New generic evolution for k-essence dark energy with $w \approx -1$

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We reexamine k-essence dark energy models with a scalar field ϕ and a factorized Lagrangian, $\mathcal{L} = V(\phi)F(X)$, with $X = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$. A value of the equation of state parameter, w, near -1 requires either $X \approx 0$ or $dF/dX \approx 0$. Previous work showed that thawing models with $X \approx 0$ evolve along a set of unique trajectories for $w(a)$, while those with $dF/dX \approx 0$ can result in a variety of different forms for $w(a)$. We show that if $dV/d\phi$ is small and $(1/V)(dV/d\phi)$ is roughly constant, then the latter models also converge toward a single unique set of behaviors for $w(a)$, different from those with $X \approx 0$. We derive the functional form for $w(a)$ in this case, determine the conditions on $V(\phi)$ for which it applies, and present observational constraints on this new class of models. We note that k-essence models with $dF/dX \approx 0$ correspond to a dark energy sound speed $c_s^2 \approx 0$.

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I. INTRODUCTION

Observational evidence [1–7] indicates that roughly 70% of the energy density in the Universe is in the form of a component called dark energy, which has negative pressure, and roughly 30% is in the form of nonrelativistic matter. The dark energy component can be parametrized in terms of its equation of state parameter, w, defined as the ratio of the dark energy pressure to its density,

$$w = p/\rho. \tag{1}$$

A cosmological constant, Λ, corresponds to the case $\rho = \text{constant}$ and $w = -1$.

While a model with a cosmological constant and cold dark matter (ΛCDM) is consistent with current observations, there are other models of dark energy that have a dynamical equation of state. The most widely investigated are quintessence models, with a time-dependent scalar field, ϕ, having potential $V(\phi)$ [8–14]. (See Ref. [15] for a review.)

While quintessence generically produces a time-varying value for w, a successful model must closely mimic ΛCDM in order to be consistent with current observations. Hence, a viable model should yield a present-day value of w close to -1. This fact has been exploited in a number of papers that explored the evolution of a scalar field subject to the constraint that w must be close to -1 [16–21]. By imposing this constraint, one can reduce an infinite number of models to a finite set of behaviors for $w(a)$.

In Ref. [22], this methodology was extended to k-essence models, which are characterized by a nonstandard kinetic term in the Lagrangian. Reference [22] found two sets of solutions that yield $w \approx -1$. The first corresponds to $\dot{\phi} \to 0$ (where a dot will refer throughout to the time derivative), and it yields a single set of behaviors for $w(a)$. The evolution of w in this case turns out to be identical to the quintessence models investigated in Refs. [17–19]. The second solution corresponds to $\dot{\phi} \to \text{constant}$. However, in the latter case, the solution is sensitive to the functional form for $V(\phi)$ and therefore fails to correspond to a single set of behaviors for $w(a)$.

In this paper, we revisit the second class of these solutions and show that, under some conditions on the potential $V(\phi)$, they do converge to a single unique set of trajectories for $w(a)$. Specifically, when $|(1/V)(dV/d\phi)|$ is small and nearly constant as ϕ evolves, then the evolution of $w(a)$ converges toward a single functional behavior. Furthermore, unlike the solutions derived in Ref. [22], the new class of solutions derived here correspond to behavior for $w(a)$ that differs from previously examined quintessence evolution.

In the next section, we briefly review previously derived results for quintessence and k-essence evolution for w near -1. In Sec. III, we present our new results for k-essence evolution, along with a discussion of the parameter ranges over which these solutions are valid. We discuss our results, including observational constraints, in Sec. IV.

II. PREVIOUS RESULTS

Before deriving our new results for k-essence, we need to present, for comparison, previously derived results for both quintessence and k-essence evolution. We assume a flat universe with the Hubble parameter given by

$$H = \left(\frac{\dot{a}}{a}\right) = \sqrt{\rho/3}. \tag{2}$$

Here a is the scale factor (with $a = 1$ at the present), ρ is the total density, and we work in units for which $8\pi G = 1$. At late times, the contribution of photons and neutrinos to the expansion can be neglected, so we take ρ to include only matter (dark matter plus baryons) with a density scaling as a^{-3}, and our unknown dark energy component, with a density which we assume to be approximately (but not exactly) constant.

