively, and $U_\varphi = g(U_0)$.

The next step is to assume more parameters simultaneously as random variables. For example, the energy E and the angular momentum L might actually be considered as random variables with mean values equal to their nominal values (say, μ_E and μ_L , respectively) and variances accounting for their uncertainties (say, σ_E^2 and σ_L^2 , respectively). The random variables E and L would be in this case independent and follow normal distributions, $E \sim \mathcal{N}(\mu_E, \sigma_E^2)$ and $L \sim \mathcal{N}(\mu_L, \sigma_L^2)$, respectively. In this case, it is also conceivable that the variables E and L are not independent. In this scenario, it could be more accurate to introduce a joint probability density function $f_{EL}(e, l)$, instead of two independent probability density functions $f_E(e)$ and $f_L(l)$ for the energy and angular momentum separately. This approach is, of course, more general, and the case of independent variables could be recovered when $f_{EL}(e, l) = f_E(e)f_L(l)$.

This idea can be generalized even further. If the energy E , the angular momentum L , and the initial position u_0 are all affected by uncertainties and should be treated as random variables, we could have a joint probability density function $f_{ELU_0}(e, l, u_0)$. In all these cases, the variable U_φ representing the quantity u at some angle φ would be a multivariate random variable, i.e., a function of more than one random variable. We shall continue to use the symbol g for this function. However, a straightforward generalization of Eq. (37) to this case does not appear possible.

In the case of two random variables, say, E and L , with joint probability density function f_{EL} , a procedure similar to the one presented for the case where M is treated as a random variable would yield

$$\begin{aligned}
 \mathcal{F}_{U_\varphi}(u_\varphi) &= \mathcal{P}(U_\varphi \leq u_\varphi) = \mathcal{P}(g(E, L) \leq u_\varphi) \\
 &= \int_{\mathcal{D}(u_\varphi)} f_{EL}(\xi, \eta) d\xi d\eta,
 \end{aligned} \tag{38}$$

where $\mathcal{D}(u_\varphi) \subset \mathbb{R}^2$ is the region of the EL plane over which $g(E, L) \leq u_\varphi$. Once the cumulative distribution function $\mathcal{F}_{U_\varphi}(u_\varphi)$ is known, the probability density function $f_{U_\varphi}(u_\varphi)$ can be finally obtained by means of the definitions of \mathcal{F}_{U_φ} and f_{U_φ} , i.e.,

$$f_{U_\varphi}(u_\varphi) = \frac{d\mathcal{F}_{U_\varphi}(u_\varphi)}{du_\varphi}. \tag{39}$$

In contrast to the case of a single random variable, this procedure for obtaining $f_{U_\varphi}(u_\varphi)$ is unlikely to be feasible

analytically, due to the complexity of both the function g and the domain of integration \mathcal{D} . It can probably be done numerically, though.

Following this approach, it should be possible to investigate the uncertainty on the orbit, described by a random variable U_φ , due to the uncertainties on the parameters E , L , and U_0 (and possibly others), and to compare this uncertainty on the orbit to the one produced by an uncertainty of the central mass M . The basis of this approach is laid on two key ingredients:

- (1) finding the (analytical or numerical) solution of the equation of motion (13) and
- (2) computing the probability density function of the orbit, once the probability density functions of the parameters are given, following the method described above.

IV. MASS UNCERTAINTY AND NONCIRCULAR ORBITS

In Sec. II B 2, solutions of the equation of motion (2) were reviewed, and qualitatively different kinds of orbit have been

identified. For these orbits, the effect of assuming the mass M as a normally distributed random variable is now investigated for orbits with set initial conditions for the position and velocity. Let us first recall that we shall be using the (dimensionful) variable u in Eq. (28), like in the previous section, and that the orbits are divided into the following four groups (u_0 represents the initial position of the particle):

- (i) If the third degree polynomial on the right-hand side of Eq. (27) has three distinct real roots ($u_1 < u_2 < u_3$), we can have the following behaviors:
 - if $0 < u_1 < u_0 < u_2 < u_3$, the graph of u versus φ oscillates between u_1 and u_2 (first case);
 - if $0 < u_1 < u_2 < u_3 < u_0$, the graph of u versus φ moves away from u_3 towards infinity (second case);
 - if $u_1 < 0 < u_0 < u_2 < u_3$, only part of an oscillation occurs, with the particle first approaching the central mass and then moving away towards infinity (third case);
- (ii) If the polynomial has one real root and two complex conjugates roots, the particle falls into the black hole

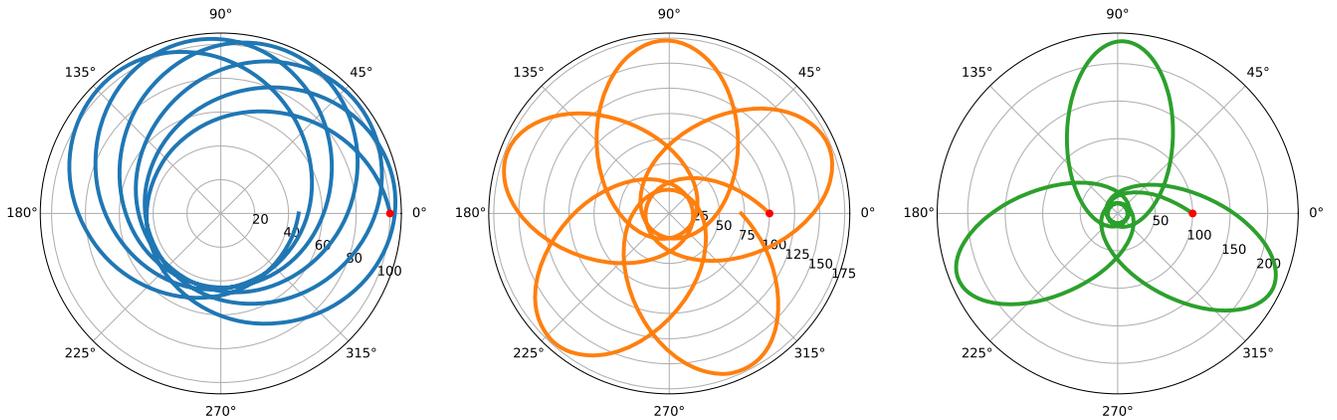


FIG. 1. (First case) Orbits for different values of $M = 1.5, 2, 2.5$ (left to right, with $G = 1, E = 0.9, L = 10$, and $u_0 = 1$.) The red dot is the starting point, and the colors are the same as in Fig. 2.

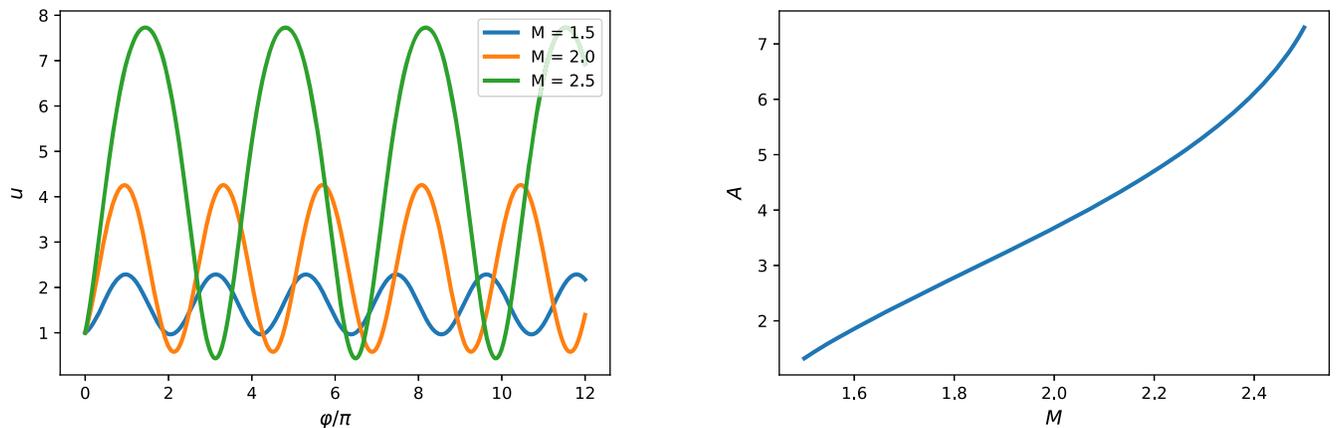


FIG. 2. (First case) Graphs of $u(\varphi)$ for different values of $M = 1.5, 2, 2.5$ (with $G = 1, E = 0.9, L = 10$, and $u_0 = 1$) (left panel) and maximum amplitude of the oscillations as a function of the mass M (right panel).

