Second-order cosmological perturbations: New conserved quantities and the general solution at super-horizon scale

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The study of long-wavelength scalar perturbations, in particular the existence of conserved quantities when the perturbations are adiabatic, plays an important role in e.g., inflationary cosmology. In this paper we present some new conserved quantities at second order and relate them to the curvature perturbation in the uniform density gauge, ζ, and the comoving curvature perturbation, $\mathcal{R}$. We also, for the first time, derive the general solution of the perturbed Einstein equations at second order, which thereby contains both growing and decaying modes, for adiabatic long-wavelength perturbations for a stress-energy tensor with zero anisotropic stresses and zero heat flux. The derivation uses the total matter gauge, but results are subsequently translated to the uniform curvature and Poisson (longitudinal, zero shear) gauges.

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I. INTRODUCTION

In this paper we consider first and second-order scalar perturbations of Friedmann-Lemaître (FL) universes subject to the following assumptions: (i) the spatial background is flat; (ii) the stress-energy tensor can be written in the form

$$T^a{}_b = (\rho + p)u^a u_b + p\delta^a{}_b, \qquad u^a u_a = -1, \quad (1)$$

thereby describing perfect fluids and scalar fields; (iii) the linear perturbation is purely scalar. This paper, which deals with perturbations on super-horizon scales, relies heavily on two previous papers which we shall refer to as UW1 [1] (a unified and simplified formulation of change of gauge formulas at second order) and UW2 [2] (five ready-to-use systems of governing equations for second-order perturbations).

Two gauge invariants that are conserved for adiabatic long-wavelength perturbations at first and second order play an important role in e.g., inflationary cosmology, namely, the curvature perturbation in the uniform density gauge, labeled ζ, and the curvature perturbation in the total matter gauge, labeled $\mathcal{R}$, also often referred to as the comoving curvature perturbation. We briefly discuss the history of these conserved quantities and give references at the beginning of Sec. IV. In this paper we present some new conserved quantities that in contrast are associated with the uniform curvature gauge. In particular, writing the perturbed Einstein equations in the super-horizon regime in that gauge suggests consideration of a gauge invariant which we denote by χ_c, defined in terms of ϕ_c, the purely temporal metric perturbation (see the next section) in the uniform curvature gauge according to

$$^{(1)}\chi_c = (1+q)^{-1}\,{}^{(1)}\phi_c, \quad {}^{(2)}\chi_c = (1+q)^{-1}({}^{(2)}\phi_c - 4\,{}^{(1)}\phi_c^2), \quad (2)$$

where q is the background deceleration parameter. At first order one of the perturbed Einstein equations shows that χ_c is a conserved quantity, while the two constraint equations relate the density and velocity perturbations algebraically to χ_c, thereby providing two more conserved quantities. In addition these equations show that χ_c in fact coincides with $\mathcal{R}$ at first order.

Unlike $\mathcal{R}$ and ζ these new quantities are not conserved at second order. However, new conserved quantities can be constructed at second order by adding a certain quadratic source term to the perturbations. In particular, we use "source compensated" second-order perturbation variables of the form that we introduced in an earlier paper UW1 [1] in order to simplify the change of gauge formulas, which, moreover, are used to relate the new conserved quantities to ζ and $\mathcal{R}$ at second order.

We then derive the general solution of the governing equations for adiabatic long-wavelength perturbations at first and second order subject to the restrictions (i)–(iii) above. We have found that the governing equations *in the*

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total matter gauge are particularly simple to solve, even when keeping both modes (growing and decaying). The time dependence of the growing mode of the first order perturbations is governed by a function $g(a)$, defined by[1]

$$g(a) = 1 - \frac{\mathcal{H}}{a^2}\int_0^a \frac{\bar{a}}{\mathcal{H}(\bar{a})}d\bar{a}, \tag{3}$$

where $\mathcal{H} = aH$, with H being the background Hubble parameter, and a is the background scale factor. A main result of this paper is to show that the simple form of the first order solution in the total matter gauge extends to second order. The conserved quantities referred to above emerge naturally in the solution process as temporal constants of integration (arbitrary spatial functions). Because of the central role played by the function $g(a)$ we shall refer to it as the *perturbation evolution function*.

The outline of the paper is as follows. In Sec. II, we introduce the notation for the metric and matter variables from UW1 [1] and UW2 [2]. In Sec. III we specialize the governing equations to second order given in UW2 [2] to long-wavelength perturbations. In Sec. IV we derive the new conserved quantities at second order and relate them to the previously known ones. In Sec. V we derive the general solution of the governing equations up to second order in the total matter gauge, and subsequently transform the results to the uniform curvature and Poisson gauges by means of gauge transformation rules, followed by some illustrative applications in Sec. VI. In Sec. VII we give a brief discussion of the history and properties of the perturbation evolution function $g(a)$. Section VIII contains the concluding remarks. In the Appendixes we give some background material from UW1 [1] and UW2 [2].

II. PERTURBATION VARIABLES

We describe scalar perturbations of a flat Robertson-Walker geometry by writing the metric in the form[2]

$$ds^2 = a^2(-(1+2\phi)d\eta^2 + \mathbf{D}_i B d\eta dx^i + (1-2\psi)\delta_{ij}dx^i dx^j), \tag{4}$$

where η is conformal time, the x^i are Cartesian background coordinates and $\mathbf{D}_i = \partial/\partial x^i$. The background geometry is described by the scale factor a which determines the conformal Hubble scalar and the deceleration parameter according to $\mathcal{H} = a'/a$ and $q = -\mathcal{H}'/\mathcal{H}^2$, where $'$ denotes differentiation with respect to η. By expanding the functions ϕ, B, ψ in a perturbation series[3] we obtain the following metric perturbations up to second order:

$$^{(r)}\phi, \mathcal{H}^{(r)}B, {}^{(r)}\psi, \quad r = 1, 2, \tag{5}$$

where the factor of $\mathcal{H}$ ensures that the B perturbation is dimensionless (see UW1 [1] and UW2 [2]).

The background matter content is described by the matter density and pressure, ρ_0 and p_0, with associated scalars $w = p_0/\rho_0$ and $c_s^2 = p_0'/\rho_0'$. We will need the fact that the background Einstein equations relate w and q according to

$$3(1+w) = 2(1+q), \tag{6}$$

(see UW2 [2]). The scalar matter perturbations are defined by expanding ρ, p, V in a perturbation series, where the scalar velocity potential V is defined in terms of the spatial *covariant* 4-velocity components by $u_i = a\mathbf{D}_i V$. As in UW2 [2], Sec. II C, we scale the density perturbations according to $^{(r)}\boldsymbol{\delta} = {}^{(r)}\rho/(\rho_0 + p_0)$, $r = 1$, 2, and replace the pressure perturbations $^{(r)}p$ by the nonadiabatic pressure perturbations $^{(r)}\Gamma$, $r = 1$, 2, which are defined to be gauge invariants with the property that they are zero for adiabatic perturbations.[4] Thus the scalar matter perturbations are described up to second order by the variables

$$\mathcal{H}^{(r)}V, {}^{(r)}\boldsymbol{\delta}, {}^{(r)}\Gamma, \quad r = 1, 2, \tag{7}$$

where the factor of $\mathcal{H}$ ensures that the V perturbation is dimensionless. In keeping with this approach we also use the background e-fold time variable $N = \ln(a/a_0)$, where a_0 denotes some reference epoch. For changing to conformal time, note that $\partial_\eta = \mathcal{H}\partial_N$, $\partial_\eta^2 = \mathcal{H}^2(\partial_N^2 - q\partial_N)$.

In this paper we will show that when studying perturbations on super-horizon scale significant simplifications arise when one makes use of the so-called source-compensated second-order perturbation variables, labeled by a hat on the kernel, that we introduced in our earlier paper UW1 [1]:

$$^{(2)}\hat{\phi} = {}^{(2)}\phi - 2{}^{(1)}\phi^2, \tag{8a}$$

$$^{(2)}\hat{\psi} = {}^{(2)}\psi + 2{}^{(1)}\psi^2, \tag{8b}$$

[1]The integral in (3) has a lengthy history in linear perturbation theory, but a standard symbol for it has not been introduced. Because of the importance of the function $g(a)$ in cosmological perturbation theory, we digress in Sec. VII to describe some of its history and properties.

[2]The scalar perturbations at first order will generate vector and tensor perturbations at second order, but we do not give these perturbation variables since we will not consider these modes in this paper.

[3]A perturbation series for a variable f is a Taylor series in a perturbation parameter ϵ, of the form $f = f_0 + \epsilon^{(1)}f + \frac{1}{2}\epsilon^{2(2)}f + \cdots$.

[4]See UW2 [2], Sec. II C, for the definitions. The details are not needed in this paper: since we are working exclusively with adiabatic perturbations, the terms in the perturbation equations that involve $^{(r)}\Gamma$, $r = 1$, 2, will be set to zero.

$$\mathcal{H}^{(2)}\hat{B} = \mathcal{H}^{(2)}B + (1+q)(\mathcal{H}^{(1)}B)^2, \tag{8c}$$

$$\mathcal{H}^{(2)}\hat{V} = \mathcal{H}^{(2)}V + (1+q)(\mathcal{H}^{(1)}V)^2, \tag{8d}$$

$$^{(2)}\hat{\boldsymbol{\delta}} = {}^{(2)}\boldsymbol{\delta} - (1+c_s^2){}^{(1)}\boldsymbol{\delta}^2. \tag{8e}$$

As regards gauge freedom, in using the line element (4) we have fixed the spatial gauge following UW1 [1]. The remaining gauge freedom is the choice of temporal gauge which we can fix to second order by setting to zero the first and second perturbations of one of the variables ψ, B, V, $\boldsymbol{\delta}$. We use the following terminology and subscripts to label the gauges as in UW1 [1]:

(i) $B = 0$, Poisson (longitudinal, zero shear) gauge, subscript $_\mathrm{p}$, e.g., ψ_p,

(ii) $\psi = 0$, uniform curvature (flat) gauge, subscript $_c$, e.g., B_c,

(iii) $V = 0$, total matter gauge, subscript $_\mathrm{v}$, e.g., ψ_v,

(iv) $\boldsymbol{\delta} = 0$, uniform density gauge, subscript $_\rho$, e.g., ψ_ρ.

