Redshift in Finsler spacetimes

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We derive and discuss a general redshift formula in Finsler spacetimes. The condition for the existence of a redshift potential is worked out. The results are illustrated with two examples, one referring to a spherically symmetric and static Finsler spacetime and the other to a cosmological Finsler spacetime.

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I. INTRODUCTION

According to Einstein's general theory of relativity, the frequency under which a standard clock in a gravitational field is seen by another standard clock undergoes a redshift. Verifying this gravitational redshift is known as "the third classical test of general relativity," in addition to the deflection of light rays and the precession of the pericenter of test particle orbits in a (spherically symmetric and static) gravitational field. The gravitational redshift, as predicted by general relativity, was measured for the first time by Pound and Rebka [1] in 1959 with gamma quanta in a building of approximately 22 m height. The accuracy of this result was considerably improved by the Gravity Probe A experiment with a hydrogen maser in a sounding rocket in 1976; see Vessot *et al.* [2]. For many years, this remained the most accurate confirmation of the gravitational redshift as predicted by general relativity. Only very recently was the accuracy improved with the help of two Galileo satellites that were accidentally placed in an eccentric orbit around Earth; see Delva *et al.* [3] and Herrmann *et al.* [4]. The prediction from general relativity is now confirmed, in the gravitational field of Earth, with an accuracy of approximately 10^{-5} at 1σ.

Redshift measurements are also of crucial relevance for cosmology. In particular, our understanding that we are living in a universe with an *accelerated* expansion is based on redshift measurements of supernovae of type Ia; see Riess *et al.* [5] and Perlmutter *et al.* [6]. These results earned Perlmutter, Riess, and Schmidt the physics Nobel prize in 2011.

In view of these facts, it seems fair to say that measurements of redshifts are among the most powerful tools for testing general relativity. To put this another way, redshift measurements can provide bounds on alternative theories of gravity. In this article, we want to provide the theoretical background for investigating the gravitational redshift in Finsler gravity. In our view, Finsler gravity is one of the most attractive alternative theories of gravity. Whereas in general relativity the spacetime geometry is given by a pseudo-Riemannian metric of Lorentzian signature, in Finsler spacetime theory it is given by a metric that has an additional dependence on the tangent vector in which it is homogeneous of degree zero. There are several motivations for considering such a generalization which we mention here only briefly. For more detailed recent discussions, we refer to Lämmerzahl and Perlick [7] and to Pfeifer [8]. In our view, the strongest motivation comes from the Ehlers-Pirani-Schild [9] axiomatic approach to spacetime theory. In this approach, light rays and freely falling particles are considered as the primitive concepts, and axioms are formulated for the behavior of these primitive concepts that, finally, establish the spacetime structure of general relativity. However, if one slightly modifies one of the axioms, one arrives at a Finsler spacetime structure; see Tavakol and Van Den Bergh [10] and Lämmerzahl and Perlick [7]. As another motivation, we mention that some approaches to a quantum theory of gravity suggest to replace, at a certain level of approximation, the pseudo-Riemannian spacetime geometry of general relativity by a Finslerian geometry; see, e.g., Girelli, Liberati, and Sindoni [11]. Moreover, Finsler geometry comes up naturally also in curved versions of very special relativity; see Gibbons, Gomis, and Pope [12] and, for the more special case where the resulting Finsler spacetime is of Berwald type, Fuster, Pabst, and Pfeifer [13], and in the Standard Model Extension, see, e.g., Kostelecký [14].

We mention that there are also spacetime theories, again motivated by ideas from a quantum theory of gravity, where the propagation of light depends on the frequency,

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i.e., where the vacuum acts like a dispersive medium; see, e.g., Amelino-Camelia *et al.* [15]. These theories, which predict a so-called dual redshift or lateshift, meaning a dependence of the travel time on the frequency, are outside of the Finslerian framework, because they violate the above-mentioned homogeneity property, and will not be considered here.

The paper is organized as follows. In Sec. II, we specify our definition of Finsler spacetimes and we discuss the notion of (conformal) Killing vector fields which will play an important role in all that follows. The definition of Finsler spacetimes (i.e., Finsler structures with an *indefinite* metric) is a subtle issue. Until now, it seems fair to say that there is no general agreement about which definition is most appropriate in view of applications to physics. We refer to Lämmerzahl and Perlick [7] for details. Here we mention only that we essentially adopt Beem's definition [16], with a slight modification that will be indicated in Sec. II. There are alternative definitions, which differ by technical but important subtleties, by Asanov [17], by Pfeifer and Wohlfarth [18,19], and by Javaloyes and Sánchez [20,21]. In Sec. III, we present a redshift formula which holds for an arbitrary emitter and an arbitrary receiver in an unspecified Finsler spacetime. This redshift formula, which generalizes the redshift formula of general relativity into a Finslerian setting, was not known before, to the best of our knowledge, and is considered by us as the main result of this paper. In Secs. IV and V, we illustrate our general redshift formula with an application to a spherically symmetric and static spacetime and to a cosmological spacetime, respectively, thereby indicating the relevance of our general result for measurements (i) in the field of Earth or the Sun and (ii) in cosmology.

II. DEFINITION OF FINSLER SPACETIMES AND (CONFORMAL) KILLING VECTOR FIELDS

For the purpose of this paper, we use the following definition of a Finsler spacetime.

Definition 1.—A Finsler spacetime is a four-dimensional manifold M with a Lagrangian function $\mathcal{L}$ that satisfies the following properties:

(a) $\mathcal{L}$ is a real-valued and sufficiently smooth function on the tangent bundle TM minus the zero section; i.e., $\mathcal{L}(x,\dot{x})$ is defined for all $(x,\dot{x})$ with $\dot{x}\neq 0$.

(b) $\mathcal{L}$ is positively homogeneous of degree two with respect to $\dot{x}$, i.e.,

$$\mathcal{L}(x,k\dot{x}) = k^2\mathcal{L}(x,\dot{x}) \quad \text{for all } k>0. \tag{1}$$

(c) The Finsler metric

$$g_{\mu\nu}(x,\dot{x}) = \frac{\partial^2\mathcal{L}(x,\dot{x})}{\partial\dot{x}^\mu\partial\dot{x}^\nu} \tag{2}$$

is well defined and has a Lorentzian signature $(-{+}{+}{+})$ for almost all $(x,\dot{x})$ with $\dot{x}\neq 0$. (As usual, "almost all" means "up to a set of measure zero.")

(d) The Euler-Lagrange equations

$$\frac{\partial\mathcal{L}(x,\dot{x})}{\partial x^\mu} - \frac{d}{ds}\frac{\partial\mathcal{L}(x,\dot{x})}{\partial\dot{x}^\mu} = 0 \tag{3}$$

admit a unique solution for every initial condition $(x,\dot{x})$ with $\dot{x}\neq 0$; at points where the Finsler metric is not well defined, this solution is to be constructed by continuous extension.

On a Finsler spacetime, we represent points in M by their coordinates $x=(x^0,x^1,x^2,x^3)$ and points in the fiber T_xM of the tangent bundle by their induced coordinates $\dot{x}=(\dot{x}^0,\dot{x}^1,\dot{x}^2,\dot{x}^3)$. We use Einstein's summation convention for Greek indices taking values 0, 1, 2, and 3.