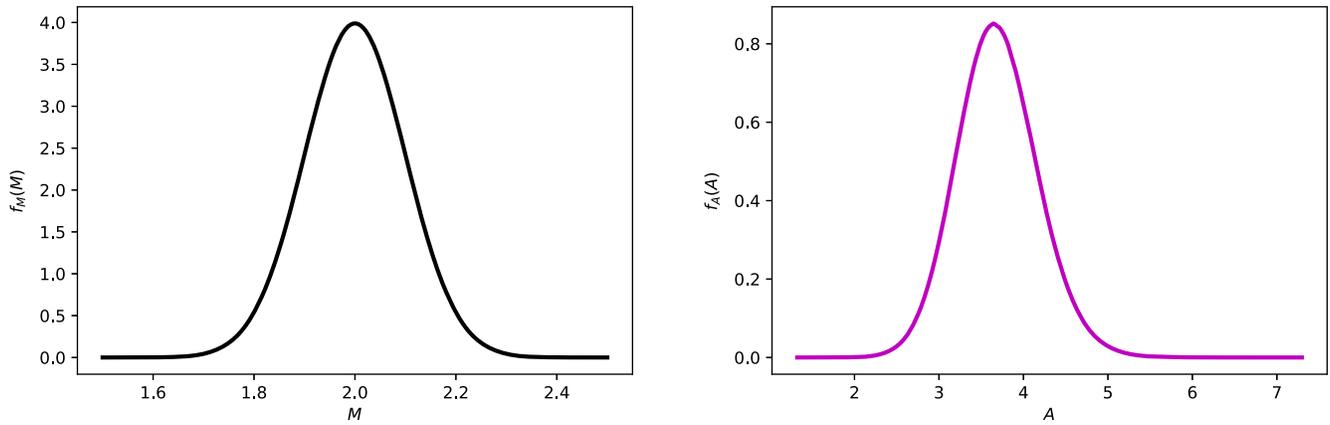


FIG. 3. (First case) Probability density function for the mass M (left panel) and for the amplitude A (right panel). The amplitude corresponding to the mean value of M is ~ 3.68 ; the value of the amplitude corresponding to the maximum of the distribution of A if ~ 3.64 . (The random variable A is not normally distributed, and its distribution is not symmetric).

and the orbit is a spiral with a finite change in φ (fourth case).

It is important to recall that, if the energy and the angular momentum of the particle are such that $u_1 < u_2 = u_3$, three behaviors are possible: the orbit may be circular at the

radius $r_{in} \sim 1/u_2 = 1/u_3$ (the inner radius), or it may spiral in asymptotically approaching the inner radius or spiral down from the inner radius to the central point. If instead $u_1 = u_2 < u_3$, a circular orbit at the radius $r_{out} \sim 1/u_1 = 1/u_2$ (the outer radius) is also possible. In all of

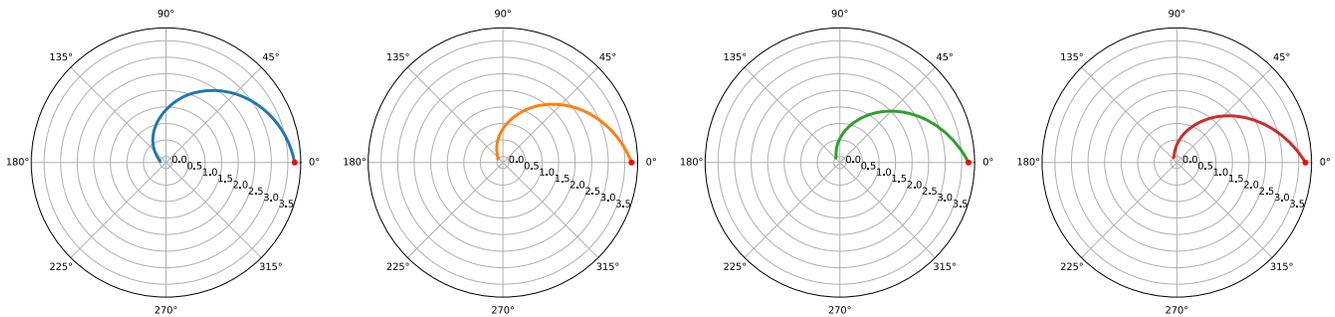


FIG. 4. (Second case) Orbits for values of $M = 1.7, 1.9, 2.1, 2.3$ (left to right, with $G = 1, E = 0.99, L = 10$, and $u_0 = 27$). The red dot is the starting point, and the colors are the same as in Fig. 5(a).

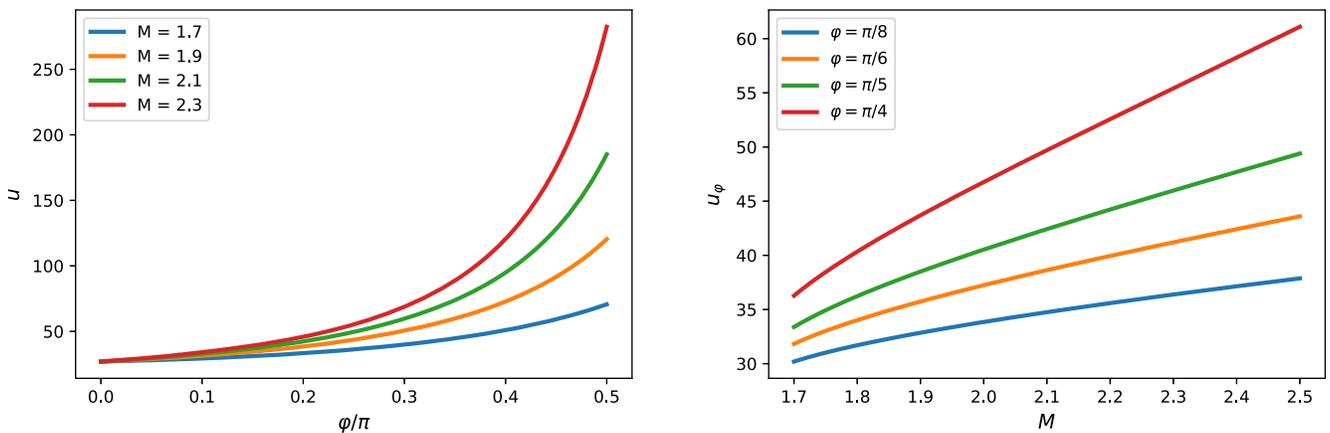


FIG. 5. (Second case) Graphs of $u(\varphi)$ for values of $M = 1.7, 1.9, 2.1, 2.3$ (left panel, with $G = 1, E = 0.99, L = 10$, and $u_0 = 27$) and u_φ as a function of the mass M for fixed angles φ (right panel).

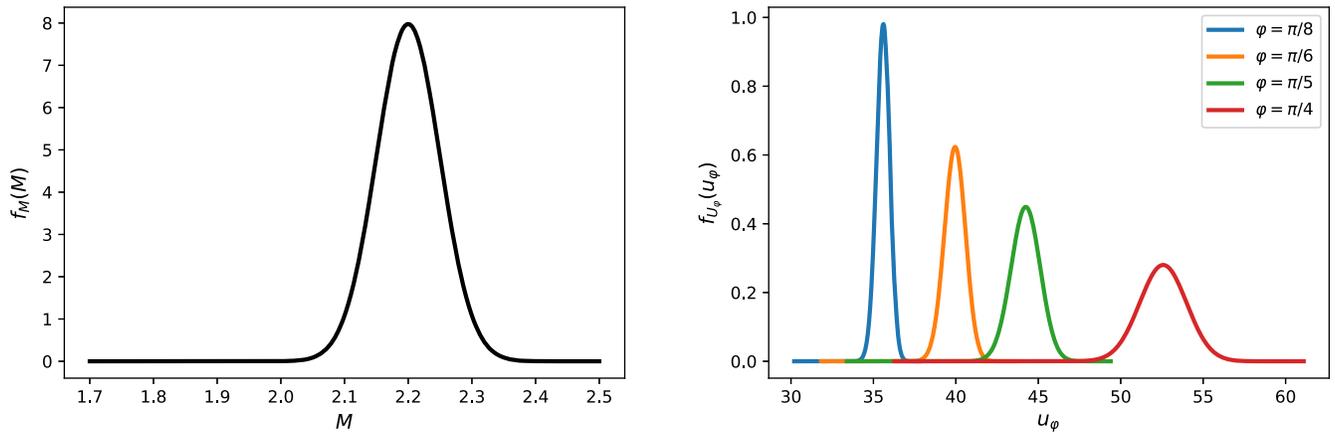


FIG. 6. (Second case) Probability density function for the mass M (left panel) and for the variable u_φ for different angles φ (right panel).

these cases, after the slightest perturbation of the central mass M , we fall back in one of the four previous cases. For example, when the stable orbit is perturbed, we fall in the first case.

In the following, we shall analyze the above four cases. For the purpose of making the results more readable, we will first show plots of the orbits corresponding to fixed

numerical values of the parameters of relevance for each distribution of the mass M and specifically chosen for the purpose of displaying clearly the qualitative features of the orbits in the case under consideration. We then display the chosen distribution for the mass M and the final distribution of the quantity of relevance again for each specific case (e.g., amplitude of the oscillation, minimum