We note in passing that on super-horizon scales the uniform density gauge is equivalent to the total matter gauge to second order (see Appendix C).

III. GOVERNING EQUATIONS IN THE SUPER-HORIZON REGIME

In this section we obtain the governing equations for perturbations at second order that we need in this paper by specializing the general equations in UW2 [2] to super-horizon scales. This is accomplished by dropping terms of degree two and higher in the dimensionless spatial differential operator $\mathcal{H}^{-1}\mathbf{D}_i$. We will use the symbol $\approx$ to indicate that two expressions are equal once such terms have been dropped.

Before continuing we digress briefly in order to clarify the meaning of the approximation symbol $\approx$ in the present context, by making the transition to Fourier space. For linear perturbations the spatial derivatives occur in the perturbation equations as the spatial Laplacian $\mathcal{H}^{-2}\mathbf{D}^2$, which is represented by $-\mathcal{H}^{-2}k^2$ in Fourier space, where k is the wave number. The super-horizon regime is defined by the requirement that the wave number satisfies $\mathcal{H}^{-2}k^2 \ll 1$, and we form the super-horizon limit of the equations by dropping terms in $\mathcal{H}^{-2}k^2$. Thus when using the super-horizon limit we expect that errors will be of order $\mathcal{H}^{-2}k^2$:

$$\mathcal{H}^{-2}\mathbf{D}^2 A \to -(\mathcal{H}^{-2}k^2)A_k = \mathcal{O}(\mathcal{H}^{-2}k^2) \quad \text{for } \mathcal{H}^{-2}k^2 \ll 1, \tag{9}$$

where $A(N, x^i)$ is any linear perturbation and $A_k(N)$ is its Fourier coefficient. It should be noted that in writing the above equality we are assuming that $A_k(N)$ is bounded as $k \to 0$, and for some range of values of N.[5] Thus, subject to this boundedness restriction,[6] a relation of the form $A \approx B$ in real space means that $A_k = B_k + \mathcal{O}(\mathcal{H}^{-2}k^2)$ as $k \to 0$ in Fourier space. For second-order perturbations the process is more complicated since one has to use the convolution theorem to take the Fourier transform of products of first order perturbations that appear in the source terms of the second-order perturbation equations. The spatial derivative operator $\mathcal{H}^{-1}\mathbf{D}^i$, which appears quadratically in the source terms, is represented by $-i\mathcal{H}^{-1}k^i$ in Fourier space, where k^i is the wave vector. For the leading order terms an analysis similar to Eq. (9) is valid, but we are not aware of analogous results for the quadratic source terms.[7]

A. The energy conservation equation

The perturbed energy conservation equation on super-horizon scales plays a central role in deriving conserved quantities in cosmological perturbation theory. By specializing the general perturbed energy conservation equation in UW2 [2] (see Sec. IV) to super-horizon scales we obtain at first order

$$\partial_N({}^{(1)}\boldsymbol{\delta} - 3{}^{(1)}\psi) + 3{}^{(1)}\Gamma \approx 0, \tag{10}$$

and at second order,

$$\partial_N({}^{(2)}\boldsymbol{\delta} - 3{}^{(2)}\psi) + 3{}^{(2)}\Gamma + \mathbb{E} \approx 0, \tag{11a}$$

where the source term is given by

$$\mathbb{E} \approx -\partial_N(6\psi^2 + (1+c_s^2)\boldsymbol{\delta}^2 + 2\boldsymbol{\delta}\Gamma) - 6\Gamma^2, \tag{11b}$$

after simplifying them using the first order equation. Here and elsewhere, in the interests of notational simplicity, we will drop the superscript $^{(1)}$ on first order perturbations, when there is no risk of confusion. This applies in particular to expressions for source terms. On introducing the hatted variables given in (8), Eq. (11a) takes the simpler form

$$\partial_N({}^{(2)}\hat{\boldsymbol{\delta}} - 3{}^{(2)}\hat{\psi}) + 3({}^{(2)}\Gamma - 2\Gamma^2) - 2\partial_N(\boldsymbol{\delta}\Gamma) \approx 0. \tag{12}$$

When specialized to the uniform density gauge (${}^{(r)}\boldsymbol{\delta} = 0$, $r = 1, 2$) we obtain

[5] For example, if $A = \psi_\mathrm{p}$ and the decaying mode is present, then A will be unbounded as $N \to -\infty (a \to 0)$. See Eq. (44) and the paragraph following Eq. (35).

[6] We note that this restriction is not satisfied in the ultra-slow-roll model of inflation. See e.g., Romano *et al.* [3], who show that ψ_ρ and ψ_v do not coincide on super-horizon scales. See the end of Sec. VII and our Eq. (25).

[7] Some references that use the convolution theorem to analyze second-order source terms are Malik [4], Huston and Malik [5] and Tram *et al.* [6].

$$\partial_N {}^{(1)}\psi_\rho \approx {}^{(1)}\Gamma, \qquad \partial_N {}^{(2)}\hat{\psi}_\rho \approx {}^{(2)}\Gamma - 2\Gamma^2. \tag{13}$$

These equations are a concise form of well-known equations in the literature [see Malik and Wands [7], Eqs. (5.34) and (5.35), and Bartolo *et al.* [8], Eqs. (144) and (147)].

B. Governing equations in the total matter gauge

We use the governing equations as given in UW2 [2] (see Sec. V C 1). At first order we have

$$ {}^{(1)}\phi_{\rm v} = -c_s^2 {}^{(1)}\boldsymbol{\delta}_{\rm v} - {}^{(1)}\Gamma, \tag{14a}$$

$$\partial_N {}^{(1)}\psi_{\rm v} = -{}^{(1)}\phi_{\rm v}, \tag{14b}$$

$$\partial_N (a^2 {}^{(1)}B_{\rm v}) = a^2 \mathcal{H}^{-1}({}^{(1)}\psi_{\rm v} - {}^{(1)}\phi_{\rm v}), \tag{14c}$$

while the second-order equations can be written as

$$ {}^{(2)}\phi_{\rm v} = -c_s^2 {}^{(2)}\boldsymbol{\delta}_{\rm v} - {}^{(2)}\Gamma - \mathbb{M}_{\rm v}, \tag{15a}$$

$$\partial_N {}^{(2)}\psi_{\rm v} = -{}^{(2)}\phi_{\rm v} + \frac{1}{2}\mathbb{G}_{\rm v}^q, \tag{15b}$$

$$\partial_N (a^2 {}^{(2)}B_{\rm v}) = a^2 \mathcal{H}^{-1}({}^{(2)}\psi_{\rm v} - {}^{(2)}\phi_{\rm v} + \mathbb{G}_{\rm v}^\pi), \tag{15c}$$

where the source terms can be obtained from UW2 [2], the Einstein terms $\mathbb{G}_{\rm v}^q$ and $\mathbb{G}_{\rm v}^\pi$ from Appendix A1 and $\mathbb{M}_{\rm v}$ from Appendix A3. In the super-horizon regime Eq. (29) below gives ${}^{(r)}\boldsymbol{\delta}_{\rm v} \approx 0$, $r = 1, 2$, which for adiabatic perturbations, ${}^{(r)}\Gamma = 0$, $r = 1, 2$, implies ${}^{(1)}\phi_{\rm v} \approx 0$ and $\partial_N {}^{(1)}\psi_{\rm v} \approx 0$ by (14). With these restrictions the source terms reduce to

$$\mathbb{M}_{\rm v} \approx 0, \qquad \mathbb{G}_{\rm v}^q \approx 0, \qquad \mathbb{G}_{\rm v}^\pi \approx 2\psi_{\rm v}^2 - 2\mathbb{D}_0(\psi_{\rm v}). \tag{16}$$

The differential operator $\mathbb{D}_0$ in (16), which we refer to as the general relativity spatial operator, is defined by[8]

$$\mathbb{D}_0(C) \coloneqq \mathcal{S}^{ij}(\mathbf{D}_i C)(\mathbf{D}_j C). \tag{17}$$

The scalar mode extraction operator $\mathcal{S}^{ij}$ is given by $\mathcal{S}^{ij} = \frac{3}{2}(\mathbf{D}^{-2})^2\mathbf{D}^{ij}$, where $\mathbf{D}_{ij} \coloneqq \mathbf{D}_{(i}\mathbf{D}_{j)} - \frac{1}{3}\gamma_{ij}\mathbf{D}^2$ and $\mathbf{D}^{-2}$ is the inverse Laplacian operator. The operator $\mathbb{D}_0$ satisfies the identity

$$\mathcal{S}^{ij}[C\mathbf{D}_{ij}C] = \frac{1}{2}C^2 - \mathbb{D}_0(C), \tag{18}$$

which is needed in simplifying the source terms to get (16).

With (16) it follows that for long-wavelength adiabatic perturbations Eqs. (14) and (15) reduce to the very simple form:

$$\partial_N {}^{(r)}\psi_{\rm v} \approx 0, \qquad {}^{(r)}\phi_{\rm v} \approx 0, \qquad {}^{(r)}\boldsymbol{\delta}_{\rm v} \approx 0, \quad r = 1, 2, \tag{19a}$$

with $B_{\rm v}$ determined by

$$\partial_N (a^2 {}^{(1)}B_{\rm v}) \approx a^2 \mathcal{H}^{-1} {}^{(1)}\psi_{\rm v}, \tag{19b}$$

$$\partial_N (a^2 {}^{(2)}B_{\rm v}) \approx a^2 \mathcal{H}^{-1}({}^{(2)}\hat{\psi}_{\rm v} - 2\mathbb{D}_0({}^{(1)}\psi_{\rm v})). \tag{19c}$$

For convenience we have incorporated part of the source term in (19c) into ${}^{(2)}\psi_{\rm v}$ to give ${}^{(2)}\hat{\psi}_{\rm v}$.