A. Quintessence

In this section, we will assume that the dark energy is provided by a minimally coupled scalar field, ϕ, with an equation of motion given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{3}$$

Equation (3) indicates that the field rolls downhill in the potential $V(\phi)$, but its motion is damped by a term proportional to H.

The pressure and density of the scalar field are given by

$$p = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{4}$$

and

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{5}$$

respectively, and the equation of state parameter, w, is given by Eq. (1).

We will consider only "thawing" models, for which the scalar field is initially at rest ($\dot{\phi} = 0$, $w = -1$) and rolls downhill in the potential $V(\phi)$ so that w increases up to the present [23]. Then Ref. [16] considered potentials satisfying the inflationary slow-roll conditions, namely

$$\left(\frac{V'}{V}\right)^2 \ll 1, \tag{6}$$

and

$$\frac{V''}{V} \ll 1, \tag{7}$$

where the prime indicates throughout derivatives with respect to the scalar field, ϕ.

Note, however, that the solutions derived here differ markedly from the inflationary slow-roll solutions. In the latter case, H in Eq. (3) contains only the density of the scalar field itself, and a solution can be derived by setting $\ddot{\phi}$ in Eq. (3) equal to zero. When both the matter and scalar field energy densities are included in H, this solution is no longer valid, as discussed in detail in Refs. [24,25].

When conditions (6) and (7) are imposed on the potential, along with the thawing initial condition ($\dot{\phi} = 0$ at early times), it is possible to derive an approximate analytic solution for $w(a)$ that is independent of $V(\phi)$. This solution is [16]

$$1 + w(a) = (1 + w_0)\frac{[G(a) - (G(a)^2 - 1)\coth^{-1}G(a)]^2}{[G(1) - (G(1)^2 - 1)\coth^{-1}G(1)]^2}, \tag{8}$$

where w_0 is the value of w at the present. The function $G(a)$ is

$$G(a) = \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}}, \tag{9}$$

where $\Omega_{\phi 0}$ is the fraction of the total density at present contributed by the scalar field, which we will take throughout to be $\Omega_{\phi 0} = 0.7$. With these definitions, $G(a) = 1/\sqrt{\Omega_\phi(a)}$ and $G(1) = 1/\sqrt{\Omega_{\phi 0}}$. Here and throughout we will not give detailed derivations of previously derived results but will instead cite the original papers; in this case, a detailed derivation of Eq. (8) is given in Ref. [16]. Note that we use different notation and express our results in a different functional form than some of the earlier works cited here, both for the sake of increased simplicity and to avoid confusion with previously adopted k-essence notation. The function given by Eq. (8) is displayed in Fig. 1 (green, long-dashed curve).

In Refs. [17–19], the condition on the potential given by Eq. (6) was retained, but condition (7) was relaxed, resulting in a wider range of possible behaviors. In this case, the evolution of w with scale factor is given by [17–19]

$$1 + w(a) = (1 + w_0)a^{3(K-1)}\frac{[(G(a) + 1)^K(K - G(a)) + (G(a) - 1)^K(K + G(a))]^2}{[(G(1) + 1)^K(K - G(1)) + (G(1) - 1)^K(K + G(1))]^2}, \tag{10}$$