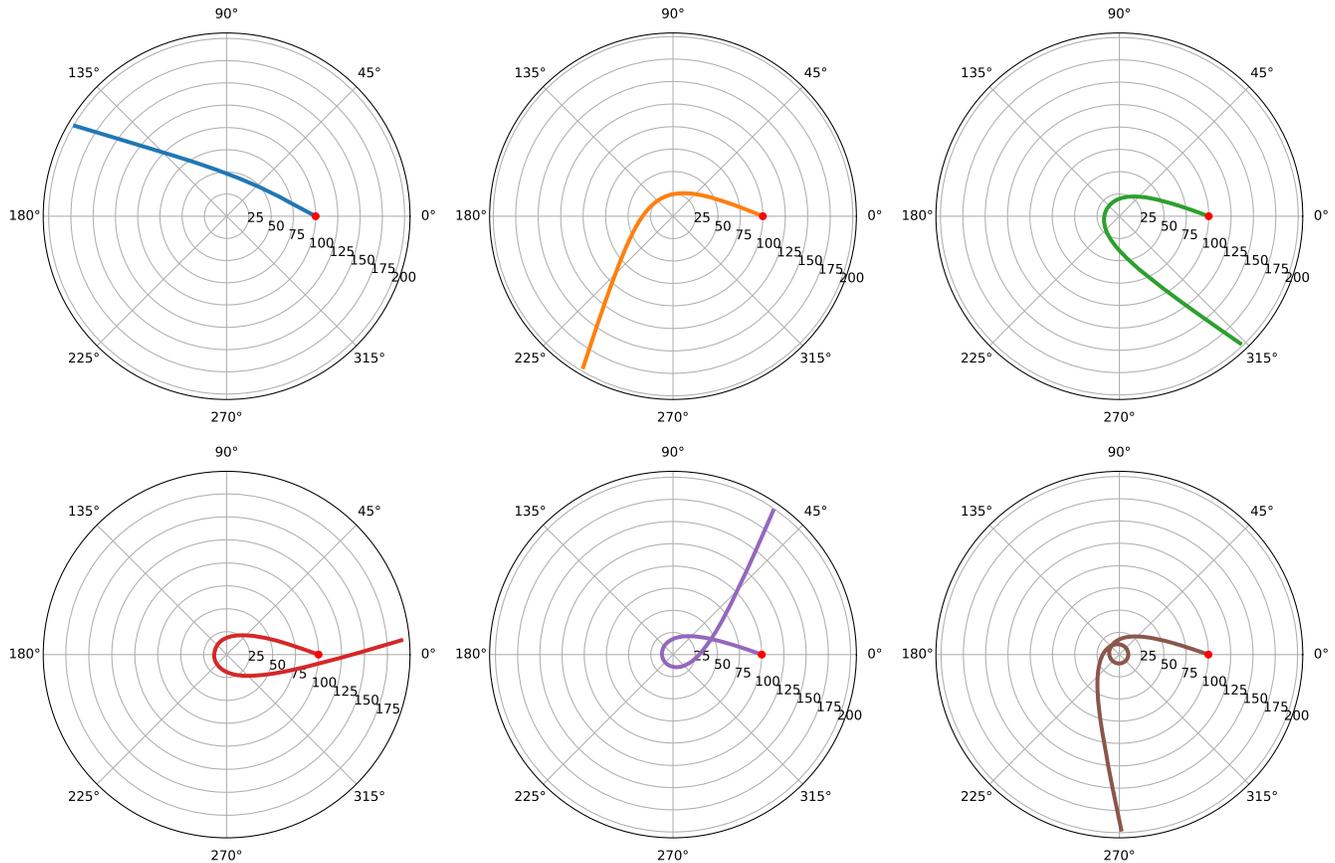


FIG. 7. (Third case) Orbits for values of M ranging from 0.2 to 2.4 (top left to bottom right, with $G = 1$, $E = 1.02$, $L = 10$, and $u_0 = 1$.) The red dot is the starting point, and the colors are the same as in Fig. 8.

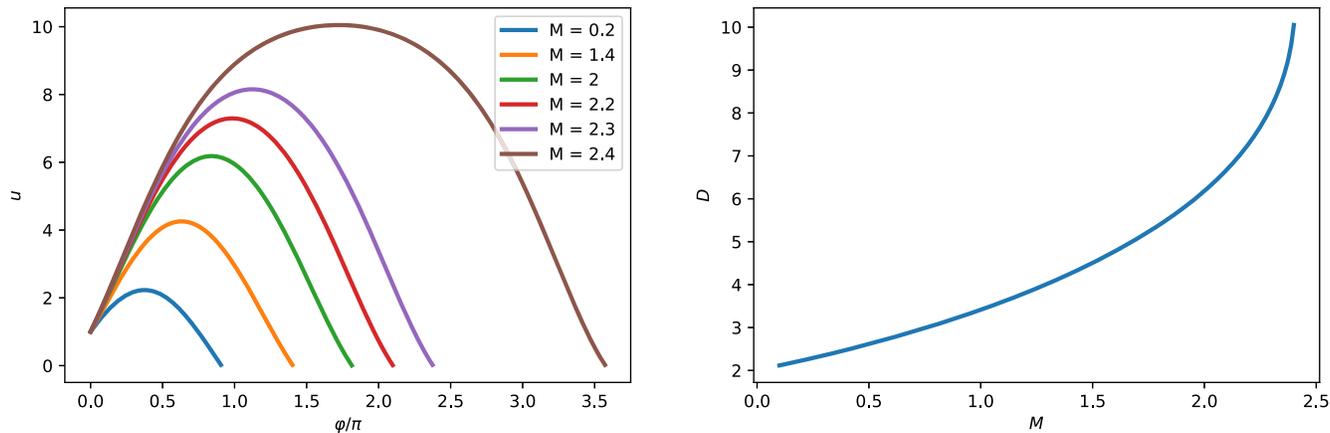


FIG. 8. (Third case) Graphs of $u(\varphi)$ for values of M ranging from 0.2 to 2.4 (left panel, with $G = 1$, $E = 1.02$, $L = 10$, and $u_0 = 1$) and maximum value D of u (inverse minimum distance of the particle from the central mass) as a function of the mass M (right panel).

distance from the center, etc.). In particular, we shall use units with $G = 1$.

A. First case

When the orbit oscillates between u_1 and u_2 , it is interesting to analyze the amplitude A (and the period) of the oscillations as a function of the random variable M . For example, the trajectories $r = r(\varphi)$ corresponding to three different values of M are shown in Fig. 1 in polar form. Cartesian plots of $u(\varphi)$ are shown in Fig. 2, together with the amplitude of the oscillations A , as a function of the mass M . (When comparing Fig. 1 with Fig. 2, it should be noted that $r = L^2/Gu = 100/u$.)

If the random variable M is normally distributed with mean value $\mu_M = 1.5$ and standard deviation $\sigma_M = 0.1$, the probability density function f_A of the random variable A can be obtained by means of Eq. (37) once the probability density function f_M and the function $A = g(M)$ are known. The probability density functions f_M and f_A are shown in Fig. 3.

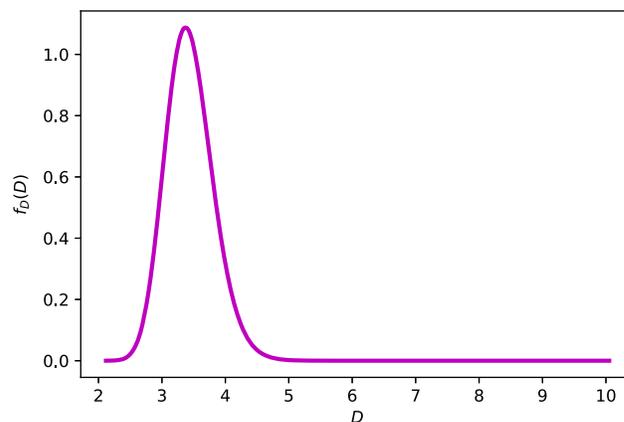
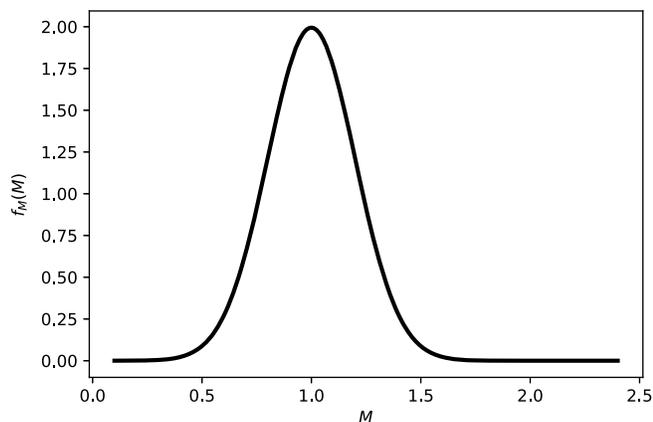


FIG. 9. (Third case) Probability density function for the mass M (left panel) and for the amplitude D inversely proportional to the minimum distance from the central mass (right panel).

B. Second case

The second possibility listed above is that the particle will monotonically move towards the central mass ($u \rightarrow \infty$), as is shown by the polar trajectories $r(\varphi)$ represented in Fig. 4. For such a type of orbits, the function $u(\varphi)$ is plotted for several values of M in Fig. 5. For this case, it is therefore interesting to analyze the behavior of u_φ as a function of M for different values of the angle φ that parameterizes the orbit.

If the random variable M is normally distributed with mean value $\mu_M = 1$ and standard deviation $\sigma_M = 0.05$, the resulting probability density function f_U of the random variable u_φ , for some exemplary values of φ , is shown in Fig. 6 along with the corresponding probability density function f_M . As expected, the variance of the probability density function of u_φ increases with φ .

C. Third case

Another type of solution is the one representing a particle coming from infinity towards the central mass and then

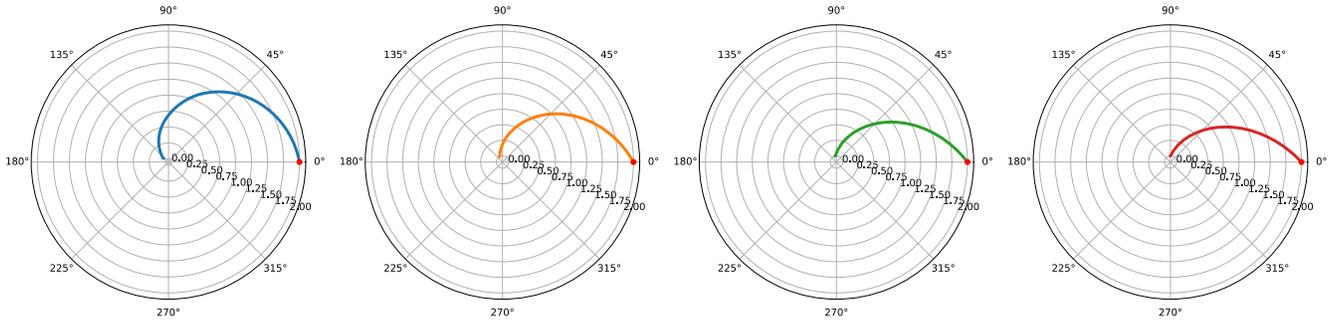


FIG. 10. (Fourth case) Orbits for values of $M = 1, 1.3, 1.6, 1.9$ (left to right, with $G = 1, E = 0.98, L = 10$, and $u_0 = 50$). The red dot is the starting point, and the colors are the same as in Fig. 11.