C. Governing equations in the uniform curvature gauge

In this section we make use of the governing equations in the uniform curvature gauge, given in UW2 [2] (see Sec. V B 1). In Appendix A we specialize these equations to the super-horizon regime [see Eqs. (A1) and (A2)]. The form of these equations suggest that we introduce the new variable $\chi_{\rm c}$ defined by (2), which at first order leads to

$$\partial_N {}^{(1)}\chi_{\rm c} \approx 0, \qquad \mathcal{H}^{(1)}V_{\rm c} = -{}^{(1)}\chi_{\rm c}, \qquad {}^{(1)}\boldsymbol{\delta}_{\rm c} \approx -3 {}^{(1)}\chi_{\rm c}. \tag{20}$$

After using these first order equations to write the source terms (A5) in terms of $\chi_{\rm c}$ the equations at second order assume the form

$$\partial_N {}^{(2)}\chi_{\rm c} \approx \partial_N[-3(1 + c_s^2)\chi_{\rm c}^2], \tag{21a}$$

$$\mathcal{H}^{(2)}V_{\rm c} \approx -{}^{(2)}\chi_{\rm c} - [3(1 + c_s^2) + (1 + q)]\chi_{\rm c}^2, \tag{21b}$$

$$ {}^{(2)}\boldsymbol{\delta}_{\rm c} \approx -3 {}^{(2)}\chi_{\rm c}. \tag{21c}$$

The form of these equations suggests that we define a hatted variable for $\chi_{\rm c}$ according to

$$ {}^{(2)}\hat{\chi}_{\rm c} = {}^{(2)}\chi_{\rm c} + 3(1 + c_s^2){\chi_{\rm c}}^2, \tag{22}$$

in analogy with the hatted variables defined in (8). On introducing these hatted variables Eq. (21) assumes the following concise form:

$$\partial_N {}^{(2)}\hat{\chi}_{\rm c} \approx 0, \quad \mathcal{H}^{(2)}\hat{V}_{\rm c} \approx -{}^{(2)}\hat{\chi}_{\rm c}, \quad {}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c} \approx -3 {}^{(2)}\hat{\chi}_{\rm c}. \tag{23}$$

[8]The *general relativity spatial operator* $\mathbb{D}_0(C)$ plays a central role in determining the spatial dependence of second-order perturbations at super-horizon scale, a general relativistic phenomenon (see, e.g., Bartolo *et al.* [9]). Usually it is written out in full which makes the source terms look unnecessarily complicated. See Appendix B of our paper UW1 [1] for some history and properties of $\mathbb{D}_0(C)$.

IV. CONSERVED QUANTITIES FOR ADIABATIC PERTURBATIONS

There are two well-known conserved quantities for long-wavelength adiabatic perturbations, the curvature perturbation in the uniform density gauge, usually denoted by ζ and the comoving curvature perturbation, usually denoted by $\mathcal{R}$. These conserved quantities were first introduced in the 1980s for linear perturbations, ζ by Bardeen *et al.* [10] [see Eqs. (2.43) and (2.45)], and $\mathcal{R}$ by Bardeen [11] [see Eqs. (5.19) and (5.21)]. They are defined in terms of the metric perturbations according to[9]

$$ {}^{(1)}\zeta = -{}^{(1)}\psi_\rho, \qquad {}^{(1)}\mathcal{R} = {}^{(1)}\psi_{\rm v}. \tag{24}$$

These conserved quantities were subsequently generalized to second order. In an important paper Malik and Wands [7] showed that ${}^{(2)}\psi_\rho$ is such a conserved quantity at second order, and moreover the conservation property depends only on the perturbed conservation of energy equation.[10] It is also known that the gauge invariant ${}^{(2)}\psi_{\rm v}$ is another conserved quantity of this type, although in this case one has to in addition use the perturbed Einstein equations in order to establish conservation.[11]

In this section we give three new conserved quantities at second order that are associated with the uniform curvature gauge and relate them to the two well-known quantities. We also derive the conservation properties in a simple, unified manner. We begin by reviewing the results at first order, most of which are known.[12] At first order the five gauge invariants ψ_ρ, $\psi_{\rm v}$, $\chi_{\rm c}$, $-\mathcal{H}V_{\rm c}$, $-\frac{1}{3}\boldsymbol{\delta}_{\rm c}$, are conserved for adiabatic perturbations on super-horizon scales, and all are equal on super-horizon scales,

$$ {}^{(1)}\psi_\rho \approx {}^{(1)}\psi_{\rm v} \approx {}^{(1)}\chi_{\rm c} \approx -\mathcal{H}{}^{(1)}V_{\rm c} \approx -\frac{1}{3}{}^{(1)}\boldsymbol{\delta}_{\rm c}, \tag{25}$$

the common value being the spatial function ${}^{(1)}C$ in the solutions in Sec. V below.[13]

At second order we have an analogous result provided one uses the gauge invariants that correspond to the hatted variables defined in Eq. (8). Specifically, the following gauge invariants are conserved and have the same value for adiabatic perturbations on super-horizon scales:

$$ {}^{(2)}\hat{\psi}_\rho \approx {}^{(2)}\hat{\psi}_{\rm v} \approx {}^{(2)}\hat{\chi}_{\rm c} \approx -\mathcal{H}{}^{(2)}\hat{V}_{\rm c} \approx -\frac{1}{3}{}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c}, \tag{26}$$

the common value being the spatial function ${}^{(2)}C$ in the solutions in Sec. V below. This statement is one of the main results of this paper.

We now give a derivation of the conservation property of these quantities, and establish the relations between them. First, we need the perturbed energy conservation equation in the super-horizon regime, Eqs. (10) and (12), which we specialize to adiabatic perturbations (${}^{(r)}\Gamma \approx 0$, $r = 1, 2$):

$$ \partial_N({}^{(1)}\boldsymbol{\delta} - 3{}^{(1)}\psi) \approx 0, \qquad \partial_N({}^{(2)}\hat{\boldsymbol{\delta}} - 3{}^{(2)}\hat{\psi}) \approx 0. \tag{27}$$

Second, in the uniform curvature gauge two of the perturbed Einstein equations are constraint equations for $\mathcal{H}{}^{(2)}V_{\rm c}$ and ${}^{(2)}\boldsymbol{\delta}_{\rm c}$ given in the super-horizon regime for adiabatic perturbations by Eq. (23), which we repeat here:

$$ \mathcal{H}{}^{(2)}\hat{V}_{\rm c} \approx -{}^{(2)}\hat{\chi}_{\rm c}, \qquad {}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c} \approx -3{}^{(2)}\hat{\chi}_{\rm c}. \tag{28}$$

Third, we specialize the constraint equation (B3) in Appendix B for ${}^{(r)}\boldsymbol{\delta}$, $r = 1, 2$, in the super-horizon regime, to the total matter gauge (${}^{(r)}V = 0$, $r = 1, 2$), which leads to

$$ {}^{(1)}\boldsymbol{\delta}_{\rm v} \approx 0, \qquad {}^{(2)}\boldsymbol{\delta}_{\rm v} \approx 0. \tag{29}$$

In other words, in the super-horizon regime the density perturbations to second order in the total matter gauge are negligible[14] (irrespective of whether the perturbations are adiabatic).

We begin by specializing Eq. (27) successively to the uniform density gauge, $\boldsymbol{\delta} = 0$, the uniform curvature gauge, $\psi = 0$, the total matter gauge, $V = 0$, and conclude that ${}^{(2)}\hat{\psi}_\rho$, ${}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c}$ and ${}^{(2)}\hat{\psi}_{\rm v}$ are conserved, where the last result also requires the property (29). It now follows from (23) that ${}^{(2)}\hat{\chi}_{\rm c}$ and $\mathcal{H}{}^{(2)}\hat{V}_{\rm c}$ are also conserved. We note that conservation of the gauge invariants ψ_ρ and $\boldsymbol{\delta}_{\rm c}$ depends only on conservation of energy while conservation of the other gauge invariants in (26) also requires the Einstein equations. Continuing, the previous manipulations also establish the approximate equality of ${}^{(2)}\hat{\chi}_{\rm c}$, $-\mathcal{H}{}^{(2)}\hat{V}_{\rm c}$ and $-\frac{1}{3}{}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c}$. Finally we can establish that ${}^{(2)}\hat{\psi}_\rho$ is equal to these variables and to ${}^{(2)}\hat{\psi}_{\rm v}$ by using a change of gauge formula in the super-horizon regime, which reads[15]

[9] See, e.g., Malik and Wands [12], Eqs. (7.61) and (7.46), and Vernizzi [13], Eq. (14).

[10] See Eqs. (4.17), (4.18), (5.34) and (5.35) in [7].

[11] See, e.g., Noh and Hwang [14], Eqs. (281) and (362), and Pitrou *et al.* [15], Eq. (3.6b).

[12] An early work that considered conserved quantities in a variety of gauges is Hwang [16] [see Eqs. (92) and (93)]. In addition to the gauges in this paper he also uses the uniform expansion gauge, but he does not include the gauge invariants $\chi_{\rm c}$ and $\mathcal{H}V_{\rm c}$.

[13] Some pairs are in fact equal on all scales as indicated by $=$ rather than $\approx$, as follows $\psi_{\rm v} = \chi_{\rm c} = -\mathcal{H}V_{\rm c}$, $\psi_\rho = -\frac{1}{3}\boldsymbol{\delta}_{\rm c} \approx -\mathcal{H}V_{\rm c}$.

[14] In accordance with the discussion prior to Sec. III A, we note that ${}^{(1)}\boldsymbol{\delta}_{\rm v} \approx 0$ can be written more precisely as ${}^{(1)}\boldsymbol{\delta}_{\rm v} = \mathcal{O}(\mathcal{H}^{-2}k^2)$ as $k \to 0$ in Fourier space.