Definition 1 is essentially Beem's definition [16] of a Finsler structure with Lorentzian signature. The only modification is in the fact that in item (c) we require the Finsler metric to be well defined and of Lorentzian signature only for *almost all* $(x,\dot{x})$ with $\dot{x}\neq 0$, whereas Beem required this for *all* such $(x,\dot{x})$. The motivation for this generalization was discussed in Lämmerzahl, Perlick, and Hasse [22].

Note that the homogeneity condition (1) of the Lagrangian implies that

$$\dot{x}^\mu\frac{\partial\mathcal{L}(x,\dot{x})}{\partial\dot{x}^\mu} = 2\mathcal{L}(x,\dot{x}), \tag{4}$$

$$\dot{x}^\mu\frac{\partial g_{\rho\sigma}(x,\dot{x})}{\partial\dot{x}^\mu} = 0, \tag{5}$$

$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}g_{\mu\nu}(x,\dot{x})\dot{x}^\mu\dot{x}^\nu. \tag{6}$$

A general-relativistic spacetime (i.e., a four-dimensional manifold with a pseudo-Riemannian metric of Lorentzian signature) is the special case of a Finsler spacetime where the $g_{\mu\nu}$ are independent of $\dot{x}$.

With the help of the Lagrangian, we classify nonzero tangent vectors as timelike $[\mathcal{L}(x,\dot{x})<0]$, lightlike $[\mathcal{L}(x,\dot{x})=0]$, or spacelike $[\mathcal{L}(x,\dot{x})>0]$. We call the solutions to the Euler-Lagrange equations (3) the affinely parametrized Finsler *geodesics*. Again, by the homogeneity condition (1) of the Lagrangian, $\mathcal{L}(x,\dot{x})$ is a constant of motion; hence, Finsler geodesics can be classified as timelike, lightlike, or spacelike. We interpret the timelike geodesics as freely falling particles and the lightlike geodesics as light rays.

We can switch to a Hamiltonian formulation by introducing canonical momenta

$$p_\mu = \frac{\partial\mathcal{L}(x,\dot{x})}{\partial\dot{x}^\mu} \tag{7}$$

and the Hamiltonian

$$\mathcal{H}(x,p) = p_\mu\dot{x}^\mu - \mathcal{L}(x,\dot{x}). \tag{8}$$

On the right-hand side of (8), $\dot{x}^\mu$ must be expressed as a function of x and p with the help of (7). With (1) and (2) from Definition 1, Eqs. (7) and (8) specify to

$$p_\mu = g_{\mu\nu}(x,\dot{x})\dot{x}^\nu \tag{9}$$

and

$$\mathcal{H}(x,p) = \frac{1}{2} g^{\mu\nu}(x,p) p_\mu p_\nu, \tag{10}$$

where $g^{\mu\nu}(x,p)$ is defined through

$$g^{\mu\nu}(x,p) g_{\nu\sigma}(x,\dot{x}) = \delta^\mu_\sigma. \tag{11}$$

Here we have used (4) and (5). As a consequence, the Hamiltonian $\mathcal{H}(x,p)$ is homogeneous of degree two with respect to the p_μ,

$$p_\mu \frac{\partial \mathcal{H}(x,p)}{\partial p_\mu} = 2\mathcal{H}(x,p), \tag{12}$$

and

$$g^{\mu\nu}(x,p) = \frac{\partial^2 \mathcal{H}(x,p)}{\partial p_\mu \partial p_\nu}. \tag{13}$$

The Finsler geodesics are the solutions to Hamilton's equations

$$\frac{dp_\mu}{ds} = -\frac{\partial \mathcal{H}(x,p)}{\partial x^\mu}, \qquad \frac{dx^\mu}{ds} = \frac{\partial \mathcal{H}(x,p)}{\partial p_\mu}, \tag{14}$$

and they are lightlike if

$$\mathcal{H}(x,p) = 0. \tag{15}$$

Interpreting the lightlike geodesics of a Finsler spacetime as light rays is justified, because they are the bicharacteristic curves (or "rays") of appropriately generalized Maxwell equations. (This was demonstrated in the Appendix of Ref. [22]; the generalized Maxwell equations were further discussed in Ref. [23].) Note that a transformation

$$\mathcal{H}(x,p) \mapsto e^{-2\Omega(x,p)}\mathcal{H}(x,p) \tag{16}$$

leaves the solutions to (14) and (15) unchanged up to parametrization. So we are free to perform such a transformation if we are interested only in lightlike geodesics. This is true with an arbitrary function $\Omega(x,p)$ which need not be homogeneous of degree zero with respect to the momenta; i.e., the transformed Hamiltonian need not be associated with a Finsler metric.

At each point of M, the tangent vectors to lightlike geodesics define the *light cone*. In the pseudo-Riemannian case, the light cone has two connected components: a future half-cone and a past half-cone. In a Finsler spacetime, there may be more components. Criteria that guarantee the existence of just two components have been worked out by Minguzzi [24]. We emphasize that our redshift formula, to be given below, is valid, in general, even if there are more than two connected components. In the examples of Secs. IV and V, however, we restrict to Finsler metrics that are small perturbations of pseudo-Riemannian metrics; then, at each point the light cone has exactly two connected components.

Symmetries of Finsler metrics are described in terms of (Finsler generalizations of) Killing vector fields. By definition, a vector field $K^\mu(x)\partial_\mu$ on a Finsler spacetime M is a *Killing vector field* if and only if its flow, if lifted to TM, leaves the Lagrangian $\mathcal{L}$ invariant. This condition can be rewritten in terms of the Finsler metric as

$$K^\mu(x)\frac{\partial g_{\rho\sigma}(x,\dot{x})}{\partial x^\mu} + \frac{\partial K^\tau(x)}{\partial x^\nu}\dot{x}^\nu \frac{\partial g_{\rho\sigma}(x,\dot{x})}{\partial \dot{x}^\tau} + \frac{\partial K^\tau(x)}{\partial x^\rho} g_{\tau\sigma}(x,\dot{x}) + \frac{\partial K^\tau(x)}{\partial x^\sigma} g_{\rho\tau}(x,\dot{x}) = 0. \tag{17}$$

The Finslerian Killing equation (17) has been known since the early days of Finsler geometry; see Knebelman [25]. In the Hamiltonian formalism, Killing vector fields are characterized by the fact that $K^\mu(x)p_\mu$ is a constant of motion, i.e.,

$$\frac{d(K^\mu(x)p_\mu)}{ds} = 0 \tag{18}$$

along any solution of Hamilton's equations (14). This is true if and only if $K^\mu(x)$ satisfies the condition

$$\{\mathcal{H}(x,p), K^\mu(x)p_\mu\} = 0, \tag{19}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket

$$\{A(x,p), B(x,p)\} = \frac{\partial A(x,p)}{\partial p_\nu}\frac{\partial B(x,p)}{\partial x^\nu} - \frac{\partial A(x,p)}{\partial x^\nu}\frac{\partial B(x,p)}{\partial p_\nu}. \tag{20}$$

With $\mathcal{H}$ inserted from (10), Eq. (19) reads

$$g^{\sigma\nu}(x,p)p_\sigma p_\mu \frac{\partial K^\mu(x)}{\partial x^\nu} - \frac{1}{2}\frac{\partial g^{\mu\sigma}(x,p)}{\partial x^\nu} p_\mu p_\sigma K^\nu(x) = 0. \tag{21}$$

Differentiating with respect to p_ρ and then with respect to p_λ gives the Hamiltonian version of the Killing equation:

$$- K^{\nu}(x)\frac{\partial g^{\rho\lambda}(x,p)}{\partial x^{\nu}} + \frac{\partial K^{\sigma}(x)}{\partial x^{\nu}} p_{\sigma} \frac{\partial g^{\rho\lambda}(x,p)}{\partial p_{\nu}} + \frac{\partial K^{\lambda}(x)}{\partial x^{\nu}} g^{\rho\nu}(x,p) + \frac{\partial K^{\rho}(x)}{\partial x^{\nu}} g^{\lambda\nu}(x,p) = 0. \quad (22)$$

We mention that Eq. (22) characterizes the symmetry of a nondegenerate Hamiltonian in general; i.e., it is true even if the Hamiltonian is not homogeneous with respect to the momenta—cf. Eq. (45) in Barcaroli *et al.* [26].