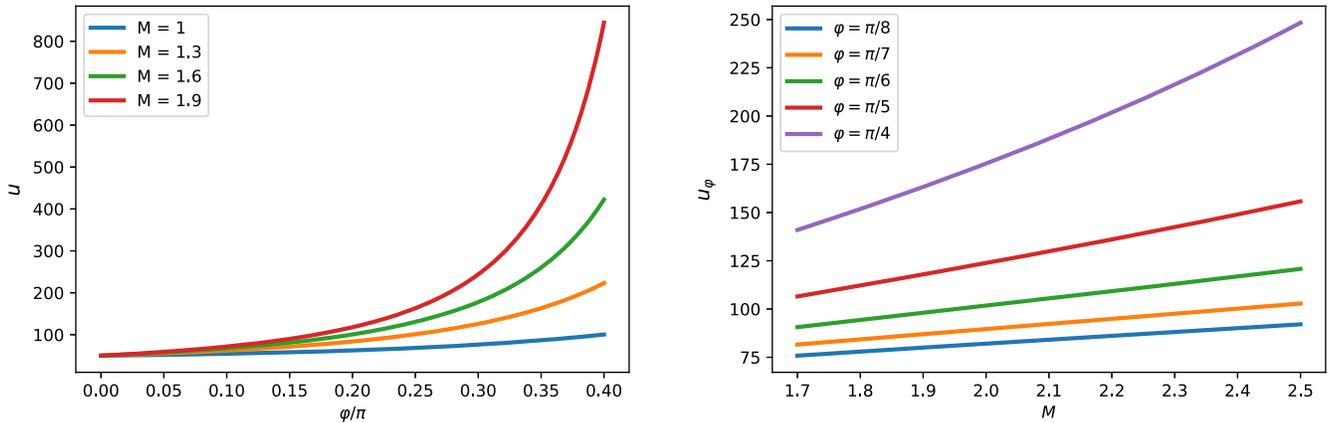


FIG. 11. (Fourth case) Graphs of $u(\varphi)$ for values of $M = 1, 1.3, 1.6, 1.9$ (left panel, with $G = 1, E = 0.98, L = 10$, and $u_0 = 50$) and maximum value D of u (inverse minimum distance of the particle from the central mass) as a function of the mass M (right panel).

moving back towards infinity ($r \rightarrow \infty$, i.e., $u \rightarrow 0$). Some typical polar trajectories $r(\varphi)$ of this kind are shown in Fig. 7, and the corresponding functions $u(\varphi)$ are displayed in Fig. 8. The maximum value D of u reached along the trajectory (proportional to the inverse of the minimum distance from the central mass) is also plotted as a function of M .

If the random variable M is normally distributed with mean value $\mu_M = 1$ and standard deviation $\sigma_M = 0.2$, the

probability density function f_D of the random variable D is again obtained by means of Eq. (37). The probability density functions f_M and f_D are shown in Fig. 9.

D. Fourth case

The last type of solution is the one representing a particle with enough energy and sufficiently low angular momentum which spirals into a black hole with a finite change

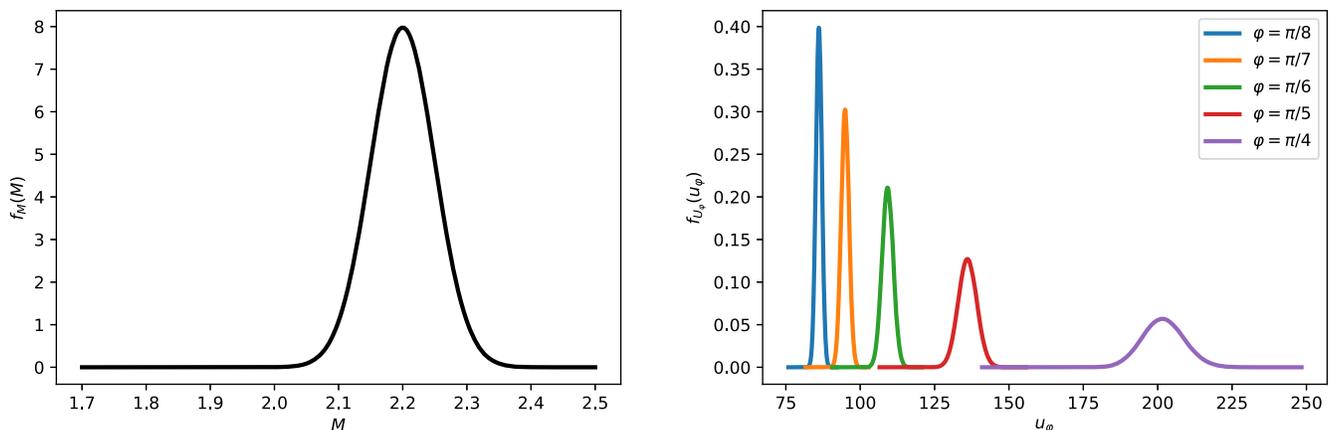


FIG. 12. (Fourth case) Probability density function for the mass M (left panel) and for the amplitude U (right panel).

in φ . Some examples of these orbits are shown in Fig. 10, and the corresponding function $u(\varphi)$ is plotted in Fig. 11, along with the behavior of u_φ as a function of M for different values of the angle φ .

For M normally distributed with mean value $\mu_M = 2.2$ and standard deviation $\sigma_M = 0.05$, the resulting probability density functions f_U for some angles φ are shown in Fig. 12.

V. PERTURBATION OF CIRCULAR ORBITS

We can now study circular orbits again using the variable u defined in Eq. (28), so that Eq. (12) reads

$$\left(\frac{du}{d\varphi}\right)^2 = \frac{E^2 L^2}{G^2} - \varepsilon \frac{L^2}{G^2} + 2\varepsilon M u - u^2 + \gamma \frac{2G^2 M}{L^2} u^3 \quad (40)$$

and Eq. (13) reads

$$\frac{d^2 u}{d\varphi^2} = \varepsilon M - u + \gamma \frac{3G^2 M}{L^2} u^2. \quad (41)$$

We wish to study the effect on circular orbits of small perturbations affecting the central mass M and the initial position of the test particle. We shall first use an analytical perturbative approach for Eq. (41) with the initial conditions

$$u(0) = u_c \equiv \frac{L^2}{G r_c}, \quad \frac{du}{d\varphi}(0) = 0, \quad (42)$$

where r_c is the radius of a circular orbit (either internal or external). Note that the initial condition on the derivative implies that an instantaneous perturbation of the mass does not instantaneously alter the tangential component of the particle velocity.

A. Mass perturbation

If we denote with δM the perturbation on the mass M , the perturbed solution can be written as

$$u = u^{(0)} + \frac{\delta M}{M} u^{(1)} + \mathcal{O}\left(\frac{\delta M^2}{M^2}\right). \quad (43)$$

Upon replacing into Eq. (41) and keeping terms up to the order of $\delta M/M$, we find

$$\begin{aligned} \frac{d^2 u^{(0)}}{d\varphi^2} + \frac{\delta M}{M} \frac{d^2 u^{(1)}}{d\varphi^2} &= \varepsilon(M + \delta M) - u^{(0)} - \frac{\delta M}{M} u^{(1)} \\ &+ \gamma \frac{3G^2}{L^2} (M + \delta M) \left(u^{(0)} + \frac{\delta M}{M} u^{(1)}\right)^2. \end{aligned} \quad (44)$$

Assuming $u^{(0)}$ satisfies the unperturbed equation (2) with fixed M , we obtain that the first-order correction must satisfy

$$\frac{d^2 u^{(1)}}{d\varphi^2} = \alpha_c u^{(1)} + \beta_c, \quad (45)$$

where

$$\alpha_c \equiv \gamma \frac{6G^2 M}{L^2} u^{(0)} - 1 \quad (46)$$

and

$$\beta_c \equiv \gamma \frac{3G^2 M}{L^2} (u^{(0)})^2 + \varepsilon M. \quad (47)$$

Moreover, the initial conditions for $u^{(1)}$ are given by

$$u^{(1)}(0) = \frac{du^{(1)}}{d\varphi}(0) = 0. \quad (48)$$

Focusing on the case $\gamma = 1$ and restricting ourselves, from now on, to the perturbation of circular orbits ($u^{(0)} = u_c$), we treat the massive and massless cases separately.

In the massive case ($\varepsilon = 1$), two circular orbits exist [see Eq. (7)]:

- (1) For the internal orbit, we have $r_c = r_{\text{in}} = r_-$ in Eq. (7), that is,

$$u_c = u_{\text{in}} = \frac{L^2 + \sqrt{L^4 - 12G^2 M^2 L^2}}{6G^2 M}, \quad (49)$$

for which $\alpha_c = \alpha > 0$ [see Eq. (7)] and $\beta_c = u_{\text{in}}$. The analytical solution of Eq. (45) is then

$$u^{(1)} = \frac{\beta_c}{\alpha_c} [\cosh(\sqrt{\alpha_c} \varphi) - 1], \quad (50)$$

and it is easy to see that, as expected, this orbit is unstable. In fact, the perturbed solution (43) to first order reads

$$u(\varphi) \simeq u_{\text{in}} \left[1 + \frac{\delta M}{M} \frac{\cosh(\sqrt{\alpha} \varphi) - 1}{\alpha} \right], \quad (51)$$

from which we can see that increasing the mass ($\delta M > 0$) leads to $r \rightarrow 0$ ($u \rightarrow \infty$). On the other hand, reducing the mass ($\delta M < 0$) leads to $r \rightarrow \infty$ ($u = 0$) for some finite value of φ .