[15] Specialize Eq. (49) in UW1 [1] to adiabatic perturbations in the super-horizon regime, and use $\partial_N{}^{(1)}\psi_\rho \approx 0$ to obtain the second order formula.

$$ {}^{(1)}\psi_\rho = {}^{(1)}\psi - \frac{1}{3}{}^{(1)}\boldsymbol{\delta}, \qquad {}^{(2)}\hat{\psi}_\rho \approx {}^{(2)}\hat{\psi} - \frac{1}{3}{}^{(2)}\hat{\boldsymbol{\delta}}. \quad (30) $$

Choose the gauge on the right side of these equations to be successively the uniform curvature gauge and the total matter gauge and use (29) to obtain

$$ {}^{(2)}\hat{\psi}_\rho \approx -\frac{1}{3}{}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c} \approx {}^{(2)}\hat{\psi}_{\rm v}. \quad (31) $$

It should be noted that if ${}^{(1)}\psi_\rho$ and ${}^{(2)}\hat{\psi}_\rho$ are conserved then so is the unhatted variable ${}^{(2)}\psi_\rho$, because the coefficient in the definition (8b) of ${}^{(2)}\hat{\psi}$ is constant. The same remark applies to ${}^{(2)}\psi_{\rm v}$. However, for the other variables in (26) conservation of the hatted variable does not imply conservation of the unhatted variable *unless q and c_s^2 are constant*.

We end this section by pointing out that there is a special class of perturbed FL cosmologies, namely the Lambda cold dark matter (ΛCDM) universes, which admit *linear* conserved quantities on super-horizon scale that remain conserved on all scales. Specifically, the linear comoving curvature perturbation ${}^{(1)}\mathcal{R} = {}^{(1)}\psi_{\rm v}$ is conserved on all scales, as are the related gauge invariants ${}^{(1)}\chi_{\rm c} = -\mathcal{H}{}^{(1)}V_{\rm c} = {}^{(1)}\psi_{\rm v}$. This conservation property follows from the fact that for a perturbed ΛCDM universe the governing equation (14a) reduces to the exact equation $\partial_N{}^{(1)}\psi_{\rm v} = 0$, since $c_s^2 = 0$ and ${}^{(1)}\Gamma = 0$. On the other hand the linear curvature perturbation in the uniform density gauge ${}^{(1)}\zeta = -{}^{(1)}\psi_\rho$ does not have this property, and neither do any of the second-order conserved quantities.

V. THE GENERAL SOLUTION FOR ADIABATIC PERTURBATIONS

In this section we derive the general solution of the governing equations for adiabatic perturbations in the super-horizon regime using the total matter gauge. We then obtain the solution in the uniform curvature gauge and the Poisson gauge by using the change of gauge formulas given in UW1 [1]. The conserved quantities described in Sec. IV emerge naturally in the solution process, beginning with ${}^{(1)}\psi_{\rm v}$ and ${}^{(2)}\hat{\psi}_{\rm v}$, and continuing with Eq. (42a).

A. Solving in the total matter gauge

The governing equations for linear perturbations in the total matter gauge when specialized to adiabatic perturbations in the super-horizon regime assume the simple form (19), which we repeat here but with N replaced by the background scale factor a as time variable. Using $\partial_N = a\partial_a$ we obtain

$$ \partial_a{}^{(1)}\psi_{\rm v} \approx 0, \quad (32a) $$

$$ \partial_a(a^2{}^{(1)}B_{\rm v}) \approx a\mathcal{H}^{-1}{}^{(1)}\psi_{\rm v}, \quad (32b) $$

with

$$ {}^{(1)}\phi_{\rm v} \approx 0, \qquad {}^{(1)}\boldsymbol{\delta}_{\rm v} \approx 0. \quad (33a) $$

It follows immediately from (32a) that

$$ {}^{(1)}\psi_{\rm v} \approx {}^{(1)}C, \quad (33b) $$

where we identify the spatial function ${}^{(1)}C(x^i)$ as the conserved quantity at first order. Solving (32b) for ${}^{(1)}B_{\rm v}$ gives

$$ \mathcal{H}{}^{(1)}B_{\rm v} \approx \left(\frac{\mathcal{H}}{a^2}\int_0^a \frac{\bar{a}}{\mathcal{H}(\bar{a})}d\bar{a}\right){}^{(1)}C + \frac{\mathcal{H}}{a^2}{}^{(1)}C_*, \quad (34) $$

where ${}^{(1)}C_* = \lim_{a\to 0} a^2{}^{(1)}B_{\rm v}$ is a second arbitrary spatial function. In terms of the perturbation growth function $g(a)$ defined in Eq. (3) we obtain

$$ \mathcal{H}{}^{(1)}B_{\rm v} \approx (1-g){}^{(1)}C + \frac{\mathcal{H}}{a^2}{}^{(1)}C_*, \quad (35) $$

which with (33) gives the general solution at first order.

We make a brief remark on the physical viability of the solution. We assume that the deceleration parameter satisfies the weak restriction $q > -2$, which implies that $\mathcal{H}/a^2$ is a decreasing function and that $\mathcal{H}/a^2 \to \infty$ as $a \to 0$. We thus refer to term $(\mathcal{H}/a^2){}^{(1)}C_*$ in the solution as the decaying mode. If the decaying mode is present (${}^{(1)}C_* \neq 0$) we impose a restriction of the form $a > a_* > 0$ on the time evolution in order to ensure that the decaying mode is sufficiently small in the time period under consideration.[16]

At second order the governing equations for adiabatic perturbations on super-horizon scales in the total matter gauge are given by Eq. (19), which we repeat here:

$$ \partial_a{}^{(2)}\hat{\psi}_{\rm v} \approx 0, \quad (36a) $$

$$ \partial_a(a^2{}^{(2)}B_{\rm v}) \approx a\mathcal{H}^{-1}({}^{(2)}\hat{\psi}_{\rm v} - 2\mathbb{D}_0({}^{(1)}\psi_{\rm v})), \quad (36b) $$

with

$$ {}^{(2)}\phi_{\rm v} \approx 0, \qquad {}^{(2)}\boldsymbol{\delta}_{\rm v} \approx 0, \quad (37a) $$

where $\mathbb{D}_0$ is defined in (17). We write the solution of (36a) as

$$ {}^{(2)}\hat{\psi}_{\rm v} \approx {}^{(2)}C, \quad (37b) $$

where we identify the spatial function ${}^{(2)}C(x^i)$ as the conserved quantity at second order. Observe that the

[16]Martin and Schwarz [17] do not impose a restriction of the form $a > a_*$ and hence argue that the decaying mode has to be excluded [see the remark following their Eq. (4.10)].

differential equation (36b) for ${}^{(2)}B_{\rm v}$ is essentially the same as Eq. (32b) for ${}^{(1)}B_{\rm v}$, with the spatial function ${}^{(1)}C$ on the right side replaced by the spatial function ${}^{(2)}C - 2\mathbb{D}_0({}^{(1)}C)$. It follows immediately on taking note of Eq. (35) that the solution for ${}^{(2)}B_{\rm v}$ is

$$\mathcal{H}{}^{(2)}B_{\rm v} \approx (1-g)({}^{(2)}C - 2\mathbb{D}_0({}^{(1)}C)) + \frac{\mathcal{H}}{a^2}{}^{(2)}C_*, \qquad (37c)$$

where ${}^{(2)}C_*$ represents the decaying mode at second order. Equation (37) gives the general solution at second order, including the decaying mode, in the total matter gauge. If ${}^{(2)}C_* \neq 0$ a restriction of the form $a > a_* > 0$ is again needed.

B. Transforming to the uniform curvature gauge

The link with the uniform curvature gauge at first order is provided by the following change of gauge formulas UW1 [1]:

$$\mathcal{H}V_{\rm c} = -\psi_{\rm v}, \qquad \mathcal{H}B_{\rm c} = \mathcal{H}B_{\rm v} - \psi_{\rm v}, \qquad (38a)$$

where we are dropping the superscript ${}^{(1)}$ on the linear solution. We also need the density and velocity constraints (20) at first order which read

$$\mathcal{H}V_{\rm c} = -\chi_{\rm c}, \qquad \boldsymbol{\delta}_{\rm c} \approx 3\mathcal{H}V_{\rm c}. \qquad (38b)$$

It follows from (33b) and (35) using (38) that

$$\chi_{\rm c} \approx C, \qquad \mathcal{H}V_{\rm c} \approx -C,$$
$$\boldsymbol{\delta}_{\rm c} \approx -3C, \qquad \mathcal{H}B_{\rm c} \approx -gC + \frac{\mathcal{H}}{a^2}C_*, \qquad (39a)$$

while by (2) we obtain

$$\phi_{\rm c} = (1+q)\chi_{\rm c} \approx (1+q)C, \qquad (39b)$$

which give the linear perturbations in the uniform curvature gauge.

The link with the uniform curvature gauge at second order is provided by the following change of gauge formulas:

$$\mathcal{H}{}^{(2)}\hat{V}_{\rm c} \approx -{}^{(2)}\hat{\psi}_{\rm v}, \qquad (40a)$$

$$\mathcal{H}{}^{(2)}\hat{B}_{\rm c} \approx \mathcal{H}{}^{(2)}\hat{B}_{\rm v} - {}^{(2)}\hat{\psi}_{\rm v} + 2\partial_N(\mathcal{H}B_{\rm v})\psi_{\rm v} - \mathcal{H}B_{\rm rem,v,c}, \qquad (40b)$$

given by Eqs. (C2a) and (C3) in Appendix C. We also need the density and velocity constraints (23) which read

$$\mathcal{H}{}^{(2)}\hat{V}_{\rm c} \approx -{}^{(2)}\hat{\chi}_{\rm c}, \qquad {}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c} \approx -3{}^{(2)}\hat{\chi}_{\rm c}. \qquad (41)$$

It immediately follows from (37b), (40a) and (41) that

$${}^{(2)}\hat{\chi}_{\rm c} \approx {}^{(2)}C, \quad \mathcal{H}{}^{(2)}\hat{V}_{\rm c} \approx -{}^{(2)}C, \quad {}^{(2)}\hat{\boldsymbol{\delta}}_{\rm c} \approx -3{}^{(2)}C, \qquad (42a)$$

where the spatial function ${}^{(2)}C(x^i)$ is the conserved quantity at second order. The metric perturbation ${}^{(2)}\phi_{\rm c}$ is determined by first finding ${}^{(2)}\chi_{\rm c}$ using (22) and then using the definition (2), which leads to

$${}^{(2)}\phi_{\rm c} \approx (1+q)({}^{(2)}C + (2(1+q) + 3(w - c_s^2))C^2). \qquad (42b)$$

Note that the decaying mode does not enter into the expressions (42a) and (42b).