More generally, $K^{\mu}(x)\partial_{\mu}$ is called a *conformal Killing vector field* if

$$\{e^{-2\Omega(x,p)}\mathcal{H}(x,p), K^{\mu}(x)p_{\mu}\} = 0 \quad (23)$$

with some function $\Omega(x,p)$. Evaluating this equation along a solution to Hamilton's equations (14) yields

$$e^{-2\Omega(x,p)}\left(\frac{d(K^{\mu}(x)p_{\mu})}{ds} - 2\mathcal{H}(x,p)\{\Omega(x,p), K^{\mu}(x)p_{\mu}\}\right) = 0, \quad (24)$$

so the conservation law (18) still holds along *lightlike* geodesics, $\mathcal{H}(x,p) = 0$.

III. THE REDSHIFT FORMULA IN FINSLER SPACETIMES

We use units making $\hbar$ equal to 1. Then the momentum p_{μ} of a light ray is the same as the wave covector. With respect to an observer, the wave covector p_{μ} can be decomposed into a spatial wave covector and a frequency. In a Finsler spacetime, an observer is determined by fixing a worldline, i.e., a curve $\gamma(\tau)$ in M with

$$g_{\mu\nu}\left(\gamma(\tau), \frac{d\gamma(\tau)}{d\tau}\right)\frac{d\gamma^{\mu}(\tau)}{d\tau}\frac{d\gamma^{\nu}(\tau)}{d\tau} = -c^2, \quad (25)$$

where c is the vacuum speed of light. The normalization condition (25) means that the worldline is parametrized by Finsler proper time. If this observer meets a light ray $x(s)$ at an event $\gamma(\tau_0) = x(s_0)$, we decompose the wave covector according to

$$p_{\mu}(s_0) = \frac{\omega(s_0)}{c^2} g_{\mu\nu}\left(\gamma(\tau_0), \frac{d\gamma}{d\tau}(\tau_0)\right)\frac{d\gamma^{\nu}}{d\tau}(\tau_0) + p_{\mu}^{\perp}(s_0), \quad (26)$$

where $p_{\mu}^{\perp}(s_0)$ is the spatial wave covector which satisfies the condition $p_{\mu}^{\perp}(s_0)\frac{d\gamma^{\mu}}{d\tau}(\tau_0) = 0$ and

$$\omega(s_0) = -p_{\mu}(s_0)\frac{d\gamma^{\mu}}{d\tau}(\tau_0) \quad (27)$$

is the frequency.

Now consider a light ray $x(s)$ that is emitted at an event $x(s_1)$ and received at an event $x(s_2)$; see Fig. 1. By (27), the emitter assigns to the light ray the frequency

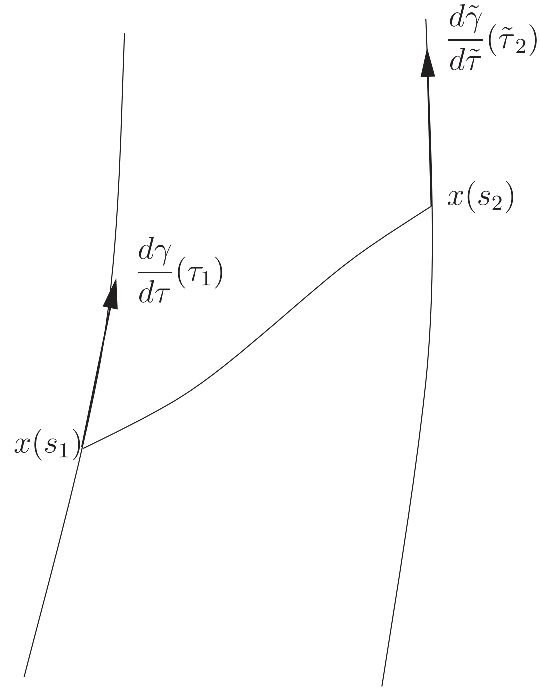

FIG. 1. Light ray $x(s)$ from an emitter to a receiver.

$$\omega_1 = -p_{\mu}(s_1)\frac{d\gamma^{\mu}}{d\tau}(\tau_1), \quad (28)$$

where $\gamma(\tau)$ is the worldline of the emitter and $\gamma(\tau_1) = x(s_1)$. Similarly, the receiver assigns to the light ray the frequency

$$\omega_2 = -p_{\mu}(s_2)\frac{d\tilde{\gamma}^{\mu}}{d\tilde{\tau}}(\tilde{\tau}_2), \quad (29)$$

where $\tilde{\gamma}(\tilde{\tau})$ is the worldline of the receiver and $\tilde{\gamma}(\tilde{\tau}_2) = x(s_2)$.

The redshift z is defined as

$$z = \frac{\omega_1 - \omega_2}{\omega_2}, \quad (30)$$

and thus

$$1 + z = \frac{p_{\mu}(s_1)\frac{d\gamma^{\mu}}{d\tau}(\tau_1)}{p_{\rho}(s_2)\frac{d\tilde{\gamma}^{\rho}}{d\tilde{\tau}}(\tilde{\tau}_2)}. \quad (31)$$

We may go back from the Hamiltonian to the Lagrangian formalism with the help of (7) and rewrite the redshift formula (31) as

$$1 + z = \frac{\frac{\partial\mathcal{L}}{\partial\dot{x}^{\mu}}(x(s_1), \dot{x}(s_1))\frac{d\gamma^{\mu}}{d\tau}(\tau_1)}{\frac{\partial\mathcal{L}}{\partial\dot{x}^{\rho}}(x(s_2), \dot{x}(s_2))\frac{d\tilde{\gamma}^{\rho}}{d\tilde{\tau}}(\tilde{\tau}_2)}. \quad (32)$$

Note that in the numerator and in the denominator of this version of the redshift formula, the expression $\partial\mathcal{L}/\partial\dot{x}^{\mu}$ is the coordinate version of the fiber derivative $F\mathcal{L}$ of the Lagrangian, which mediates between the Lagrangian and the Hamiltonian form; see, e.g., Abraham and Marsden [27], Def. 3.5.2. Also note that we have not explicitly used the homogeneity property of the Lagrangian for deriving the redshift formula (32). However, we have used that light rays are solutions of the Euler-Lagrange equation (3) with $\mathcal{L}(x,\dot{x})=0$; if the Lagrangian is not homogeneous (of any degree), $\mathcal{L}$ is not, in general, a constant of motion, so solutions with $\mathcal{L}(x,\dot{x})=0$ need not exist.