- (2) For the external orbit, we have $r_c = r_{\text{out}} = r_+$ in Eq. (7), or

$$u_c = u_{\text{out}} = \frac{L^2 - \sqrt{L^4 - 12G^2 M^2 L^2}}{6G^2 M}, \quad (52)$$

so that $\alpha_c = -\alpha < 0$ and $\beta_c = u_{\text{out}}$. The analytical solution of Eq. (45) is now

$$u^{(1)} = \frac{\beta_c}{\alpha_c} [\cos(\sqrt{|\alpha_c|}\varphi) - 1], \quad (53)$$

which implies that the outer orbit is stable, with $u^{(1)}$ oscillating between 0 and $2\beta_c/|\alpha_c|$. The perturbed solution to first order is given by

$$u(\varphi) \simeq u_{\text{out}} \left[1 + \frac{\delta M}{M} \frac{1 - \cos(\sqrt{\alpha}\varphi)}{\alpha} \right], \quad (54)$$

so that $\delta M > 0$ leads to periodic oscillations internal ($u \geq u_{\text{out}}$) to the stable orbit $r = r_{\text{out}}$, whereas $\delta M < 0$ leads to periodic oscillations external ($u \leq u_{\text{out}}$) to the stable orbit $r = r_{\text{out}}$. This is potentially quite interesting in practical terms, because the oscillations are proportional to the perturbation δM (assuming that the oscillations are large enough to be detectable).

For the massless case ($\varepsilon = 0$), there is one possible circular orbit $r_c = r_{\text{ph}}$ in Eq. (6), or

$$u_c = u_{\text{ph}} = \frac{L^2}{3G^2M}. \quad (55)$$

In this case, Eq. (45) becomes

$$\begin{aligned} \frac{d^2u^{(0)}}{d\varphi^2} + \frac{\delta u_0}{u_0} \frac{d^2u^{(1)}}{d\varphi^2} = \gamma \frac{3G^2M}{L^2} \left[(u^{(0)})^2 + \frac{\delta u_0}{u_0} (u^{(1)})^2 + 2 \frac{\delta u_0}{u_0} u^{(0)}u^{(1)} \right] \\ + \varepsilon M - u^{(0)} - \frac{\delta u_0}{u_0} u^{(1)} \end{aligned} \quad (60)$$

so that the first-order perturbation must satisfy

$$\frac{d^2u^{(1)}}{d\varphi^2} = \alpha_c u^{(1)}, \quad (61)$$

with α_c still given by Eq. (46). Moreover, the proper initial conditions are given by

$$u^{(1)}(0) = \delta u_0, \quad \frac{du^{(1)}}{d\varphi}(0) = 0. \quad (62)$$

Similarly to the case of mass perturbations, we shall consider only circular orbits ($u^{(0)} = u_c = u_0$ and

$$\frac{d^2u^{(1)}}{d\varphi^2} = u^{(1)} + u_{\text{ph}}, \quad (56)$$

and its analytical solution is

$$u^{(1)} = u_c [\cosh(\varphi) - 1]. \quad (57)$$

It is clear that, since $u^{(1)} \rightarrow \infty$ as $\varphi \rightarrow \infty$, the orbit

$$u(\varphi) \simeq u_{\text{ph}} \left\{ 1 + \frac{\delta M}{M} [\cosh(\varphi) - 1] \right\} \quad (58)$$

is unstable, and it will fall into the singularity ($u \rightarrow \infty$) if $\delta M > 0$ or escape to infinity ($u \rightarrow 0$) if $\delta M < 0$.

B. Initial position perturbation

Let us now consider a perturbation δu_0 on the initial position u_0 , so that

$$u = u^{(0)} + \frac{\delta u_0}{u_0} u^{(1)} + \mathcal{O}\left(\frac{\delta u_0^2}{u_0^2}\right). \quad (59)$$

From Eq. (41), we have

$\delta u_0 \equiv \delta u_c$) for $\gamma = 1$ and distinguish the massive and massless cases.

In the massive case ($\varepsilon = 1$), we distinguish the internal and the external orbits:

(1) For the internal orbit $u_c = u_{\text{in}}$, we can write the perturbation as

$$u^{(1)} = \delta u_c \cosh(\sqrt{\alpha_c}\varphi), \quad (63)$$

where $\alpha_c = \alpha$. The perturbed solution

$$u(\varphi) \simeq u_{\text{in}} \left[1 + \frac{\delta u_c}{u_{\text{in}}} \cosh(\sqrt{\alpha}\varphi) \right] \quad (64)$$

is therefore unstable.

TABLE I. Overview of perturbed solutions.

ε	u_c	$u(\varphi; \delta M)$	$u(\varphi; \delta u_0)$
1	u_{in} in Eq. (49)	$u_{\text{in}} \left[1 + \frac{\delta M}{M} \frac{\cosh(\sqrt{\alpha}\varphi) - 1}{\alpha} \right]$	$u_{\text{in}} \left[1 + \frac{\delta u_c}{u_{\text{in}}} \cosh(\sqrt{\alpha}\varphi) \right]$
1	u_{out} in Eq. (52)	$u_{\text{out}} \left[1 + \frac{\delta M}{M} \frac{1 - \cos(\sqrt{\alpha}\varphi)}{\alpha} \right]$	$u_{\text{out}} \left[1 + \frac{\delta u_c}{u_{\text{out}}} \cos(\sqrt{\alpha}\varphi) \right]$
0	u_{ph} in Eq. (55)	$u_{\text{ph}} \left\{ 1 + \frac{\delta M}{M} [\cosh(\varphi) - 1] \right\}$	$u_{\text{ph}} \left[1 + \frac{\delta u_c}{u_{\text{ph}}} \cosh(\varphi) \right]$

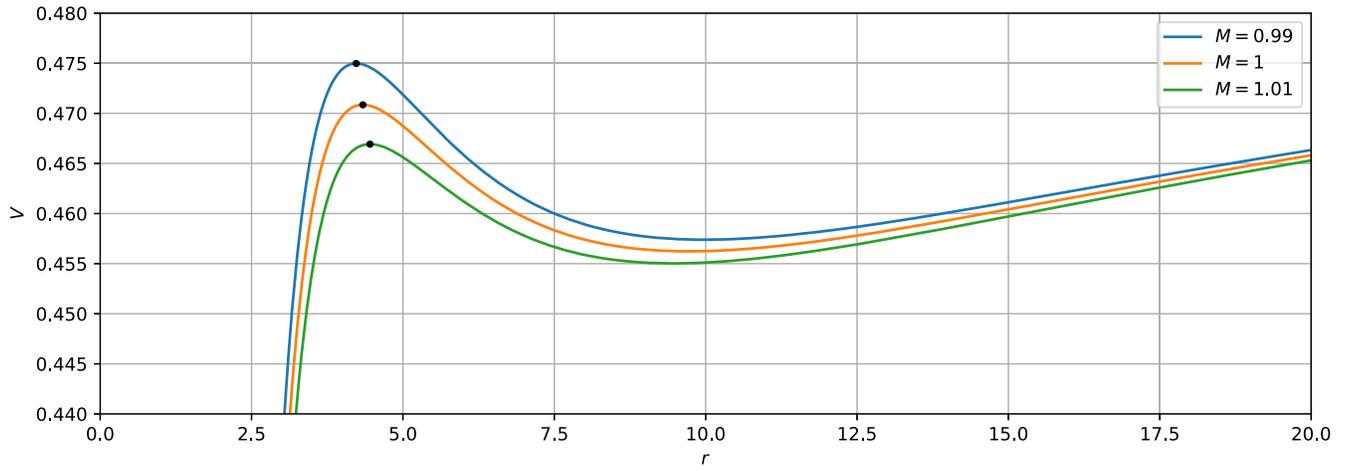


FIG. 13. Potential V as a function of the radial coordinate r for values of the central mass $M = 0.99, 1, 1.01$ (with $G = 1$ and $L = 3.75$). Black dots represent the local maximum of the potential, corresponding to the unstable circular orbits.