We finally use (40b) in conjunction with (37b) and (37c) and the definitions of the hatted variables (8) to obtain an expression for $\mathcal{H}{}^{(2)}B_{\rm c}$. This necessitates using the first order solution that is given by (33), (35), (39) and (43) to evaluate the complicated source term $\mathcal{H}B_{\rm rem,v,c}$ given by Eq. (C3b) in Appendix C. At this stage, in the interests of simplicity, we drop the decaying mode. The final result is

$$\mathcal{H}{}^{(2)}B_{\rm c} \approx -g{}^{(2)}C + (g - (1+q)(g+1))C^2 + 2(q-1)g\mathbb{D}_0(C). \qquad (42c)$$

In summary Eq. (42) gives the solution at second order in the uniform curvature gauge, with the decaying mode set to zero in (42c). If needed the decaying mode terms can be worked out without difficulty.

C. Transforming to the Poisson gauge

It turns out that the super-horizon solution has its most complicated form when expressed in the Poisson gauge. At first order the link with the Poisson gauge is provided by the following change of gauge formulas (UW1 [1], Sec. III):

$$\psi_{\rm p} = \psi_{\rm v} - \mathcal{H}B_{\rm v}, \qquad \mathcal{H}V_{\rm p} = -\mathcal{H}B_{\rm v}, \qquad \boldsymbol{\delta}_{\rm p} = \boldsymbol{\delta}_{\rm v} - 3\mathcal{H}B_{\rm v}, \qquad (43a)$$

and the perturbed Einstein equations give

$$\phi_{\rm p} = \psi_{\rm p}. \qquad (43b)$$

It follows from (33) and (35) using (43) that

$$\psi_{\rm p} \approx gC - \frac{\mathcal{H}}{a^2}C_*, \qquad \mathcal{H}V_{\rm p} \approx -(1-g)C - \frac{\mathcal{H}}{a^2}C_*,$$
$$\boldsymbol{\delta}_{\rm p} \approx -3(1-g)C - 3\frac{\mathcal{H}}{a^2}C_*, \qquad (44)$$

which give the linear perturbations in the Poisson gauge.

The link with the Poisson gauge at second order is obtained by generalizing the change of gauge formulas (43a) to second order, as in Eqs. (C4), (C2b) and

(C2c). We use (C4) to first calculate ${}^{(2)}\hat{\psi}_{\rm p}$ in terms of ${}^{(2)}\hat{\psi}_{\rm v}$ and ${}^{(2)}\hat{B}_{\rm v}$, and then set $\phi_{\rm v} = 0$ in (C2b) and (C2c) to get $\mathcal{H}^{(2)}\hat{V}_{\rm p} \approx {}^{(2)}\hat{\psi}_{\rm p} - {}^{(2)}C$ and ${}^{(2)}\hat{\boldsymbol{\delta}}_{\rm p} \approx 3\mathcal{H}^{(2)}\hat{V}_{\rm p}$. The only use of the perturbed Einstein equations is to relate ${}^{(2)}\hat{\phi}_{\rm p}$ to ${}^{(2)}\hat{\psi}_{\rm p}$ as in Eq. (C5). The results for the unhatted variables, obtained using (8), are as follows:

$$ {}^{(2)}\psi_{\rm p} \approx g^{(2)}C + ((1+q)(1-g)^2 - g^2 - g)C^2 + 4g(1-g)\mathbb{D}_0(C), \tag{45a} $$

$$ {}^{(2)}\phi_{\rm p} \approx {}^{(2)}\psi_{\rm p} + 4g^2C^2 - 4((1+q)(1-g)^2 + g^2)\mathbb{D}_0(C), \tag{45b} $$

$$ \mathcal{H}^{(2)}V_{\rm p} \approx -(1-g)^{(2)}C - g(1-g)(C^2 - 4\mathbb{D}_0(C)), \tag{45c} $$

$$ {}^{(2)}\boldsymbol{\delta}_{\rm p} \approx 3\mathcal{H}^{(2)}V_{\rm p} + 3[3(1+c_s^2) + (1+q)](1-g)^2C^2, \tag{45d} $$

with the decaying mode set to zero (${}^{(r)}C_* = 0$, $r = 1, 2$) in the interest of simplicity. Note that the decaying mode would appear in each of these expressions.

VI. APPLICATIONS

The solution of the governing equations for adiabatic long-wavelength perturbations given in Sec. V A using the total matter gauge [see Eq. (37)] is general in the sense that it is valid for any stress-energy tensor of the form (1) (zero anisotropic stress and heat flux), and also includes the decaying mode. The spatial dependence of the solution is determined by four spatial functions, the two functions ${}^{(1)}C$ and ${}^{(2)}C$, which determine the growing mode and represent the conserved quantities, and the two functions ${}^{(1)}C_*$ and ${}^{(2)}C_*$, which determine the decaying mode. The dependence in time of the growing mode at first and second order is determined solely by the perturbation growth function $g(a)$. Indeed the solution as derived in the total matter gauge has a remarkably simple form. In the uniform curvature gauge [see Eq. (42)] and Poisson gauge [see Eq. (45)], however, the perturbations at second order also depend on the matter variables w and c_s^2.

Before giving some examples we briefly digress to relate the arbitrary functions ${}^{(r)}C$, $r = 1, 2$, to the usual conserved quantities ${}^{(r)}\zeta \equiv -{}^{(r)}\psi_\rho$ and ${}^{(r)}\mathcal{R} \equiv {}^{(r)}\psi_{\rm v}$, which are approximately equal but opposite in sign for adiabatic perturbations in the super horizon regime. In our derivation of the solutions we introduced ${}^{(1)}C$ as ${}^{(1)}\psi_{\rm v}$, and ${}^{(2)}C$ as ${}^{(2)}\hat{\psi}_{\rm v}$. It follows that

$$ {}^{(1)}C \equiv {}^{(1)}\mathcal{R} \approx -{}^{(1)}\zeta, \tag{46a} $$

$$ {}^{(2)}C \equiv {}^{(2)}\mathcal{R} + 2^{(1)}\mathcal{R}^2 \approx -({}^{(2)}\zeta - 2^{(1)}\zeta^2), \tag{46b} $$

since ${}^{(2)}\hat{\psi} = {}^{(2)}\psi + 2^{(1)}\psi^2$. We mention that in inflationary cosmology it is customary to parametrize the primordial non-Gaussianity level in terms of the conserved curvature perturbation ζ according to

$$ {}^{(2)}\zeta = 2a_{\rm NL}{}^{(1)}\zeta^2, \tag{47} $$

where the parameter $a_{\rm NL}$ depends on the physics of the type of inflation [see, e.g., Bartolo *et al.* [18], Eq. (38)]. In terms of our conserved quantity C the relation (47) reads

$$ {}^{(2)}C = 2(1 - a_{\rm NL})^{(1)}C^2. \tag{48} $$

For standard single field inflation $a_{\rm NL} \approx 1$ and hence ${}^{(2)}C = 0$.

The general solution that we derived in Sec. V A applies to the case of a perfect fluid with a barotropic equation of state $p = p(\rho)$ since then the adiabaticity conditions ${}^{(r)}\Gamma = 0$, $r = 1, 2$, are satisfied. In this case the scalars w and c_s^2 are determined by the equation of state. In the special case of a linear equation of state $p = w\rho$ with w constant and $w > -\frac{5}{3}$, it follows that $q > -2$ is constant and integrating $a\partial_a\mathcal{H} = -q\mathcal{H}$ gives

$$ \mathcal{H}(a) = \mathcal{H}_0(a/a_0)^{-q}, \tag{49} $$

where $\mathcal{H}_0 = \mathcal{H}(a_0)$, where a_0 is a fixed reference epoch.[17] On substituting this expression in the definition (3) of the perturbation growth function $g(a)$ we obtain

$$ g(a) = \frac{1+q}{2+q} = \frac{3(1+w)}{5+3w}, \tag{50} $$

i.e., $g(a)$ is constant. Note that $g(a) = \frac{3}{5}$ for dust and $g(a) = \frac{2}{3}$ for radiation. In this case the solution in the Poisson gauge given by (45a) and (45b) simplifies considerably, resulting in

$$ {}^{(2)}\hat{\psi}_{\rm p} \approx g^{(2)}C + 4g(1-g)\mathbb{D}_0({}^{(1)}C), \tag{51a} $$

$$ {}^{(2)}\hat{\phi}_{\rm p} \approx g^{(2)}C - 4g^2\mathbb{D}_0({}^{(1)}C). \tag{51b} $$

The general solution also applies to long-wavelength perturbations in a two-fluid universe with the matter described as a single fluid with barotropic equation of state, so that the perturbations are adiabatic. The two fluids are assumed to be noninteracting, each with a linear equation of state, with parameters w_1, w_2 satisfying $w_2 < w_1$. Two cases of particular interest are the

[17] Here and in the rest of this section we are temporarily suspending our convention of using $_0$ to denote a background quantity and are instead using it to refer to the value of some quantity at a fixed reference epoch denoted by a_0.

radiation-matter universe with $w_2 = 0$, $w_1 = \frac{1}{3}$ and the ΛCDM universe with $w_2 = -1$, $w_1 = 0$. The former case arises when deriving an expression for the second-order early integrated Sachs-Wolfe effect in the anisotropy of the CMB on large scales ([9] Eqs. (3.9)–(3.10), Sec. III C, and Appendix C.)