With the help of (9), the redshift formula (31) in a Finsler spacetime can be written more specifically as

$$1+z=\frac{g_{\mu\nu}(x(s_1),\dot{x}(s_1))\dot{x}^{\nu}(s_1)\frac{d\gamma^{\mu}}{d\tau}(\tau_1)}{g_{\rho\sigma}(x(s_2),\dot{x}(s_2))\dot{x}^{\sigma}(s_2)\frac{d\tilde{\gamma}^{\rho}}{d\tilde{\tau}}(\tilde{\tau}_2)}. \tag{33}$$

It looks exactly the same as the familiar redshift formula in a general-relativistic spacetime (see, e.g., Straumann [28]), with the only difference that now the $g_{\mu\nu}$ depend also on the tangent vector of the light ray.

The redshift formula (33) takes a particularly simple form if γ and $\tilde{\gamma}$ are integral curves of a vector field $V^{\mu}(x)\partial_{\mu}$ that is proportional to a conformal Killing vector field $K^{\mu}(x)\partial_{\mu}$:

$$K^{\mu}(x)=e^{f(x)}V^{\mu}(x). \tag{34}$$

Then (31) can be rewritten as

$$1+z=\frac{p_{\mu}(s_1)e^{-f(x(s_1))}K^{\mu}(x(s_1))}{p_{\rho}(s_2)e^{-f(x(s_2))}K^{\rho}(x(s_2))}. \tag{35}$$

Because of the conservation law (18), this simplifies to

$$\ln(1+z)=f(x(s_2))-f(x(s_1)), \tag{36}$$

where ln denotes the natural logarithm. In this situation, we say that f is a *redshift potential*. From general-relativistic spacetimes, it is known [29] that the existence of a timelike conformal Killing vector field $K^{\mu}(x)\partial_{\mu}$ implies the existence of a redshift potential (36) for observers whose worldlines are (reparametrized) integral curves of $K^{\mu}(x)\partial_{\mu}$. We have now demonstrated that this result carries over to the Finsler case.

IV. REDSHIFT IN A SPHERICALLY SYMMETRIC STATIC FINSLER SPACETIME

As our first example, we consider the same type of spherically symmetric and static spacetime with a Finsler perturbation as in Lämmerzahl, Perlick, and Hasse [22]. The Lagrangian for the geodesics is of the form

$$2\mathcal{L}=(1+\phi_0(r))h_{tt}(r)\dot{t}^2+(1+\phi_1(r))h_{rr}(r)\dot{r}^2 +r^2(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2)+\frac{\phi_2(r)h_{rr}(r)r^2\dot{r}^2(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2)}{h_{rr}(r)\dot{r}^2+r^2(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2)}, \tag{37}$$

where $h_{\mu\nu}$ is the Schwarzschild metric,

$$h_{tt}(r)=-F(r), \qquad h_{rr}(r)=\frac{c^2}{F(r)}, \tag{38}$$

and

$$F(r)=c^2\left(1-\frac{2GM}{c^2r}\right). \tag{39}$$

Here G is Newton's gravitational constant, c is the vacuum speed of light, and M is the mass of the central body in the unperturbed Schwarzschild spacetime. We refer to the functions $\phi_A(r)$ as to the "perturbation coefficients" and we assume that they are so small that all equations can be linearized with respect to them. ϕ_0 and ϕ_1 change the time measurement and the radial length measurement, respectively, without affecting the pseudo-Riemannian character of the spacetime geometry. By contrast, a nonzero ϕ_2 destroys the spatial isotropy in each tangent space which results in a genuinely Finslerian geometry. We refer to ϕ_2 as to the "Finslerity."

The Hamiltonian corresponding to the Lagrangian (37) reads

$$2\mathcal{H}=(1-\phi_0(r))\frac{p_t^2}{h_{tt}(r)}+(1-\phi_1(r))\frac{p_r^2}{h_{rr}(r)} +\frac{1}{r^2}\left(p_{\vartheta}^2+\frac{p_{\varphi}^2}{\sin^2\vartheta}\right)-\frac{p_r^2(p_{\vartheta}^2+\frac{p_{\varphi}^2}{\sin^2\vartheta})\phi_2(r)}{r^2p_r^2+h_{rr}(r)(p_{\vartheta}^2+\frac{p_{\varphi}^2}{\sin^2\vartheta})}. \tag{40}$$

We observe that $\partial_t+\Omega\partial_{\varphi}$ is a Killing vector field, for any constant Ω:

$$\{\mathcal{H},p_t+\Omega p_{\varphi}\}=-\frac{\partial\mathcal{H}}{\partial t}-\Omega\frac{\partial\mathcal{H}}{\partial\varphi}=0. \tag{41}$$

We want to calculate the redshift for the case that the emitter and receiver are in circular (in general, nongeodesic) uniform motion in the equatorial plane. If we use coordinate time t for the parametrization, their worldlines are given as

$$\text{emitter: } r(t)=r_1, \quad \varphi(t)=\varphi_{01}+\Omega_1t, \quad \vartheta(t)=\frac{\pi}{2}; \tag{42}$$

$$\text{receiver: } r(t)=r_2, \quad \varphi(t)=\varphi_{02}+\Omega_2t, \quad \vartheta(t)=\frac{\pi}{2}. \tag{43}$$

If reparametrized with proper time, these worldlines are integral curves of the vector fields

$$V_a^\mu \partial_\mu = e^{-f_a(r)}(\partial_t + \Omega_a \partial_\varphi) \tag{44}$$

with

$$e^{f_a(r)} = \frac{1}{c}\sqrt{F(r) - \Omega_a^2 r^2}\left(1 + \frac{\phi_0(r)F(r)}{2(F(r) - \Omega_a^2 r^2)}\right) \tag{45}$$

for $a = 1$ and $a = 2$, respectively.

By (31), the redshift is

$$\begin{aligned} 1 + z &= \frac{p_t + \Omega_1 p_\varphi}{p_t + \Omega_2 p_\varphi} e^{f_2(r_2) - f_1(r_1)} \\ &= \frac{1 - \Omega_1 b}{1 - \Omega_2 b} e^{f_2(r_2) - f_1(r_1)}, \end{aligned} \tag{46}$$

where

$$b := \frac{p_\varphi}{-p_t} \tag{47}$$

is the impact parameter of the light ray that connects the emitter and receiver. Geometrically, b determines the angle under which the light ray arrives at the receiver. For evaluating (46) we have to determine for each observation event the impact parameter b of the particular light ray that arrives from the emitter at this observation event. This makes (46) difficult to use.

There is only one special case where this problem does not exist, namely if $\Omega_1 = \Omega_2 =: \Omega$, i.e., if the emitter rigidly corotates with the receiver. In this case we may think of the receiver as a station on Earth and of the emitter as a geostationary satellite. Then we have a redshift potential $f_1(r) = f_2(r) =: f(r)$ and the redshift is given as

$$\begin{aligned} 1 + z &= e^{f(r_2) - f(r_1)} \\ &= \sqrt{\frac{F(r_2) - \Omega^2 r_2^2}{F(r_1) - \Omega^2 r_1^2}} \\ &\quad \times \left(1 + \frac{\phi_0(r_2)F(r_2)}{2(F(r_2) - \Omega^2 r_2^2)} - \frac{\phi_0(r_1)F(r_1)}{2(F(r_1) - \Omega^2 r_1^2)}\right). \end{aligned} \tag{48}$$

This equation takes a particularly simple form for $\Omega = 0$ (observers at rest) because then only the difference $\phi_0(r_2) - \phi_0(r_1)$ occurs. More generally, we see that according to (48) the Finslerity ϕ_2 (and also the perturbation function ϕ_1) has no influence on the redshift. This result remains true even if we consider a Finsler perturbation beyond the linearization: As the vector fields (44) have no components in the direction of ∂_r, the functions (45) are insensitive to terms in the Lagrangian that involve a factor $\dot{r}$. Therefore, if we want to use redshift measurements in the gravitational field of Earth or the Sun as a genuine Finsler test, we have to consider the case $\Omega_1 \neq \Omega_2$.