(2) For the external orbit $u_c = u_{out}$, the perturbation is given by

$$u^{(1)} = \delta u_c \cos(\sqrt{|\alpha_c|}\varphi), \quad (65)$$

with $\alpha_c = -\alpha$. The perturbed solution reads

$$u(\varphi) = u_{out} \left[1 + \frac{\delta u_c}{u_{out}} \cos(\sqrt{\alpha}\varphi) \right], \quad (66)$$

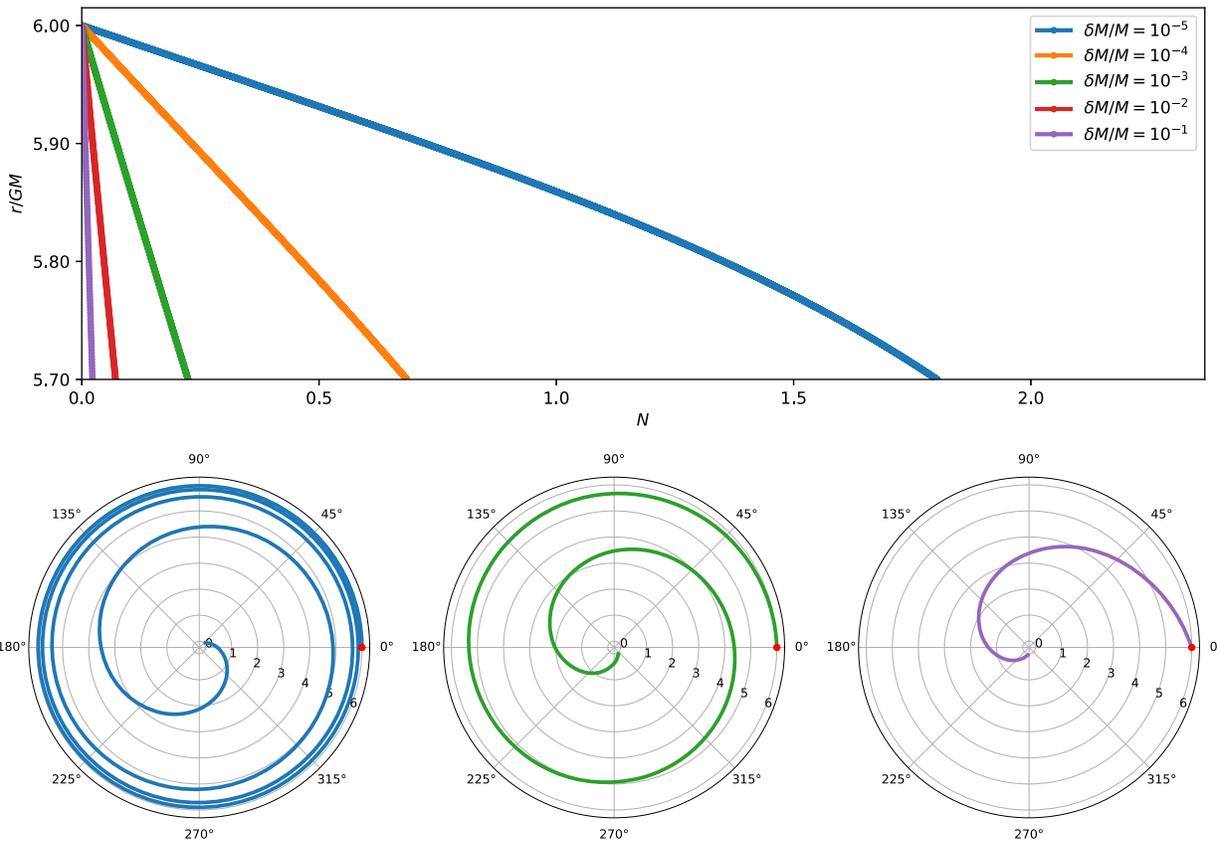


FIG. 14. Radial coordinate $\tilde{r} = r/GM$ of trajectories subject to perturbations δM starting from the unstable circular orbit for $\tilde{L} = L/GM = \sqrt{12}$ (top panel); trajectories subject to the perturbation $\delta M/M = 10^{-5}$ (bottom left panel), $\delta M/M = 10^{-3}$ (bottom center panel), and $\delta M/M = 10^{-1}$ (bottom right panel).

which is stable, with oscillations of amplitude $2\delta u_c$ around the circular orbit.

For the massless case ($\varepsilon = 0$), there is one possible circular orbit (6) with $u_c = u_{\text{ph}} = L^2/3G^2M$, and Eq. (61) reads

$$\frac{d^2u^{(1)}}{d\varphi^2} = u^{(1)}, \quad (67)$$

with the solution

$$u^{(1)} = \delta u_c \cosh(\varphi). \quad (68)$$

As expected, the orbit is unstable:

$$u(\varphi) \simeq u_{\text{ph}} \left[1 + \frac{\delta u_c}{u_{\text{ph}}} \cosh(\varphi) \right]. \quad (69)$$

We can now easily compare the effect of a perturbation in the initial position with the one produced by a perturbation in the central mass by looking at Table I. It appears that the two effects are very similar, in fact, and in the following we shall focus on the first case in the table.

VI. UNSTABLE CIRCULAR ORBITS

In this section, we want to analyze in more detail the unstable inner circular orbit r_{in} for massive particles. Since the analysis will be performed numerically, it is convenient to introduce the dimensionless variables

$$\tilde{L} \equiv \frac{L}{GM}, \quad \tilde{r}_{\text{in}} \equiv \frac{r_{\text{in}}}{GM}, \quad (70)$$

and recall that (two) circular orbits exist under the condition $\tilde{L} \geq \sqrt{12}$. In particular, as discussed in Sec. II A 2, the inner radius

$$3 = \tilde{r}_{\text{in}}(\tilde{L} \rightarrow \infty) \leq \tilde{r}_{\text{in}} \leq \tilde{r}_{\text{in}}(\tilde{L} = \sqrt{12}) = 6. \quad (71)$$

Since the energy and the angular momentum of test particles moving on the internal circular orbits are connected by the relation

$$E = 108[36 - \tilde{L}^{-1}(\tilde{L}^2 - 12)^{3/2} + \tilde{L}], \quad (72)$$

the effects of a perturbation of the mass M can be determined by considering L as the only free parameter.

It is worth noticing that, in our analysis, an instantaneous reduction of the mass M , i.e., a perturbation $\delta M < 0$, leads to a smaller radius for the circular orbit and to an increase of

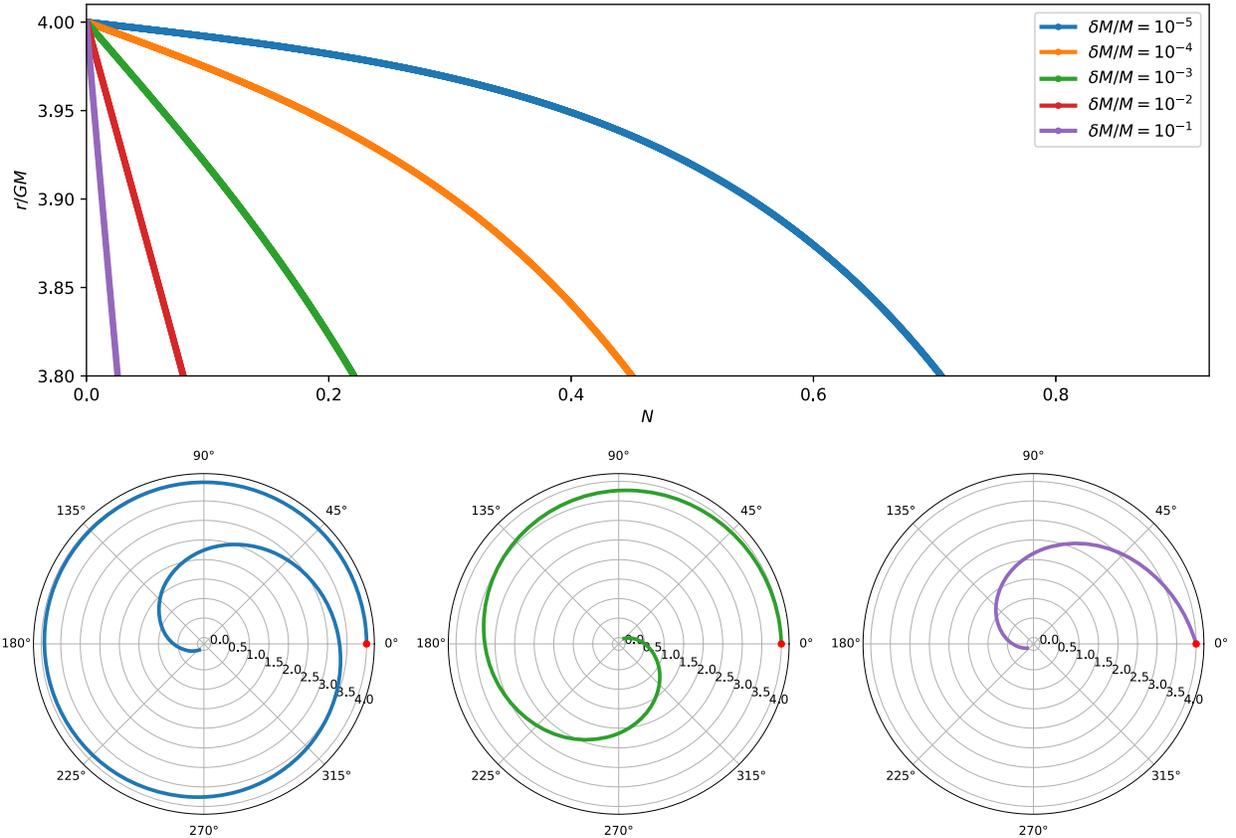


FIG. 15. Radial coordinate $\tilde{r} = r/GM$ of trajectories subject to perturbations δM starting from the unstable circular orbit for $\tilde{L} = L/GM = 4$ (top panel); trajectories subject to the perturbation $\delta M/M = 10^{-5}$ (bottom left panel), $\delta M/M = 10^{-3}$ (bottom center panel), and $\delta M/M = 10^{-1}$ (bottom right panel).