In order to calculate $g(a)$ we need an expression for $\mathcal{H}(a)$. Conservation of energy for each fluid leads to $\rho_A/\rho_{A,0} = x^{-3(1+w_A)}$, $x = a/a_0$, $A = 1, 2$, where ρ_A, $A = 1, 2$ are the background densities of the fluids and $\rho_{A,0} = \rho_A(a_0)$. It follows that the individual density parameters $\Omega_A = \rho_A/(3H^2)$, $A = 1, 2$ are given by

$$\Omega_A = \Omega_{A,0} x^{-(1+3w_A)} \left(\frac{\mathcal{H}_0}{\mathcal{H}}\right)^2, \tag{52}$$

where $\Omega_{A,0} = \rho_{A,0}/(3H_0)^2$, $A = 1, 2$. Since the background is flat, we have $\Omega_1 + \Omega_2 = 1$ and (52) leads to

$$\left(\frac{\mathcal{H}}{\mathcal{H}_0}\right)^2 = \Omega_{1,0} x^{-(1+3w_1)} + \Omega_{2,0} x^{-(1+3w_2)}, \qquad x = a/a_0, \tag{53}$$

where $\Omega_{1,0} + \Omega_{2,0} = 1$. We can now substitute (53) in (3) to obtain an explicit expression for $g(a)$ which determines all the first order perturbations, and in the case of the total matter gauge, also the second-order perturbations. The matter parameters w and c_s^2 for the combined fluid are given by

$$w = w_1\Omega_1 + w_2\Omega_2, \quad c_s^2 = \frac{w_1(1+w_1)\Omega_1 + w_2(1+w_2)\Omega_2}{1+w}, \tag{54}$$

where the Ω_A are given by (52). As an example the curvature perturbation ψ_p in the Poisson gauge is given by Eqs. (45a) and (8b):

$$^{(1)}\psi_\mathrm{p} \approx g^{(1)}C, \tag{55a}$$

$$^{(2)}\psi_\mathrm{p} \approx g^{(2)}C + \left(\frac{3}{2}(1+w)(1-g)^2 - g^2 - g\right){}^{(1)}C^2 + 4g(1-g)\mathbb{D}_0({}^{(1)}C). \tag{55b}$$

At second order the leading order term is determined by g alone while the source terms depend also on w.

For all values of w_1 and w_2 it has been shown by Hu and Eisenstein [19] that the integral in (3) that determines g for these two-fluid models can be expressed in terms of the incomplete beta function, and that if $(5 + 3w_1)/3(w_1 - w_2)$ is an integer then $g(a)$ can be expressed in elementary form (see page 12 in [19]). We now consider a radiation-matter universe ($w_1 = \frac{1}{3}$, $w_2 = 0$), which satisfies this condition.

In this case it is convenient to choose $a_0 = a_\mathrm{eq}$, the epoch of matter-radiation equality. It follows that $\Omega_{1,0} = \Omega_{2,0} = \frac{1}{2}$, and (53) simplifies to give

$$\mathcal{H}(a) = \mathcal{H}_\mathrm{eq}\frac{\sqrt{x+1}}{\sqrt{2}x}, \qquad x = a/a_\mathrm{eq}. \tag{56}$$

It is a simple matter to evaluate the integral (3) for $g(a)$ to obtain

$$g(a) = \frac{1}{15}x^{-3}(9x^3 + 2x^2 - 8x - 16 + 16\sqrt{1+x}), \quad x = a/a_\mathrm{eq}. \tag{57a}$$

In addition (53) and (52) lead directly to

$$w = \frac{1}{3(1+x)} \qquad 3c_s^2 = \frac{4}{3x+4}. \tag{57b}$$

As expected it follows that $\lim_{a\to 0} g(a) = \frac{2}{3}$ (radiation) and $\lim_{a\to\infty} g(a) = \frac{3}{5}$ (pressure-free matter). The curvature perturbation ψ_p in the Poisson gauge, given by (55), can now be calculated using (57). The first order expression has been given, e.g., by Hu and Eisenstein [19] [see Eq. (67)].[18] To the best of our knowledge the second-order expression is new.[19]

The second special case of importance is the perturbed ΛCDM universe given by $w_2 = -1$, $w_1 = 0$. It follows from (53) that

$$\mathcal{H}^2 = \mathcal{H}_0^2(\Omega_{m,0}x^{-1} + \Omega_{\Lambda,0}x^2) \qquad x = a/a_0, \tag{58}$$

which when substituted into (3) gives $g(a)$ for the ΛCDM universe.[20] From (52) and (54) we obtain

$$1 + w = \Omega_m = \Omega_{m,0}x^{-1}\left(\frac{\mathcal{H}_0}{\mathcal{H}}\right)^2. \tag{59}$$

With these expressions one can use (55) to calculate the long-wavelength curvature perturbation ψ_p in the Poisson gauge, and any other perturbations for the ΛCDM universe using the results of Sec. V. In this case, however, one can do more: since $c_s^2 = 0$ and $\Gamma = 0$ for the perturbed ΛCDM universe the full (i.e., nontruncated) equations (14) at linear

[18] This expression for $g(a)$ has also been given by Kodama and Sasaki [20] [see Eqs. (IV.4.11) and (IV.4.14) with $z = 1 + x$], Dodelson [21] [see Eq. (7.32), up to a constant multiplicative factor], and Bartolo *et al.* [9] [see Eq. (5.19)]. Mukhanov [22] gives an expression for $g(\eta)$, see Eq. (7.71).

[19] An expression for $^{(2)}\psi_\mathrm{p}$ for the radiation-matter universe has been given by Bartolo *et al.* [9] [see Eq. (3.48)], but the source term was left as a complicated integral.

[20] We note that the function $g(a)$ for ΛCDM can be represented in different ways and has been studied extensively, as described in Sec. VII [see Eqs. (74) and (78)].

order in the total matter gauge can be solved explicitly as in the super-horizon case, giving the *exact* expressions

$$\psi_{\rm v} = {}^{(1)}C, \qquad \phi_{\rm v} = 0, \qquad \mathcal{H}B_{\rm v} = (1-g)^{(1)}C, \tag{60}$$

where ${}^{(1)}C$ is the conserved quantity. The new feature is the exact expression for the density perturbation which we can calculate using

$$\boldsymbol{\delta}_{\rm v} = \frac{2}{3}(1+w)^{-1}\mathcal{H}^{-2}\mathbf{D}^2(\psi_{\rm v} - \mathcal{H}B_{\rm v}). \tag{61}$$

It follows from (59) and (60) that

$$\boldsymbol{\delta}_{\rm v} = \frac{2}{3}m^{-2}xg\mathbf{D}^{2\,(1)}C, \qquad x = a/a_0, \tag{62}$$

where m^2 is a constant given by $m^2 = \mathcal{H}_0^2\Omega_{m,0}$.

Furthermore, the full (nontruncated) equations (15) at second order in the total matter gauge can likewise be solved explicitly, and one finds that the evolution of the perturbations ${}^{(2)}\psi_{\rm v}$, ${}^{(2)}\phi_{\rm v}$, $\mathcal{H}{}^{(2)}B_{\rm v}$ and ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ is again determined by $g(a)$, partly algebraically and partly through an integral involving $g(a)$. We will give details elsewhere. We note, however, that the density perturbation ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ has been previously determined in an indirect way and this expression shows the role played by $g(a)$ [see Uggla and Wainwright [23], Eqs. (10), (13) and (16)]. Our simple method of integration using the total matter gauge confirms the earlier result.

VII. THE PERTURBATION EVOLUTION FUNCTION

The function $g(a)$ is defined by Eq. (3), which we repeat here:

$$g(a) = 1 - \frac{\mathcal{H}}{a^2}\int_0^a \frac{\bar{a}}{\mathcal{H}(\bar{a})}d\bar{a}. \tag{63}$$

This function first emerged in this paper when we solved the governing equations in the total matter gauge at first order to obtain the metric perturbation $B_{\rm v}$. We subsequently showed that it determines the evolution of the perturbations at first order in all the standard gauges. In particular, in the Poisson gauge which plays an important role in applications, g determines the growing mode of the curvature perturbation $\psi_{\rm p}$ at first order in the long-wavelength limit according to[21]

$$\psi_{\rm p}/\mathcal{R} \approx g. \tag{64}$$

In other words g represents the growth of the nonconserved Poisson curvature perturbation $\psi_{\rm p}$ relative to the conserved comoving curvature perturbation $\mathcal{R}$. We note that the ratio $\psi_{\rm p}/\mathcal{R}$ has been emphasized by Hu and Eisenstein [19], who derived the following expression for long-wavelength adiabatic perturbations with negligible anisotropic stress:[22]

$$\psi_{\rm p}/\mathcal{R} \approx 1 - \frac{\sqrt{\rho}}{a}\int_0^a \frac{d\bar{a}}{\sqrt{\rho(\bar{a})}}, \tag{65}$$

where ρ denotes the background matter density. The relation $\rho = 3H^2$, valid in a flat background, shows that the integral in (65) is equal to the integral in (63).

We now derive some properties of g, first noting that g can also be expressed as a function of t or of η by making a change of variable in the integral, leading to

$$g(t) = 1 - \frac{H}{a}\int_0^t a(\bar{t})d\bar{t}, \qquad g(\eta) = 1 - \frac{\mathcal{H}}{a^2}\int_0^\eta a(\bar{\eta})^2 d\bar{\eta}. \tag{66}$$

The initial singularity is given by $a = 0$, with the clock time translated so that $t = 0$ when $a = 0$. We assume that $H > 0$ and that $q > -2$ for all $t > 0$. It follows from the first of Eq. (66) that $g(t) < 1$ for $t > 0$.

As regards asymptotic behavior, if $H/a \to \infty$, $q \to q_{\rm sing}$ as $t \to 0$ and $H/a \to 0$, $q \to q_\infty$ as $t \to \infty$, where $q_{\rm sing}, q_\infty > -2$, then it follows from the first of Eq. (66) that[23]

$$\lim_{t\to 0} g(t) = \frac{1+q_{\rm sing}}{2+q_{\rm sing}} = \frac{3(1+w_{\rm sing})}{5+3w_{\rm sing}}, \qquad \lim_{t\to\infty} g(t) = \frac{1+q_\infty}{2+q_\infty} = \frac{3(1+w_\infty)}{5+3w_\infty}, \tag{67}$$

are finite. By integrating the identity

$$\partial_t\left(\frac{a}{H}\right) - a = a(1+q), \tag{68}$$

we can write $g(t)$ in the alternate form

$$g(t) = \frac{H}{a}\int_0^t a(\bar{t})(1+q(\bar{t}))d\bar{t}, \tag{69}$$

which implies that if $1+q > 0$ then $g(t) > 0$ for $t > 0$. A final property that follows from (63) is

$$\partial_a(ag) = (1+q)(1-g). \tag{70}$$

Thus if $q > -1$ then $ag(a)$ is an increasing function.