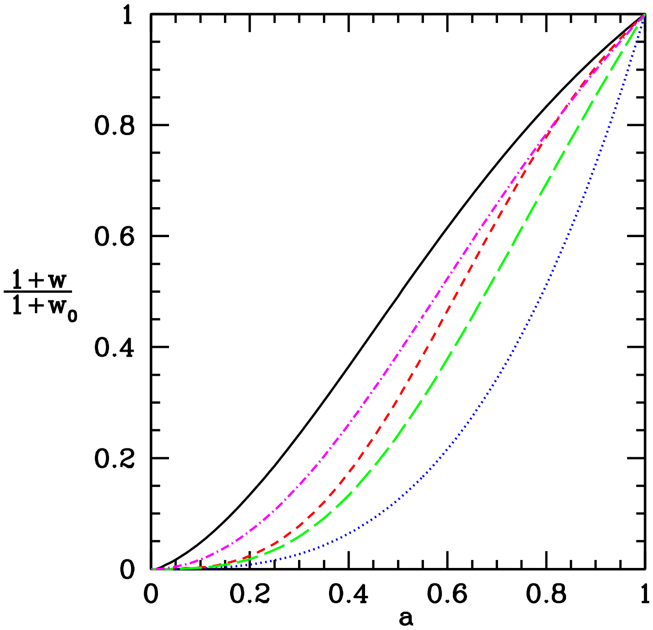

FIG. 1. Evolution of $1 + w$ relative to its value at the present, $1 + w_0$, as a function of the scale factor a for the analytic predictions discussed in this paper. Solid (black) curve is for k-essence with $F_X \approx 0$ (the new result of this paper). Blue (dotted) curve and red (short-dashed) curve are for k-essence with $X \approx 0$ or quintessence with non-negligible curvature in the potential, for $K = 2$ and $K \to 0$, respectively. Green (long-dashed) curve is for quintessence in a nearly flat potential. Magenta (dot-dashed) curve is for noncanonical quintessence with $\alpha = 2$.

where the constant K is a function of V''/V,

$$K = \sqrt{1 - (4/3)V''(\phi_*)/V(\phi_*)}, \tag{11}$$

evaluated at ϕ_*, which can be taken to be the initial value of ϕ [18]. Now instead of a single functional form for $w(a)$ for a given value of w_0, Eq. (10) provides a family of solutions that depend on K. As K becomes large, these solutions thaw more slowly; i.e., w remains close to -1 until later in the evolution [17]. In the opposite limit, as $K \to 1$, the solution in Eq. (10) approaches the evolution given in Eq. (8). For $K \to 0$, w increases more rapidly than in Eq. (8). This behavior is illustrated in Fig. 1, where $(1 + w)/(1 + w_0)$ is displayed as a function of a for $K = 2$ (blue, dotted curve) and $K \to 0$ (red, short-dashed curve).

B. k-essence

Now consider k-essence models with w near -1. In general, k-essence can be defined as any scalar field ϕ with a noncanonical kinetic term, so that the Lagrangian is of the form $\mathcal{L}(X, \phi)$, where

$$X = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi. \tag{12}$$

In practice, only a few special classes of such models have been explored in detail. The most widely investigated class of models (and the one examined in detail here and in Ref. [22]) is taken to have a Lagrangian in factorized form,

$$\mathcal{L} = V(\phi)F(X). \tag{13}$$

Such models were first introduced for inflation [26,27], and later extended to possible models for dark energy [28–34].

Before considering such models in detail, we briefly mention a second class of models, for which the Lagrangian has the form,

$$\mathcal{L} = X^\alpha - V(\phi). \tag{14}$$

These models have been dubbed "noncanonical quintessence" and have been previously examined as models both for inflation [35,36] and for dark energy [37–42]. For these models, Li and Scherrer [42] showed that when both slow-roll conditions on the potential (Eqs. (6) and (7)) are satisfied, the equation of state is well-approximated by

$$1 + w(a) = (1 + w_0)\frac{[G(a) - (G(a)^2 - 1)\coth^{-1} G(a)]^{2\alpha/(2\alpha-1)}}{[G(1) - (G(1)^2 - 1)\coth^{-1} G(1)]^{2\alpha/(2\alpha-1)}}. \tag{15}$$

As expected, this expression for $1 + w(a)$ reduces to the corresponding quintessence result [Eq. (8)] when $\alpha = 1$, which corresponds to quintessence with a standard kinetic term. The behavior of $(1 + w)/(1 + w_0)$ as a function of a for the representative case $\alpha = 2$ is shown in Fig. 1 (magenta, dot-dashed curve).