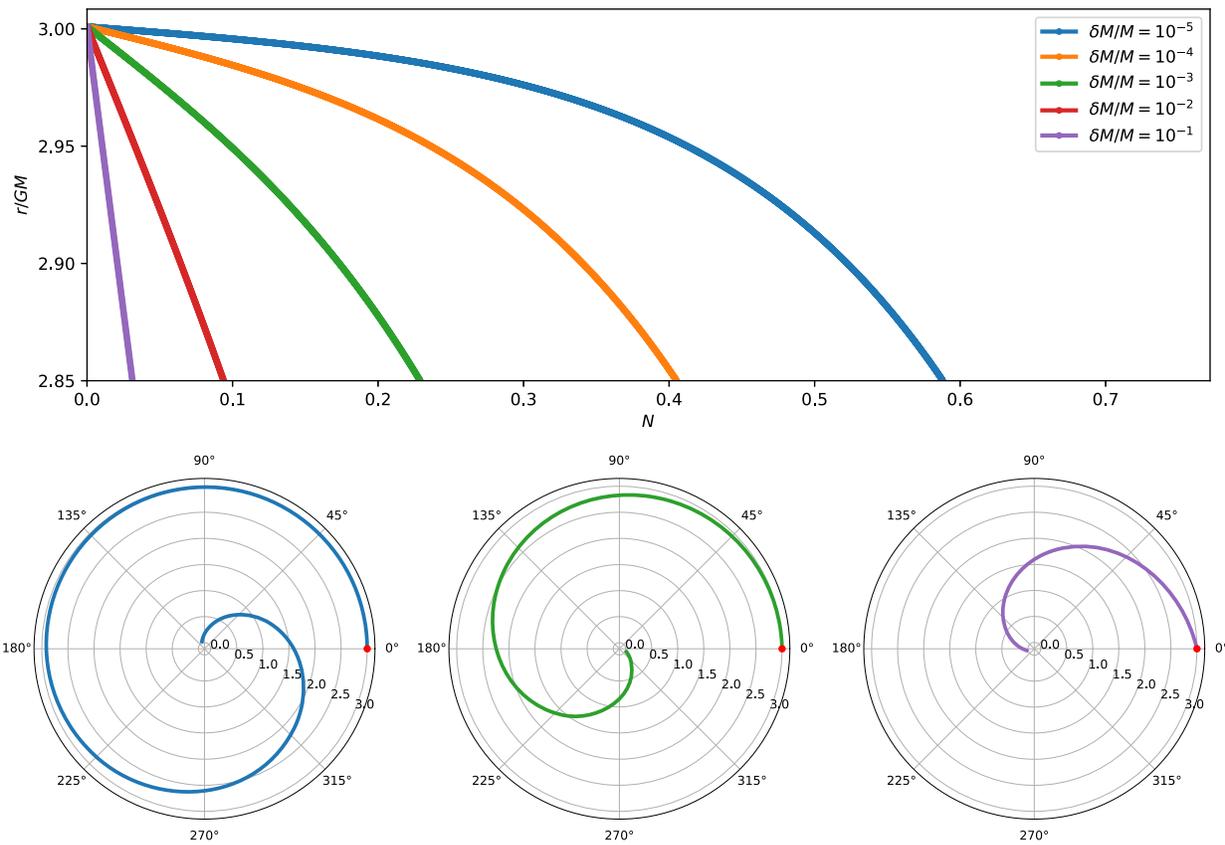


FIG. 16. Radial coordinate $\tilde{r} = r/GM$ of trajectories subject to perturbations δM starting from the unstable circular orbit for $\tilde{L} = L/GM = 100$ (top panel); trajectories subject to the perturbation $\delta M/M = 10^{-5}$ (bottom left panel), $\delta M/M = 10^{-3}$ (bottom center panel), and $\delta M/M = 10^{-1}$ (bottom right panel).

the corresponding potential V (see Fig. 13). After the perturbation, the particle orbiting on the circular orbit would happen to have an energy lower than the potential energy V . We shall therefore consider only perturbations $\delta M > 0$, i.e., instantaneous increases of the mass M . [Note that, in the analytical perturbation method, solutions are

found under the condition $du/d\varphi(0) = 0$; see Eq. (42) and the following comment.]

The change of the radial coordinate of the particle as a function of the angle φ following an instantaneous perturbation $\delta M > 0$ are shown in Figs. 14–16 for three values of the angular momentum, namely,

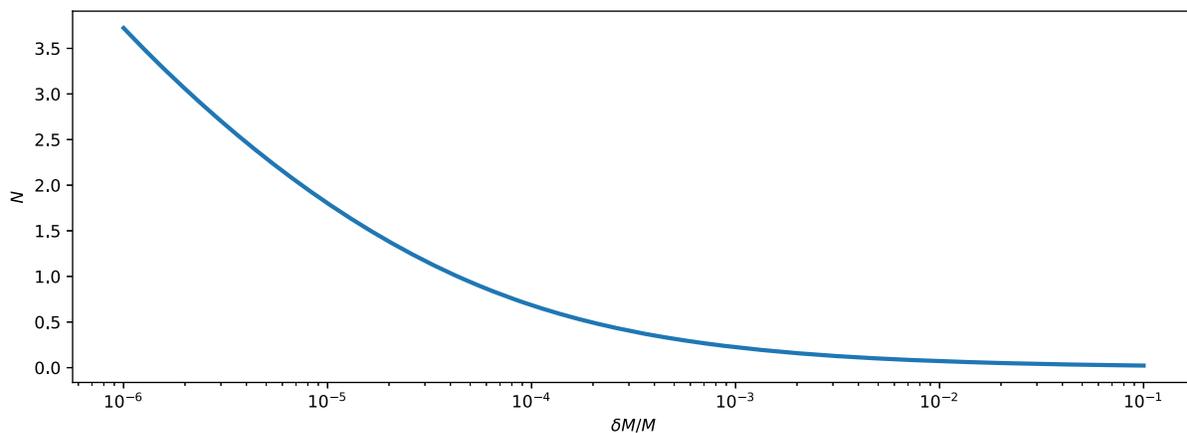


FIG. 17. Number of revolutions N corresponding to a change of 5% in the radial coordinate as a function of the perturbation $\delta M/M$ ranging from 10^{-6} to 10^{-1} .

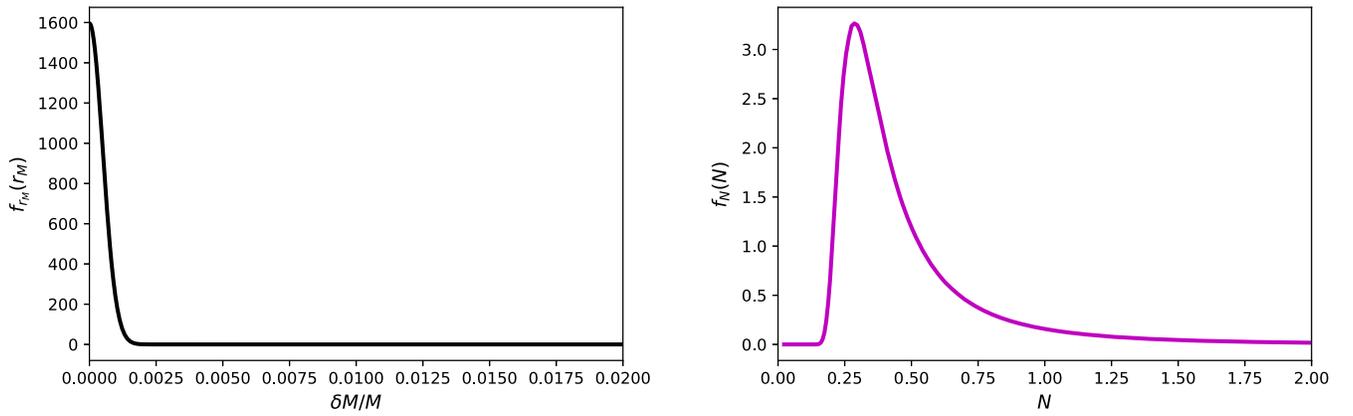


FIG. 18. Probability density function of the perturbation $\delta M \sim \mathcal{N}^*(\mu_{\delta M} = 0, \sigma_{\delta M}^2 = 0.0005^2)$ (left panel) and corresponding probability density function of the number of revolutions N after which the variation of the radial position of the particle amounts to 5% of the initial value (right panel).

- (1) $\tilde{L} = \sqrt{12}$, which is the minimum possible value for which the circular orbit exists—the radius of the circular orbit in this case is $\tilde{r}_{\text{in}} = 6$ (Fig. 14);
- (2) $\tilde{L} = 4$, for which the radius of the circular orbit is $\tilde{r}_{\text{in}} = 4$ (Fig. 15);
- (3) $\tilde{L} = 100$, for which the radius of the circular orbit is $\tilde{r}_{\text{in}} \approx 3.0009$, close to the minimum possible value $\tilde{r}_{\text{in}} = 3$ (Fig. 16).

For all three cases analyzed here, the radial position of the particle as a function of the number N of revolutions ($N = \varphi/2\pi$) is shown for several values of the perturbation $\delta M/M$, ranging from 10^{-5} to 10^{-1} . The results show that the number of revolutions after which the particle decreases its radial coordinate by 5% of the original value is always less than 2 and decreases as the perturbation increases or the angular momentum increases. For each of the three cases, the polar representations of the orbits are shown for three selected values of the perturbation, namely, $\delta M/M = 10^{-5}, 10^{-3}, 10^{-1}$. The number of revolutions N

after which the variation of the radial position of the particle amounts to 5% of the initial value has then been systematically investigated for values of the perturbation parameter $\delta M/M$ varying between 10^{-6} and 10^{-1} . The result is shown in Fig. 17.