Since 1985 the integrals that appear in the expressions (63) and (66) for the function g have appeared in many papers on linear perturbation theory, usually giving the Bardeen potential $\psi_{\rm p}$ for adiabatic long-wavelength

[21]This follows from (33b) and (44), noting that $\psi_{\rm v} = \mathcal{R}$.

[22]See Eq. (59) in [19], dropping the decaying mode, neglecting the second term and noting that Φ and ζ correspond to our $\psi_{\rm p}$ and $\mathcal{R}$.

[23]Write $g(t) = 1 - \frac{\int_0^t a(\bar{t})d\bar{t}}{a/H}$ and apply l'Hôpital's rule to the indeterminate ratio using (68).

perturbations. However, a notation for the function g has not been introduced. We have already mentioned that Hu and Eisenstein [19] effectively introduced the integral expression for $g(a)$ in this context. In order to relate our function g to other work we consider our expression (44) for $\psi_{\rm p}$ for adiabatic long-wavelength perturbations, which we write here using t as follows:[24]

$$\psi_{\rm p}(t) \approx Cg(t) - C_* \frac{H}{a} = C\left(1 - \frac{H}{a}\int_0^t a(\bar{t})d\bar{t}\right) - C_* \frac{H}{a}. \tag{71}$$

Here C and C_* are arbitrary spatial functions. The solution with $C_* = 0$ is the growing mode, and is the unique solution which is bounded as $a \to 0$. The solution with $C = 0$ is the decaying mode and is unbounded as $a \to 0$.

If $C \neq 0$ then one can incorporate C_* into the lower bound of the integral as follows:

$$\psi_{\rm p}(t) \approx C\left(1 - \frac{H}{a}\int_{t_*}^t a(\bar{t})d\bar{t}\right), \tag{72}$$

where t_* is a spatial function. This is the form in which the expression for $\psi_{\rm p}$ is usually given in the literature. In some references the expression (72) is derived by assuming a particular matter content, e.g., a perfect fluid with an arbitrary equation of state [Hwang [25], see Eq. (55), Mukhanov [22], see Eq. (7.69)] or a minimally coupled scalar field [Mukhanov [26], Eq. (13), Mukhanov *et al.* [27], see Eq. (6.56), Hwang [16], see Eq. (94)]. It is known, however, that one can derive (71) or (72) without specifying the matter content in detail, as we have done. We refer to Hu and Eisenstein [19], Eq. (59), Bertschinger [28], Eq. (24) with (10) and (11), noting that his κ corresponds to our $\mathcal{R}$, and Weinberg [29], Eqs. (5.4.16) and (5.4.20). We note that these authors identify the arbitrary function C in (71) with the comoving curvature perturbation $\mathcal{R}$, thereby completing the solution.

We showed in Sec. VI that the function g as defined by (63) or (69) also arises in a perturbed ΛCDM cosmology, in which case it describes the perturbations exactly and on all scales. In this context, however, it was introduced in a completely different way, namely, by finding the function $D(a)$, called the growth factor, that is the appropriately normalized growing solution of the evolution equation for the linear density perturbation:

$$\left(\partial_\eta{}^2 + \mathcal{H}\partial_\eta - \frac{3}{2}\Omega_m\mathcal{H}^2\right)\boldsymbol{\delta}_{\rm v} = 0. \tag{73}$$

[24]Several authors have used the expression (69) with $1+q = -\dot{H}/H^2$ for $g(t)$ in (71), e.g., Martin and Schwarz [17], Eq. (4.26) and Malik and Wands [24] Eq. (3.38).

This function has the following integral expression:[25]

$$D(a) = \frac{5}{2}\mathcal{H}_0^2\Omega_{m,0}\frac{\mathcal{H}}{a}\int_0^a \frac{1}{\mathcal{H}(\bar{a})^3}d\bar{a}, \tag{74}$$

where $\mathcal{H}^2$ is given by (58). The numerical factor $\frac{5}{2}$ was determined by requiring that

$$\lim_{a\to 0}\left(\frac{D(a)}{a/a_0}\right) = 1. \tag{75}$$

We now relate $D(a)$ to $g(a)$. We begin by writing the general expression (69) for $g(t)$ in terms of a, obtaining:

$$g(a) = \frac{\mathcal{H}}{a^2}\int_0^a \frac{\bar{a}(1+q(\bar{a}))}{\mathcal{H}(\bar{a})}d\bar{a}. \tag{76}$$

In a ΛCDM universe it follows from (59) using $1+q = \frac{3}{2}(1+w)$ that

$$(a/a_0)(1+q) = \frac{3}{2}\mathcal{H}_0^2\Omega_{m,0}\mathcal{H}^{-2}. \tag{77}$$

We now specialize the expression (76) to the ΛCDM universe by substituting (77). On comparing the result with (74) we obtain

$$g(a) = \frac{3}{5}\left(\frac{D(a)}{a/a_0}\right). \tag{78}$$

In the ΛCDM context the function g was first defined in terms of D in this way, i.e., $g(a)$ is proportional to $D(a)/a$. The function g then determines the Bardeen potential according to $\psi_{\rm p} = g(a)\psi_0(x^i)$ where $\psi_0(x^i)$ is an arbitrary spatial function. See, e.g, Bartolo *et al.* [9] [in the text following Eq. (2.3)] and Villa and Rampf [32] [in the text following Eq. (5.12)]. The factor $\frac{3}{5}$ in (78) implies that $\psi_0 = \mathcal{R}$. In the above references this factor is omitted, which implies that $\psi_0 = \frac{3}{5}\mathcal{R}$.

VIII. DISCUSSION

In this paper we have considered scalar perturbations of flat FL cosmologies up to second order, subject to the assumption that at first order the vector and tensor modes are zero. The metric perturbations are described by the spatially gauge fixed variables $\phi, \psi, \mathcal{H}B$. The perturbations of the stress-energy tensor, which is assumed to have zero anisotropic stresses and zero heat flux, are described by the

[25]See Eisenstein [30], Eqs. (3) and (4). Note that his a and H correspond to our a/a_0 and H/H_0. This result was first given by Heath [31] using unfamiliar notation. See also Villa and Rampf [32] Eqs. (5.7) and (5.12)–(5.13), where their a corresponds to our a/a_0. Matsubara [33] gives a different representation of D, see Eqs. (8) and (10).

variables $\boldsymbol{\delta}$, $\mathcal{H}V$, Γ. The background stress-energy tensor is characterized by the scalars w, c_s^2 and the background dynamics by $\mathcal{H}$, q, where $1+q=\frac{3}{2}(1+w)$.

Within this framework we have given for the first time the general explicit solution of the governing equations up to second order for adiabatic perturbations on super-horizon scales [see Eq. (37)]. We showed that in the total matter gauge the governing equations can be integrated very easily, leading to a solution that has a remarkably simple form: the three matter perturbations are zero and of the three metric perturbations, one is zero, one is constant in time and the remaining one has an increasing mode and a decreasing mode[26] with time dependence proportional to $1-g(a)$ and $\mathcal{H}/a^2$, respectively, at both first and second order. In other words, the perturbation evolution function $g(a)$, which is determined by the background dynamics through Eq. (3), completely determines the evolution of the growing mode up to second order for adiabatic perturbations on super-horizon scale. Going beyond the initial scope of this paper we showed in addition that the function $g(a)$ for ΛCDM determines the growing mode of perturbations of these models *on all scales* to second order.

Having derived the solutions using the total matter gauge we also obtained the solution in the uniform curvature gauge and the Poisson gauge by using the change of gauge formulas. There is an increasing complexity in the solution as one progresses to the uniform curvature gauge and then to the Poisson gauge, with the decaying mode adding significantly to the complexity. Moreover, in these gauges the background scalars w (or q) and c_s^2 also play a role in determining the evolution.

In a subsequent related paper [35] we consider second-order perturbations of a flat Friedmann-Lemaître universe whose stress-energy content is a single minimally coupled scalar field with an arbitrary potential. We apply the methods used in this paper to derive the general solution of the perturbed Einstein equations in explicit form for this class of models when the perturbations are in the super-horizon regime. As a by-product we obtain a new conserved quantity for long-wavelength perturbations of a single scalar field at second order.

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APPENDIX A: GOVERNING EQUATIONS IN THE UNIFORM CURVATURE GAUGE

On super-horizon scale the governing equations in the uniform curvature gauge simplify significantly: the source terms are independent of $B_{\rm c}$ and hence the evolution equation for $B_{\rm c}$ decouples from the other equations. This equation will, however, not be needed in this paper. The remaining equations, assuming that the perturbations are adiabatic (${}^{(r)}\Gamma \approx 0,\ r=1,\ 2$), have the following form (specialize the equations in UW2 [2], Sec. V B 1):

$$(1+q)\partial_N((1+q)^{-1\,(1)}\phi_{\rm c}) \approx 0, \tag{A1a}$$

$$\mathcal{H}^{(1)}V_{\rm c} = -(1+q)^{-1\,(1)}\phi_{\rm c}, \tag{A1b}$$

$${}^{(1)}\boldsymbol{\delta}_{\rm c} \approx 3\mathcal{H}^{(1)}V_{\rm c}, \tag{A1c}$$

while at second order we obtain

$$(1+q)\partial_N((1+q)^{-1\,(2)}\phi_{\rm c}) \approx -\frac{1}{2}\mathbb{S}_{\rm c}^{\Gamma}, \tag{A2a}$$

$$\mathcal{H}^{(2)}V_{\rm c} \approx -(1+q)^{-1}\left({}^{(2)}\phi_{\rm c} - \frac{1}{2}\mathbb{S}_{\rm c}^{q}\right), \tag{A2b}$$

$${}^{(2)}\boldsymbol{\delta}_{\rm c} \approx 3\mathcal{H}^{(2)}V_{\rm c} + \frac{1}{2}(1+q)^{-1}(\mathbb{S}_{\rm c}^{\rho} - 3\mathbb{S}_{\rm c}^{q}). \tag{A2c}$$