Then we have to solve the geodesic equation for the light rays. Starting out from the equation $\mathcal{H} = 0$ in the equatorial plane, where the Hamiltonian is given by (40), we find that the momentum coordinate p_r of each light ray is given by

$$\begin{aligned} p_r = \pm \frac{c\sqrt{r^2 p_t^2 - F(r)p_\varphi^2}}{rF(r)} \Bigg(1 &- \frac{\phi_0(r) r^2 p_t^2}{2(r^2 p_t^2 - F(r)p_\varphi^2)} \\ &+ \frac{\phi_1(r)}{2} + \frac{\phi_2(r)F(r)p_\varphi^2}{2r^2 p_t^2}\Bigg). \end{aligned} \tag{49}$$

Inserting this expression for p_r into Hamilton's equations

$$\frac{dt}{ds} = \frac{\partial \mathcal{H}}{\partial p_t}, \qquad \frac{d\varphi}{ds} = \frac{\partial \mathcal{H}}{\partial p_\varphi}, \qquad \frac{dr}{ds} = \frac{\partial \mathcal{H}}{\partial p_r} \tag{50}$$

yields

$$\frac{dt}{dr} = \Phi(r), \qquad \frac{d\varphi}{dr} = \Psi(r), \tag{51}$$

where

$$\begin{aligned} \Phi(r) = \frac{\pm cr}{F(r)\sqrt{r^2 - b^2 F(r)}} \Bigg(1 &- \frac{\phi_0(r)(r^2 - 2b^2 F(r))}{2(r^2 - b^2 F(r))} \\ &+ \frac{\phi_1(r)}{2} - \frac{\phi_2(r) b^2 F(r)}{2r^2}\left(1 - \frac{2b^2 F(r)}{r^2}\right)\Bigg), \end{aligned} \tag{52}$$

$$\begin{aligned} \Psi(r) = \frac{\pm cb}{r\sqrt{r^2 - b^2 F(r)}} \Bigg(1 &+ \frac{\phi_0(r) r^2}{2(r^2 - b^2 F(r))} + \frac{\phi_1(r)}{2} \\ &- \phi_2(r)\left(1 - \frac{3b^2 F(r)}{2r^2}\right)\Bigg). \end{aligned} \tag{53}$$

In (49), (52), and (53) the upper sign is valid if $r_2 > r_1$ and the lower sign is valid if $r_1 > r_2$. Note that p_t is negative if the light rays are future oriented, $dt/ds > 0$.

Integration of (51) from the emitter worldline to the receiver worldline results in

$$t_2 - t_1 = \int_{r_1}^{r_2} \Phi(r) dr, \tag{54}$$

$$\varphi_{20} + \Omega_2 t_2 - \varphi_{10} - \Omega_1 t_1 = \int_{r_1}^{r_2} \Psi(r) dr. \tag{55}$$

If r_1, r_2, Ω_1, Ω_2, φ_{10} and φ_{20} and t_2 are known, Eqs. (54) and (55) determine t_1 and b. Inserting into (46) then gives the redshift as a function of the observation time t_2. In contrast to the case $\Omega_1 = \Omega_2$, the redshift now depends on the Finslerity ϕ_2. Note that, by (37), our radius coordinate has a geometric meaning: A circle $r = \text{const}$ in the equatorial plane has circumference $2\pi r$. Also, the angles φ_{10} and φ_{20} are measurable quantities and the frequencies Ω_1 and Ω_2 can be determined from measuring the rotation periods in terms of proper time and converting into coordinate time with the help of the functions $f_1(r)$ and $f_2(r)$, respectively. In this sense, the results of this section give a method for experimentally detecting possible Finsler deviations in the gravitational field of Earth or of the Sun with satellites in circular orbits. For applications to satellites in noncircular orbits, such as the two Galileo satellites that have gone astray [3,4], the relevant equations are considerably more involved. We are planning to work this out in a follow-up paper.

V. REDSHIFT IN A COSMOLOGICAL FINSLER SPACETIME

As a second example, we consider a cosmological model with a Finsler perturbation. As the unperturbed spacetime, we choose a kinematical Robertson-Walker model with scale factor $S(t)$ and spatial curvature parameter k; the latter takes the value $+1$, 0 or -1, depending on whether the spatial sections are positively curved, flat or negatively curved. The Lagrangian for the geodesics in the unperturbed spacetime is

$$2\mathcal{L}_0 = -c^2\dot{t}^2 + S(t)^2(\dot{r}^2 + \Sigma(r)^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)), \qquad (56)$$

where

$$\Sigma(r)^2 = \begin{cases} k^{-1}\sin^2(\sqrt{k}r) & \text{for } k > 0, \\ r^2 & \text{for } k = 0, \\ |k|^{-1}\sinh^2(\sqrt{|k|}r) & \text{for } k < 0. \end{cases} \qquad (57)$$

We want to preserve spatial isotropy and spatial homogeneity. Then we may choose any point in space as the spatial origin of the coordinate system and we must have spherical symmetry about this point. According to the analysis of McCarthy and Rutz [30,31] this implies that the Finsler-perturbed Lagrangian must be independent of φ and that r, ϑ, $\dot{r}$, $\dot{\vartheta}$ and $\dot{\varphi}$ may enter into the Lagrangian only in terms of the combination

$$u := \sqrt{\dot{r}^2 + \Sigma(r)^2(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2)}. \qquad (58)$$

As a consequence, any term in the Lagrangian that is positively homogeneous of degree zero with respect to $\dot{x}^\alpha$ must be some function of the two variables t and $u/\dot{t}$, provided that $\dot{t} \neq 0$. Thus, on the subset of the tangent bundle TM where $\dot{t} \neq 0$ the Lagrangian can be written as

$$2\mathcal{L} = -c^2\dot{t}^2\ell\left(\frac{u}{c\dot{t}}, t\right) \qquad (59)$$

with some function ℓ. [As a subtlety, we remark that ℓ may depend, in addition, explicitly on the sign of $\dot{t}$ because in (1) we required homogeneity only for positive k.] Note that in the unperturbed spacetime t gives proper time for the observers at rest (i.e., for observers with $u = 0$). Without loss of generality, we require that also in the perturbed spacetime the time coordinate t measures (Finsler) proper time for observers at rest. Then the function ℓ has to satisfy

$$\ell(0, t) = 1 \qquad (60)$$

for all t.

Clearly, by (59), the function ℓ has to vanish on lightlike vectors. In the following we will restrict to the case that the equation $\ell = 0$ can be solved for the spatial direction; i.e., we require that a function b of t is implicitly defined by the equation

$$\ell(b(t), t) = 0. \qquad (61)$$

[Up to here, we followed the same line of argument as Hohmann and Pfeifer [32] who treat observables in cosmological Finsler spacetimes in terms of the geodesic spray; our equations (60) and (61) are analogous to their equations (12) and (47), respectively. Note, however, that their definition of a Finsler spacetime is slightly different from ours.]