Let us next assume the perturbation $\delta M > 0$ is a random variable with distribution $\mathcal{N}^*(\mu_{\delta M}, \sigma_{\delta M})$, i.e., a random variable with probability density function

$$f_{\delta M}(x) = \begin{cases} 0 & x < 0, \\ \frac{2}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu_{\delta M})^2}{2\sigma_{\delta M}^2}\right] & x \geq 0. \end{cases} \quad (73)$$

Setting $\mu_{\delta M} = 0$ and $\sigma_{\delta M} = 0.0005$, the corresponding random variable representing the number of revolutions N after which the variation of the radial coordinate amounts to 5% of the initial value is shown in Fig. 18. The same analysis for a 10 times larger value of $\sigma_{\delta M} = 0.005$ is also shown in Fig. 19. The cumulative distribution functions of

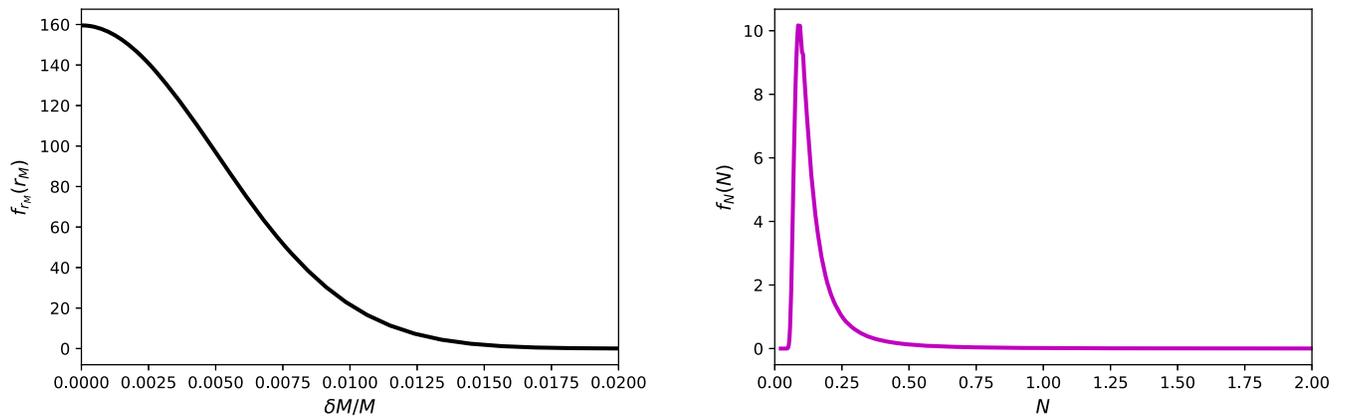


FIG. 19. Probability density function of the perturbation $\delta M \sim \mathcal{N}^*(\mu_{\delta M} = 0, \sigma_{\delta M}^2 = 0.005^2)$ (left panel) and corresponding probability density function of the number of revolutions N after which the variation of the radial position of the particle amounts to 5% of the initial value (right panel).

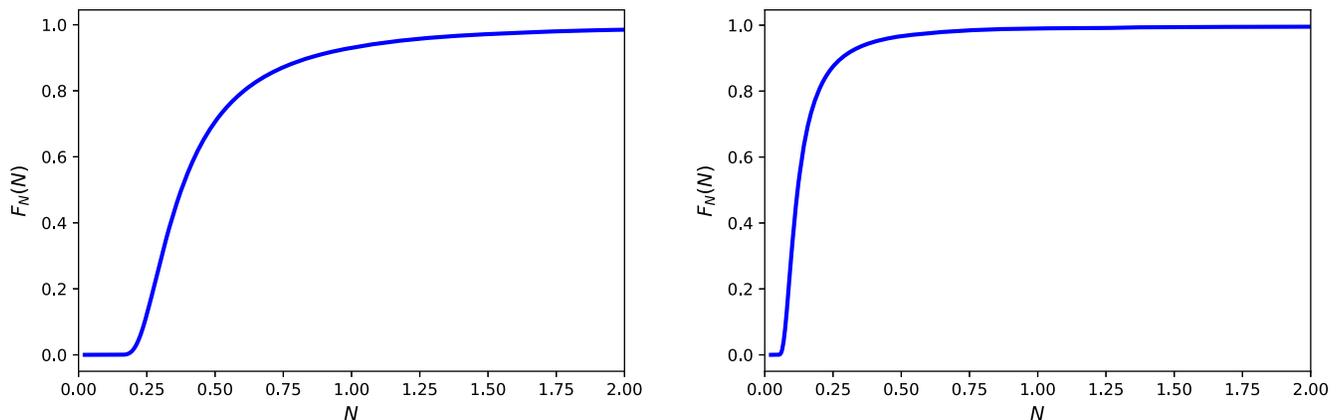


FIG. 20. Cumulative distribution functions of the random variable N resulting from a probability density function of the perturbation $\delta M \sim \mathcal{N}^*(\mu_{\delta M} = 0, \sigma_{\delta M}^2 = 0.0005^2)$ (left panel) and $\mathcal{N}^*(\mu_{\delta M} = 0, \sigma_{\delta M}^2 = 0.005^2)$ (right panel).

the random variable N for the two cases $\sigma_{\delta M} = 0.0005$ and $\sigma_{\delta M} = 0.005$ are shown in Fig. 20.

VII. DISCUSSION AND OUTLOOK

In this work, we started to consider in detail the effects of uncertainties in the determination of the black hole mass on geodesics, mainly motivated by studies of quantum aspects of black holes [23–27]. In particular, we analyzed the consequences of a randomly distributed mass on qualitatively different orbits, and we were interested in the possible differences between such effects and those simply stemming from the experimental uncertainties in the measurements of masses and positions. It is, in fact, important to be able to tell apart the two sources of uncertainties if one eventually wishes to look for experimental evidence of quantum gravity in black hole physics.

A comparison between the two kinds of uncertainties for circular orbits was considered in Sec. V, where we showed that they appear functionally very similar, which points to the fact that it would be very difficult to identify quantum effects from experimental data [31] about such orbits of test bodies obtained, for instance, by the Event Horizon Telescope [32] and BlackHoleCam [33,34]. In the search for more significant signatures of quantum effects, beside considering more explicitly models of quantum black holes [11–14,35–38], we also intend to study other effects occurring on black hole space-times. In particular, it will be interesting to investigate the redshift of signals emitted by sources either falling towards the black hole along (perturbed and unperturbed geodesics) as well as following perturbed unstable circular orbits. Given the precision with which the redshift can be measured, this might be a more promising route towards experimental quantum gravity.

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APPENDIX: EQUIVALENT ANALYTIC SOLUTIONS

Introducing the change of variable $\tilde{x} = (3\rho x - 1)/12$, Eq. (15) can be written as [39,40]

$$\left(\frac{d\tilde{x}}{d\varphi}\right)^2 = 4\tilde{x}^3 - g_2\tilde{x} - g_3, \quad (\text{A1})$$

with

$$g_2 = \frac{1}{12} - \frac{\rho^2}{4\alpha^2}, \quad g_3 = \frac{1}{216} + \left(\frac{1}{24\alpha^2} - \frac{1}{16\beta^2}\right)\rho^2, \quad (\text{A2})$$

whose general solution is given by the Weierstrass elliptic function $\wp(z; g_2, g_3)$ of parameters (g_2, g_3) . Hence, the general solution of Eq. (15) can be expressed as

$$x(\varphi) = \frac{1}{\rho} \left[4\wp(\varphi + \tilde{\delta}; g_2, g_3) + \frac{1}{3} \right], \quad (\text{A3})$$

where the constant $\tilde{\delta}$ is again obtained from the initial condition $x(\varphi_0) = x_0$.

Moreover, the Weierstrass elliptic function \wp can be written in terms of Jacobi's elliptic function sn as

$$\wp(z) = e_1 + \frac{e_3 - e_1}{\text{sn}^2(z\sqrt{e_3 - e_1}, \bar{k})}, \quad \text{with } \bar{k} = \sqrt{\frac{e_2 - e_1}{e_3 - e_1}}, \quad (\text{A4})$$

where e_1, e_2 , and e_3 are the roots of the polynomial on the right-hand side of Eq. (A1), i.e.,

$$4\tilde{x}^3 - g_2\tilde{x} - g_3 = 4(\tilde{x} - e_1)(\tilde{x} - e_2)(\tilde{x} - e_3), \quad (\text{A5})$$

with $e_1 + e_2 + e_3 = 0$. Taking into account that, for $j = 1, 2, 3$,

$$e_j = \frac{\rho}{4}x_j - \frac{1}{12}, \quad (\text{A6})$$

we see that $\bar{k} = k$ in Eq. (23). Making use of the relation [39]

$$\begin{aligned} \wp(z) &= e_1 + \frac{e_3 - e_1}{\text{sn}^2(z\sqrt{e_3 - e_1}, k)} \\ &= e_1 + \frac{e_2 - e_1}{k^2 \text{sn}^2(z\sqrt{e_3 - e_1}, k)}, \end{aligned} \quad (\text{A7})$$

the solution (A1) can be also written as

$$x(\varphi) = x_1 + \frac{x_2 - x_1}{k^2 \text{sn}^2\left(\frac{\varphi}{2} \sqrt{\rho(x_3 - x_1)} + \bar{\delta}, k\right)}, \quad (\text{A8})$$

where $\bar{\delta} = \tilde{\delta} \sqrt{\rho(x_3 - x_1)}/2$. Finally, making use of the relation [39]

$$\wp(z) = e_1 + (e_2 - e_1) \text{sn}^2((z - \omega_3)\sqrt{e_1 - e_3}, k), \quad (\text{A9})$$

where $\wp(\omega_3/2) = e_3$, Eq. (A1) can be finally written as

$$x(\varphi) = x_1 + (x_2 - x_1) \text{sn}^2\left(\frac{\varphi}{2} \sqrt{\rho(x_3 - x_1)} + \delta^*, k\right), \quad (\text{A10})$$

where $\delta^* = (\tilde{\delta} - \omega_3)\sqrt{\rho(x_3 - x_1)}/2$. Of course, the three expressions (22), (A8), and (A10) are perfectly equivalent.

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