The source terms with kernel $\mathbb{S}_{\rm c}$ are given by (see UW2 [2], Sec. V B 1)

$$\mathbb{S}_{\rm c} = \mathbb{G}_{\rm c} - 3(1+w)\mathbb{T}_{\rm c}, \tag{A3}$$

where the Einstein tensor source terms are

$$\mathbb{G}_{\rm c}^{\Gamma} \approx -8\mathcal{L}_1\phi_{\rm c}^2 = -8(1+q)\partial_N((1+q)^{-1}\phi_{\rm c}^2),$$
$$\mathbb{G}_{\rm c}^{q} \approx 8\phi_{\rm c}^2, \qquad \mathbb{G}_{\rm c}^{\rho} \approx 24\phi_{\rm c}^2, \tag{A4}$$

[see Eq. (34a) in UW2 [2] for the definition of the differential operator $\mathcal{L}_1$] and the stress-energy source terms are

$$\mathbb{T}_{\rm c}^{\Gamma} \approx -\frac{1}{3}(\partial_N c_s^2)\boldsymbol{\delta}_{\rm c}^2, \tag{A5a}$$

$$\mathbb{T}_{\rm c}^{q} \approx 2\mathcal{S}^i[((1+c_s^2)\boldsymbol{\delta}_{\rm c} - \phi_{\rm c})\mathbf{D}_i(\mathcal{H}V_{\rm c})], \tag{A5b}$$

$$\mathbb{T}_{\rm c}^{\rho} \approx 0. \tag{A5c}$$

The scalar mode extraction operator $\mathcal{S}^i$ in (A5b) is given by $\mathcal{S}^i = \mathbf{D}^{-2}\mathbf{D}^i$, where $\mathbf{D}^{-2}$ is the inverse spatial Laplacian.

Here and elsewhere in this Appendix, in order to simplify the notation we have dropped the superscript ${}^{(1)}$ on the linear perturbations in the source terms.

APPENDIX B: THE DENSITY PERTURBATION CONSTRAINT

We restrict the general expression for the density perturbations ${}^{(r)}\boldsymbol{\delta}$, $r=1,\ 2$, valid in any temporal gauge, given in UW2 [2] [see Eq. (40)] to super-horizon scales:

[26] In cosmological perturbation theory at second order the decaying mode is usually set to zero. One exception is Christopherson *et al.* [34].

$$ {}^{(1)}\boldsymbol{\delta} \approx 3\mathcal{H}^{(1)}V, \tag{B1a} $$

$$ {}^{(2)}\boldsymbol{\delta} \approx 3\mathcal{H}^{(2)}V + \mathbb{S}^{\rho} - 3\mathbb{S}^{q}, \tag{B1b} $$

where

$$ \mathbb{S}^{\rho} = \mathbb{G}^{\rho} - 3(1+w)\mathbb{T}^{\rho}, \quad \mathbb{S}^{q} = \mathbb{G}^{q} - 3(1+w)\mathbb{T}^{q}. \tag{B1c} $$

On specializing the source terms $\mathbb{G}$ and $\mathbb{T}$ to super-horizon scales and using the equation $(1+q)\mathcal{H}^{(1)}V = -(\partial_N {}^{(1)}\psi + {}^{(1)}\phi)$ we obtain

$$ \mathbb{S}^{\rho} - 3\mathbb{S}^{q} \approx 3(1+q)(\mathcal{H}V)^2 + (1+c_s^2)\boldsymbol{\delta}^2 + 6\mathcal{S}^i(\Gamma \mathbf{D}_i(\mathcal{H}V)). \tag{B2} $$

On introducing the hatted variables as defined by Eq. (8), Eq. (B1b) assumes the concise form

$$ {}^{(2)}\hat{\boldsymbol{\delta}} \approx 3\mathcal{H}^{(2)}\hat{V} + 6\mathcal{S}^i(\Gamma \mathbf{D}_i(\mathcal{H}V)), \tag{B3} $$

valid for any temporal gauge, where $\mathcal{S}^i$ is defined following (A5) in Appendix A.

APPENDIX C: CHANGE OF GAUGE FORMULAS

We require the following change of gauge formulas for long-wavelength perturbations that can be obtained from UW1 [1] (specialize the formulas at the end of Sec. III by dropping terms of order two or higher in $\mathbf{D}_i$):

$$ {}^{(2)}\hat{\square}_{\rm v} = {}^{(2)}\hat{\square} - \mathcal{H}^{(2)}\hat{V} + 2(\partial_N \square_{\rm v})\mathcal{H}V + \square_{\rm rem,v} + 2\mathcal{S}^i[\phi_{\rm v}(\mathbf{D}_i\mathcal{H}V)], \tag{C1a} $$

$$ {}^{(2)}\hat{\square}_{\rm p} = {}^{(2)}\hat{\square} - \mathcal{H}^{(2)}\hat{B} + 2(\partial_N \square_{\rm p})\mathcal{H}B + \square_{\rm rem,p} - \mathcal{H}B_{\rm rem,p}, \tag{C1b} $$

where the kernel $\square$ can be one of ψ, $\mathcal{H}B$, $\mathcal{H}V$ or $\frac{1}{3}\boldsymbol{\delta}$, and the gauge on the right side can be one of the standard choices. The quantities $\square_{\rm rem}$ are given by Eq. (32) in UW1 [1], and the operator $\mathcal{S}^i$ is defined following (A5) in Appendix A. Equation (C1a) can be specialized to give the following generalizations of some of the first order gauge formulas[27]:

$$ {}^{(2)}\hat{\psi}_{\rm v} \approx -\mathcal{H}^{(2)}\hat{V}_{\rm c} - 2\mathcal{S}^i[(\mathbf{D}_i\phi_{\rm v})\mathcal{H}V_{\rm c}], \tag{C2a} $$

$$ {}^{(2)}\hat{\psi}_{\rm v} \approx {}^{(2)}\hat{\psi}_{\rm p} - \mathcal{H}^{(2)}\hat{V}_{\rm p} - 2\mathcal{S}^i[(\mathbf{D}_i\phi_{\rm v})\mathcal{H}V_{\rm p}], \tag{C2b} $$

$$ {}^{(2)}\hat{\boldsymbol{\delta}}_{\rm p} \approx 3\mathcal{H}^{(2)}\hat{V}_{\rm p} - 6\mathcal{S}^i[\phi_{\rm v}(\mathbf{D}_i\mathcal{H}V_{\rm p})]. \tag{C2c} $$

These formulas simplify further and match the corresponding first order formulas if the perturbations are also adiabatic and the Einstein equations hold since then ${}^{(1)}\phi_{\rm v} \approx 0$.

Next choose $\square = \mathcal{H}B$ in (C1a) with the uniform curvature gauge on the right side. On using (C2a), the relation ${}^{(1)}\psi_{\rm v} = -\mathcal{H}^{(1)}V_{\rm c}$ and the first order solution (33) we obtain the following more complicated relation:

$$ \mathcal{H}^{(2)}\hat{B}_{\rm c} \approx \mathcal{H}^{(2)}\hat{B}_{\rm v} - {}^{(2)}\hat{\psi}_{\rm v} + 2\partial_N(\mathcal{H}B_{\rm v})\psi_{\rm v} - \mathcal{H}B_{\rm rem,v,c}, \tag{C3a} $$

where

$$ \mathcal{H}B_{\rm rem,v,c} \approx (\partial_N + 2q)(\mathbb{D}_0(\mathcal{H}B_{\rm v}) - \mathbb{D}_0(\mathcal{H}B_{\rm c})) + 2\mathcal{S}^i[(\phi_{\rm v}+\phi_{\rm p})\mathbf{D}_i\mathcal{H}B_{\rm v} - (\phi_{\rm c}+\phi_{\rm p})\mathbf{D}_i\mathcal{H}B_{\rm c}]. \tag{C3b} $$

We recall that the differential operator $\mathbb{D}_0$ is defined in (17). Next choose $\square = \psi$ in (C1b) and use the total matter gauge on the right side to obtain

$$ {}^{(2)}\hat{\psi}_{\rm p} \approx {}^{(2)}\hat{\psi}_{\rm v} - \mathcal{H}^{(2)}\hat{B}_{\rm v} + 2(\partial_N\psi_{\rm p})\mathcal{H}B_{\rm v} + (\partial_N + 2q)\mathbb{D}_0(\mathcal{H}B_{\rm v}) + 2\mathcal{S}^i[(\phi_{\rm p}+\phi_{\rm v})\mathbf{D}_i(\mathcal{H}B_{\rm v})]. \tag{C4} $$

In addition the perturbed Einstein equations in the Poisson gauge UW2 [2] [introduce hatted variables in Eq. (48b) in [2]] yield

$$ {}^{(2)}\hat{\phi}_{\rm p} \approx {}^{(2)}\hat{\psi}_{\rm p} - 4[\mathbb{D}_0(\psi_{\rm p}) + (1+q)\mathbb{D}_0(\mathcal{H}V_{\rm p})]. \tag{C5} $$

The source terms in Eqs. (C3) and (C4) can be evaluated using the first order solutions in Secs. V A–V C, and the derivative $\partial_N g = (1+q)(1-g) - g$ which follows from (70).

Finally we show that the uniform density gauge is equivalent to the total matter gauge on super-horizon scales to second order. This is a consequence of the relations (29), (B1a) and (B3), which imply that ${}^{(r)}V_{\rho} \approx 0$, $r = 1, 2$, and the fact that the metric perturbations $f = (\phi, \psi, B)$ satisfy ${}^{(r)}f_{\rho} \approx {}^{(r)}f_{\rm v}$ for $r = 1, 2$, on super-horizon scales when the perturbed Einstein equations at linear order hold, where the latter result follows from UW1 [1].[28]

[27]Choose $\square = \psi$ with first the uniform curvature gauge on the right side and then the Poisson gauge and use (14b) ($\partial_N\psi_{\rm v} = -\phi_{\rm v}$). Then choose $\square = \frac{1}{3}\boldsymbol{\delta}$ with the Poisson gauge on the right side and use (29) ($\boldsymbol{\delta}_{\rm v} \approx 0$).

[28]Choose the total matter gauge in Eq. (41d) which yields $\xi^N_{\rho,{\rm v}} \approx 0$, and then use (39a), (39b) and (40b).

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