We will now discuss properties of lightlike geodesics and, in particular, the redshift in our Finsler-perturbed cosmological spacetimes. Owing to spatial homogeneity, we know all lightlike geodesics in the spacetime if we know the lightlike geodesics through one particular point in space which we may choose as the spatial origin of the coordinate system. Therefore, it suffices to consider radial lightlike geodesics ($\dot{\vartheta} = 0$ and $\dot{\varphi} = 0$.) They satisfy

$$cb(t) = \frac{u}{\dot{t}} = \frac{|\dot{r}|}{\dot{t}} = \pm\frac{dr}{dt}, \qquad (62)$$

where the sign depends on whether the light signal moves in the direction of increasing or decreasing r coordinate. For an emitter and an observer, both at rest ($u = 0$) at r_1 and r_2, respectively, we have

$$|r_2 - r_1| = c\int_{t_1}^{t_2} b(t)dt. \qquad (63)$$

Here we consider a light ray that is emitted at time t_1 and observed at time t_2. The spacetime geometry determines t_2 as a function of t_1. As r_1 and r_2 are kept fixed, differentiation of (63) with respect to t_1 yields

$$0 = b(t_2)\frac{dt_2}{dt_1} - b(t_1). \qquad (64)$$

Since, by construction, t is proper time for observers at rest, this gives the redshift

$$1+z=\frac{\omega_1}{\omega_2}=\frac{dt_2}{dt_1}=\frac{b(t_1)}{b(t_2)}. \tag{65}$$

Comparison of this equation with the standard redshift formula in Robertson-Walker spacetimes, $1+z=S(t_2)/S(t_1)$, reveals that, as far as the redshift formula is concerned, the function

$$\hat{S}(t)\coloneqq\frac{1}{b(t)} \tag{66}$$

may be viewed as the Finsler generalization of the scale factor $S(t)$. This becomes even more evident if we introduce on the spacetime the real-valued function

$$\hat{f}\coloneqq\ln(\hat{S}\circ t). \tag{67}$$

Here t is to be viewed as the function which assigns to each point in the spacetime the value of its t coordinate, the ring denotes composition of maps and ln is the natural logarithm. Then it is readily verified that $\hat{f}$ is a redshift potential for the observers at rest; see (36).

Here it is important to realize that in an unperturbed Robertson-Walker universe the scale factor $S(t)$ does not only give the redshift but also the growth rate of distances, as measured with the purely spatial part of the metric, between two observers at rest. As to the latter property, our function $\hat{S}$ must *not* be viewed as the Finsler generalization of the scale factor. This can be seen by considering the Finslerian arclength s of a segment of an r coordinate line parametrized by r itself, $r_1\le r\le r_2$ or $r_2\le r\le r_1$. Along such a segment $u=|\dot{r}|=1$, $\dot{\vartheta}=\dot{\varphi}=0$ and $\dot{t}=0$. Therefore, we find this arclength s from (59) by a limit procedure:

$$s=\left|\int_{r_1}^{r_2}\sqrt{2\mathcal{L}}dr\right|=\sqrt{\lim_{B\to\infty}\frac{|\ell(B,t)|}{B^2}}|r_2-r_1|. \tag{68}$$

This implies that the function

$$\bar{S}(t)\coloneqq\sqrt{\lim_{B\to\infty}\frac{|\ell(B,t)|}{B^2}} \tag{69}$$

has to be viewed as the Finsler generalization of the scale factor as far as the growth rate of distances is concerned.

We summarize these findings in the following way. In standard general relativity a spatially homogeneous and isotropic cosmological model is completely determined by one function of cosmic time, provided that the spatial curvature parameter k has been fixed. This is the scale factor $S(t)$ which determines the redshift, the growth rate of spatial distances and all the other geometric features of the model. By contrast, in the case of a spatially homogeneous and isotropic Finsler model the redshift and the growth rate of spatial distances are given by two different functions, $\hat{S}(t)$ and $\bar{S}(t)$.

On the basis of this observation it should not come as a surprise that the relations between the redshift and certain distance measures in a cosmological Finsler model are more complicated than in a standard Robertson-Walker model. In the following we will work out these relations for the two most important distance measures, the area distance and the luminosity distance. For this part we will restrict to a special class of cosmological Finsler spacetimes which are small perturbations of standard Robertson-Walker spacetimes. It will then be possible to operate with explicit expressions, to compare with the unperturbed Robertson-Walker model and, in doing so, to demonstrate the applicability of our redshift formula.

In analogy to the procedure in the preceding example, we consider a perturbed Lagrangian of the form

$$\begin{aligned}2\mathcal{L}=&-c^2\dot{t}^2(1+\phi_0(t))\\&+S(t)^2(\dot{r}^2+\Sigma(r)^2(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2))(1+\phi_1(t))\\&+\frac{\phi_2(t)S^2c^2\dot{t}^2(\dot{r}^2+\Sigma(r)^2(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2))}{S(t)^2(\dot{r}^2+\Sigma(r)^2(\dot{\vartheta}^2+\sin^2\vartheta\dot{\varphi}^2))+c^2\dot{t}^2}.\end{aligned} \tag{70}$$

In contrast to the example of Sec. IV, where we had perturbation coefficients depending on r, now we have perturbation coefficients ϕ_A that are functions of t. Clearly, ϕ_0 changes the time measurement, ϕ_1 changes the length measurement in all spatial directions, and ϕ_2 is a genuine Finsler perturbation.

It is easy to verify that the Lagrangian (70) is of the form of (59) with

$$\begin{aligned}\ell(B,t)=&1+\phi_0(t)-S(t)^2B^2(1+\phi_1(t))\\&-\phi_2(t)\frac{S(t)^2B^2}{S(t)^2B^2+1},\end{aligned} \tag{71}$$

where B is a place holder for the first argument of the function ℓ. Our condition (60) implies that

$$\phi_0(t)\equiv 0. \tag{72}$$

Note that, in addition, we could transform $\phi_1(t)$ to zero by redefining the scale factor, $S(t)^2\mapsto S(t)^2(1+\phi_1(t))$. This is, of course, related to the fact that $\phi_1(t)$ describes a perturbation within the class of standard Robertson-Walker models and not a genuine Finsler perturbation. However, we will not make use of the freedom to transform $\phi_1(t)$ to zero because we want to compare our cosmological Finsler spacetime with a *prescribed* unperturbed Robertson-Walker model; i.e., we want to consider $S(t)$ as a given function which is fixed.

As in the preceding section, we linearize all expressions with respect to the perturbations ϕ_A. To derive the function $\hat{S}(t)$ which was defined in (66) we insert (71) with (72) into (61). This gives the quadratic equation

$$(1+\phi_1)(S^2b^2)^2 + (\phi_1+\phi_2)(S^2b^2) - 1 = 0 \qquad (73)$$

for S^2b^2 (where the argument t of the functions S, b, ϕ_1 and ϕ_2 has been omitted). After the above-mentioned linearization, the solution reads $S^2b^2 = 1 - \phi_1 - \phi_2/2$ which yields

$$\hat{S}(t) = \frac{1}{b(t)} = S(t)(1+\hat{\phi}(t)), \qquad \hat{\phi} = \frac{\phi_1}{2} + \frac{\phi_2}{4}. \qquad (74)$$

Thus, a redshift potential is given by

$$\hat{f} = \ln(\hat{S}\circ t) = f + \hat{\phi}\circ t, \qquad (75)$$

where $f = \ln(S\circ t)$ is a redshift potential for the unperturbed spacetime.