Now we direct our attention to factorizable k-essence models with the Lagrangian given by Eq. (13); such models are what is usually meant by the term "k-essence." The pressure in these models is simply given by Eq. (13), while the energy density is

$$\rho = V(\phi)[2XF_X - F], \tag{16}$$

where $F_X \equiv dF/dX$. Therefore, the equation of state parameter is

$$w = \frac{F}{2XF_X - F}. \tag{17}$$

The sound speed, which is relevant for the growth of density perturbations, is

$$c_s^2 = \frac{F_X}{2XF_{XX} + F_X}, \tag{18}$$

with $F_{XX} \equiv d^2F/dx^2$. In the flat Robertson-Walker metric, the equation for the evolution of the k-essence field takes the form,

$$(F_X + 2XF_{XX})\ddot{\phi} + 3HF_X\dot{\phi} + (2XF_X - F)\frac{V'}{V} = 0. \quad (19)$$

Chiba *et al.* [22] noted that $w \approx -1$ in Eq. (17) requires

$$|XF_X| \ll |F|, \quad (20)$$

which can be satisfied when either (i) $X \approx 0$ or (ii) $F_X \approx 0$. Note that these two conditions are sufficient, but not necessary to produce $w \approx -1$; one can derive other functional forms for $F(X)$ for which Eq. (20) is satisfied for arbitrary X. For example, if $F = X^{-\alpha}$ we obtain

$$w = -\frac{1}{2\alpha + 1}, \quad (21)$$

and $\alpha \ll 1$ corresponds to $w \approx -1$. Here we will consider only the two cases examined in Ref. [22], since these both converge toward unique sets of behaviors for $w(a)$.

Consider first case (i). For this case, Chiba *et al.* showed that the resulting evolution for w is given by

$$1 + w(a) = (1 + w_0)a^{3(K-1)}\frac{[(G(a)+1)^K(K - G(a)) + (G(a)-1)^K(K + G(a))]^2}{[(G(1)+1)^K(K - G(1)) + (G(1)-1)^K(K + G(1))]^2}, \quad (22)$$

where now,

$$K = \sqrt{1 - \frac{4}{3}\frac{V''(\phi_i)}{F_X(0)V(\phi_i)^2}}. \quad (23)$$

This result is identical to the corresponding quintessence result in Eq. (10). Thus, these two models are observationally indistinguishable. The behavior of $(1+w)/(1+w_0)$ as given by Eq. (22) is displayed in Fig. 1 for $K = 2$ (blue, dotted curve) and $K \to 0$ (red, short-dashed curve).

For case (ii), Chiba *et al.* derived a functional form for $w(a)$, but the result depends on $V(\phi)$ and is therefore considerably less interesting. It is this second case that we will revisit in the next section, showing that there are some conditions under which it produces a single functional behavior for $w(a)$ that is independent of $V(\phi)$.

III. EVOLUTION OF w FOR k-ESSENCE MODELS WITH $F_X \approx 0$

Consider a k-essence model for which $F_X \approx 0$. Following Ref. [22], we will expand $F(X)$ around the extremum in F, which we will take to occur at $X = X_m$. Then taking

$$X = X_m + \Delta, \quad (24)$$

where $\Delta \ll X_m$, we can write $F(X)$ as

$$F(X) = F(X_m) + \frac{1}{2}F_{XX}(X_m)\Delta^2, \quad (25)$$

so that

$$F_X(X) = F_{XX}(X_m)\Delta, \quad (26)$$

$$F_{XX}(X) = F_{XX}(X_m). \quad (27)$$

Then Eq. (17) can be expanded to linear order in Δ to yield

$$1 + w = \left[\frac{2X_mF_{XX}(X_m)}{F(X_m)}\right]\Delta. \quad (28)$$

In order to solve for $w(a)$, we first need to reexpress Eq. (19) in terms of Δ instead of ϕ. Using Eqs. (24)—(27), we can rewrite Eq. (19), up to linear order in Δ, as

$$\dot{\Delta} + 3H\Delta + \left(\sqrt{2X_m}\frac{V'}{V}\right)\Delta - \left(\sqrt{2X_m}\frac{V'}{V}\right)\left(\frac{F(X_m)}{4X_m^2F_{XX}(X_m)}\right)\Delta - \left(\sqrt{2X_m}\frac{V'}{V}\right)\left(\frac{F(X_m)}{2X_mF_{XX}(X_m)}\right) = 0. \quad (29)$$