To derive the function $\bar{S}(t)$ which was defined in (69) we divide (71) by B^2 and send B to infinity. This results in

$$\bar{S}(t) = S(t)\left(1 + \frac{\phi_1(t)}{2}\right). \qquad (76)$$

According to (74) and (75), for emitters and observers at rest a light signal emitted at time t_1 and observed at time t_2 will show a redshift of

$$1+z = \frac{\hat{S}(t_2)}{\hat{S}(t_1)} = \frac{S(t_2)}{S(t_1)}(1+\hat{\phi}(t_2) - \hat{\phi}(t_1)). \qquad (77)$$

From (77) we will now derive the relation between the redshift z, the area distance D_A and the luminosity distance D_L. Recall that the area distance D_A is defined by the property that, for a thin pencil of light rays with a vertex at the observer, the cross-sectional area increases with D_A^2. In our cosmological Finsler spacetime the most convenient way of calculating the area distance is by placing the observer in the origin, $r_2 = 0$, and utilizing the isotropy. Intersecting the past light cone of the observation event with the hypersurface $t = t_1$ gives a sphere of constant coordinate radius $r = R$. From (70) we read that this sphere has area $4\pi S(t_1)^2\Sigma(R)^2(1+\phi_1(t_1))$. Equating this expression to $4\pi D_A^2$ determines the area distance:

$$D_A = S(t_1)\Sigma(R)\left(1+\frac{\phi_1(t_1)}{2}\right), \qquad (78)$$

$$R = \int_{t_1}^{t_2} \frac{(1-\hat{\phi}(t))c\,dt}{S(t)}. \qquad (79)$$

Here the expression for R follows from (63) and (74) with $r_2 = 0$ and $r_1 = R$.

Now we consider the luminosity distance D_L. As a preliminary first step, one usually introduces the so-called corrected luminosity distance D_C, which is defined quite analogously to D_A, but now for a pencil with a vertex at the emitter. For calculating D_C in our cosmological Finsler spacetime it is most convenient to place the emitter in the origin of the coordinate system, $r_1 = 0$. In analogy to (78) we then find

$$D_C = S(t_2)\Sigma(R)\left(1+\frac{\phi_1(t_2)}{2}\right), \qquad (80)$$

where R is again given by (79), but this time we have to use (63) and (74) with $r_1 = 0$ and $r_2 = R$. The (uncorrected) luminosity distance D_L is defined as

$$D_L = (1+z)D_C. \qquad (81)$$

Whereas D_C is a purely geometrical quantity, describing for a pencil with a vertex at the emitter how the cross-sectional area changes, D_L carries an additional redshift factor; thereby, D_L is defined such that the radiated energy flux decreases with D_L^2. From (78), (80) and (81) we find that

$$D_L = (1+z)\frac{S(t_2)}{S(t_1)}\left(1+\frac{\phi_1(t_2)}{2} - \frac{\phi_1(t_1)}{2}\right)D_A. \qquad (82)$$

With (77), this result can be rewritten as

$$D_L = (1+z)^2\left(1-\frac{\phi_2(t_2)}{4} + \frac{\phi_2(t_1)}{4}\right)D_A. \qquad (83)$$

In the unperturbed case, (83) reduces to Etherington's [33] reciprocity law $D_L = (1+z)^2 D_A$, which is well known to hold in *any* general-relativistic spacetime; for a proof and a discussion see, e.g., Perlick [34]. Equation (83) shows how Etherington's law is modified in our cosmological Finsler spacetime. Note that ϕ_1 does not enter; i.e., only the genuine Finsler perturbation ϕ_2 has an effect.

Finally, we derive the relation between the redshift and the (area or luminosity) distance in our cosmological Finsler model. To that end we introduce the distance D_T measured in terms of the travel time of light:

$$D_T = c(t_2 - t_1). \qquad (84)$$

Taylor expansion of (77) yields

$$\begin{aligned} 1+z &= \frac{S(t_2)(1+\hat{\phi}'(t_2)\frac{D_T}{c} + O(D_T^2))}{S(t_2) - S'(t_2)\frac{D_T}{c} + O(D_T^2)} \\ &= 1 + \left(\frac{S'(t_2)}{S(t_2)} + \hat{\phi}'(t_2)\right)\frac{D_T}{c} + O(D_T^2) \qquad (85)\end{aligned}$$

and thus

$$D_T = \frac{cS(t_2)}{S'(t_2)}\left(1 - \frac{S(t_2)}{S'(t_2)}\hat{\phi}'(t_2)\right)z + O(z^2). \quad (86)$$

In the unperturbed case, (86) reduces of course to the familiar Lemaître-Hubble law.

For deriving the relation between D_A and z we observe that, by (79),

$$R = \frac{1 - \hat{\phi}(t_2)}{S(t_2)} D_T + O(D_T^2). \quad (87)$$

From (57) we read that, for any value of k,

$$\Sigma(R) = \frac{1 - \hat{\phi}(t_2)}{S(t_2)} D_T + O(D_T^2). \quad (88)$$

With $S(t_1) = S(t_2) + O(D_T)$ and $\phi_1(t_1) = \phi_1(t_2) + O(D_T)$ we find from (78), (86) and (88) that

$$D_A = \frac{cS(t_2)}{S'(t_2)}\left(1 - \frac{\phi_2(t_2)}{4} - \frac{S(t_2)}{S'(t_2)}\hat{\phi}'(t_2)\right)z + O(z^2). \quad (89)$$

By (83), we have the same relation between D_L and z:

$$D_L = \frac{cS(t_2)}{S'(t_2)}\left(1 - \frac{\phi_2(t_2)}{4} - \frac{S(t_2)}{S'(t_2)}\hat{\phi}'(t_2)\right)z + O(z^2); \quad (90)$$

i.e., the linear Lemaître-Hubble law is modified for D_A and for D_L in the same way. In principle, the relation (90) can be tested with standard candles such as type Ia supernovae.

It was the purpose of this section to illustrate our general redshift formula with a cosmological example. To that end we restricted to Finsler spacetimes that are small perturbations of Robertson-Walker spacetimes. For a discussion of the distance-redshift relation in other cosmological Finsler models we refer to Hohmann and Pfeifer [32].

VI. CONCLUSIONS

In this paper we have presented a redshift formula that holds for emitters and receivers on arbitrary worldlines in an unspecified Finsler spacetime. We have illustrated the physical relevance of this formula with two examples: a Finsler-perturbed Schwarzschild spacetime, that may be used for applying our formula to tests in the gravitational field of Earth or the Sun, and a Finsler-perturbed Robertson-Walker spacetime, that may be used for cosmological redshift tests of Finsler geometry. In both cases we have restricted to the simplest nontrivial examples because it was our purpose just to illustrate the general features of our redshift formula. In view of applications, more sophisticated examples are certainly of interest. In particular, instead of just considering circular orbits in the gravitational field of a spherically symmetric and static body, as we did in Sec. IV, it would certainly desirable to consider noncircular orbits. This would make it possible to use the two Galileo satellites that have gone astray for testing possible Finsler deviations of our spacetime geometry. We are planning to do this in a follow-up article.

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APPENDIX: A GEOMETRIC DERIVATION OF THE REDSHIFT FORMULA

Our derivation of the general redshift formula (32) was based on the formal definition of the frequency in terms of the canonical momentum of the light ray, (27). In this Appendix we demonstrate that the same formula can be derived by a more geometrical procedure. The derivation follows closely Brill's derivation [35] of the redshift formula for general-relativistic spacetimes; cf. Straumann [28].

The only assumptions used in the following derivation are that the spacetime is a (four-dimensional) manifold and that light rays are the solutions to the Euler-Lagrange equations (3) with $\mathcal{L} = 0$.

We consider two curves

$$\gamma\colon I \to M, \qquad \tau \mapsto \gamma(\tau) \quad (A1)$$

and

$$\tilde{\gamma}\colon \tilde{I} \to M, \qquad \tilde{\tau} \mapsto \tilde{\gamma}(\tilde{\tau}), \quad (A2)$$

where I and $\tilde{I}$ are real intervals. We refer to γ as to the worldline of the emitter and to $\tilde{\gamma}$ as to the worldline of the receiver. For our application to Finsler geometry, they should be timelike curves parametrized by Finsler proper time; the following mathematical consideration, however, holds for arbitrarily parametrized curves.

Assume that in the events $\gamma(\tau)$ and $\gamma(\tau + \Delta\tau)$ two light rays are emitted. They will be received in two events $\tilde{\gamma}(\tilde{\tau})$ and $\tilde{\gamma}(\tilde{\tau} + \Delta\tilde{\tau})$; see Fig. 2. Then we define the frequency ratio

$$\frac{d\tilde{\tau}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\Delta\tilde{\tau}}{\Delta\tau} = \frac{\omega_1}{\omega_2} = 1 + z. \quad (A3)$$

Here ω_1 and ω_2 refer to the emitted and received frequency, respectively, as measured with clocks whose reading is given by the chosen parametrizations. Mathematically, this defines the redshift factor z for any parametrizations.

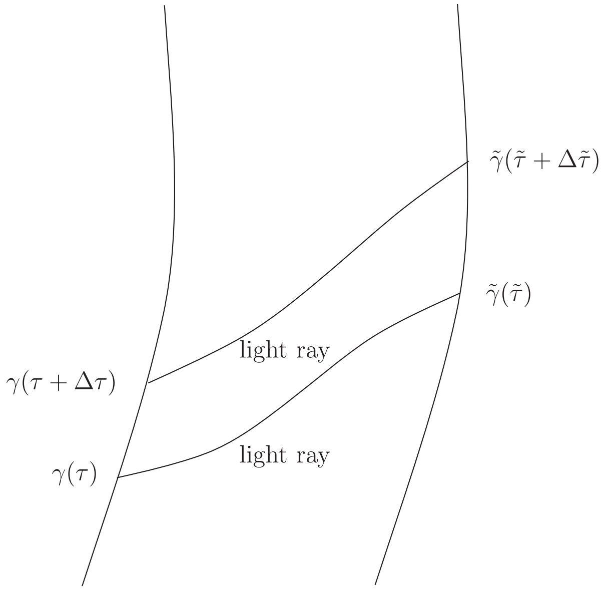

FIG. 2. Two light rays connecting an emitter worldline with a receiver worldline.

We want to derive a formula for the frequency ratio (A3). To that end we consider a variation

$$\mu\colon [s_1, s_2] \times I \to M, \qquad (s,\tau) \mapsto \mu(s,\tau)$$

such that $\mu(s_1,\tau) = \gamma(\tau)$, $\mu(s_2,\tau) = \tilde{\gamma}(\tilde{\tau}(\tau))$ and $\mu(\cdot,\tau)$ is a solution to the Euler-Lagrange equation (3) with $\mathcal{L} = 0$ for all $\tau \in I$; see Fig. 3.

Then, by assumption,

$$0 = \mathcal{L}(\mu(s,\tau), \partial_s\mu(s,\tau)) \tag{A4}$$

for all s and τ. Calculating the total derivative with respect to τ yields

$$\begin{aligned} 0 = {} & \frac{\partial\mathcal{L}}{\partial x^\rho}(\mu(s,\tau), \partial_s\mu(s,\tau))\partial_\tau\mu^\rho(s,\tau) \\ & + \frac{\partial\mathcal{L}}{\partial\dot{x}^\rho}(\mu(s,\tau), \partial_s\mu(s,\tau))\partial_\tau\partial_s\mu^\rho(s,\tau). \end{aligned} \tag{A5}$$

After commuting the partial derivatives ∂_s and ∂_τ and using the product rule we find

$$\begin{aligned} 0 = {} & \left(\frac{\partial\mathcal{L}}{\partial x^\rho}(\mu(s,\tau), \partial_s\mu(s,\tau)) - \partial_s\frac{\partial\mathcal{L}}{\partial\dot{x}^\rho}(\mu(s,\tau), \partial_s\mu(s,\tau))\right) \\ & \times \partial_\tau\mu^\rho(s,\tau) + \partial_s\left[\frac{\partial\mathcal{L}}{\partial\dot{x}^\rho}(\mu(s,\tau), \partial_s\mu(s,\tau))\partial_\tau\mu^\rho(s,\tau)\right]. \end{aligned} \tag{A6}$$

The first term vanishes because we assume that all curves $\mu(\cdot,\tau)$ satisfy the Euler-Lagrange equation. So the term in the square brackets takes the same value at $s = s_1$ and at $s = s_2$. We evaluate this equality for the light ray $x(s) = \mu(s,\tau_1)$, where τ_1 is a particular value of the parameter τ, and we write $\tilde{\tau}(\tau_1) = \tilde{\tau}_2$. With

$$\partial_\tau\mu^\rho(s_1,\tau_1) = \frac{d\gamma^\rho}{d\tau}(\tau_1), \qquad \partial_\tau\mu^\rho(s_2,\tau_2) = \frac{d\tilde{\gamma}^\rho}{d\tilde{\tau}}(\tilde{\tau}_2)\frac{1}{1+z}, \tag{A7}$$

where (A3) has been used; this results indeed in our redshift formula (32).

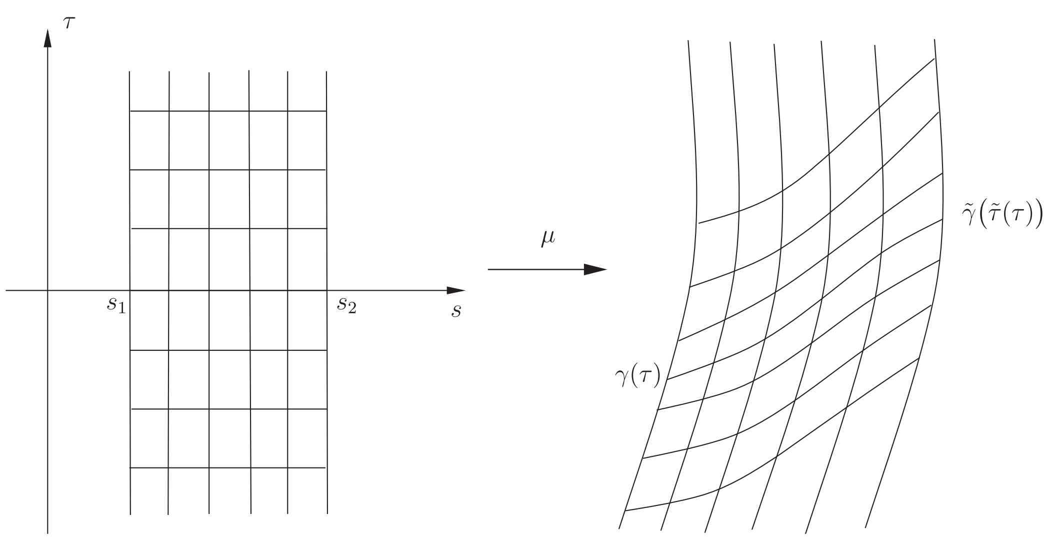

FIG. 3. The variation μ.

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