The ratio of the third term to the final term is [from Eq. (28)] equal to $1 + w$, which we take to be $\ll 1$. The ratio of the fourth term to the final term is $\Delta/2X_m$, and we have assumed that $\Delta/X_m \ll 1$. Thus, the third and fourth terms in Eq. (29) are negligible compared to the final term in that equation. Then Eq. (29) simplifies to

$$\dot{\Delta} + 3H\Delta - \left(\sqrt{2X_m}\frac{V'}{V}\right)\left(\frac{F(X_m)}{2X_mF_{XX}(X_m)}\right) = 0. \quad (30)$$

To solve this equation, we make one final assumption: that V'/V is roughly constant as the k-essence field evolves through the period of interest. With this assumption, Eq. (30) can be solved exactly to yield

$$\Delta = \frac{C}{\sqrt{3\rho_{\phi 0}}}[G(a) - [G(a)^2 - 1]\coth^{-1}G(a)], \quad (31)$$

where C is the negative of the third term in Eq. (30), now taken to be constant,

$$C = \left(\sqrt{2X_m}\frac{V'}{V}\right)\left(\frac{F(X_m)}{2X_m F_{XX}(X_m)}\right). \tag{32}$$

Then Eq. (28) gives the value of $1+w$,

$$1 + w = \sqrt{\frac{2X_m}{3\rho_{\phi 0}}\frac{V'}{V}}[G(a) - (G(a)^2 - 1)\coth^{-1} G(a)]. \tag{33}$$

We can reexpress this in terms of the w_0 as in Eqs. (8), (10), (15), and (22) to give

$$1 + w(a) = (1 + w_0)\frac{G(a) - (G(a)^2 - 1)\coth^{-1}G(a)}{G(1) - (G(1)^2 - 1)\coth^{-1}G(1)}. \tag{34}$$

Equation (34) is the main result of this paper.

In Fig. 1, we show the behavior of $w(a)$ given by Eq. (34) (solid black curve), along with the corresponding behavior for the models examined previously. Note that, unlike the solution for k-essence with $X \approx 0$, the result here does not resemble any corresponding quintessence model, although it does correspond to the limiting behavior of noncanonical quintessence (Eq. (15)) in the limit where $\alpha \to \infty$. This correspondence is not surprising, as $\alpha \to \infty$ in noncanonical quintessence corresponds to the limit $X \to$ constant [40], the same behavior as in the k-essence models considered here.

The difference between this result for k-essence and the corresponding behavior for quintessence [Eq. (8)] is particularly clear if we examine these results in the $w - w'$ plane [23,43], where $w' \equiv a(dw/da)$, in the limit $a \ll 1$. In that limit, Eq. (8) reduces to $w' = 3(1 + w)$ for quintessence (see also Ref. [25]), while Eq. (34) gives $w' = \frac{3}{2}(1 + w)$ for k-essence.

Now consider the conditions on the model parameters necessary for Eq. (34) to represent a good approximation to the evolution of w. The conditions we imposed to derive Eq. (34) are (i) $1 + w \ll 1$, (ii) $\Delta \ll X_m$, and (iii) V'/V is approximately constant as ϕ evolves.

Clearly, if all of the other parameters in the k-essence models are $\sim\mathcal{O}(1)$, then conditions (i) and (ii) can be satisfied by choosing $(V'/V)^2 \ll 1$ as in Eq. (6); this follows directly from Eqs. (31)–(33). Condition (iii) indicates that V'/V evolves only a small amount compared to its initial value as ϕ evolves. This will be the case as long as $(V'/V)'/(V'/V)\delta\phi \ll 1$, where $\delta\phi$ is the total change in ϕ between $a = 0$ and $a = 1$.

In Fig. 2, we compare the analytic approximation of Eq. (34) to a numerical integration of the equation for k-essence evolution, where the parameters of these models are chosen to satisfy $(V'/V)^2 \ll 1$ and $(V'/V)'/(V'/V)\delta\phi \ll 1$; these conditions can be satisfied for all three potentials by taking the initial value of ϕ to be sufficiently large. For all of these cases we take $F(X) = F_0 + F_2(X - X_m)^2$. The fit to our analytic expression is very good in all three cases, and nearly exact for the exponential potential. The latter is not surprising, as the exponential potential has $V'/V =$ constant by construction.

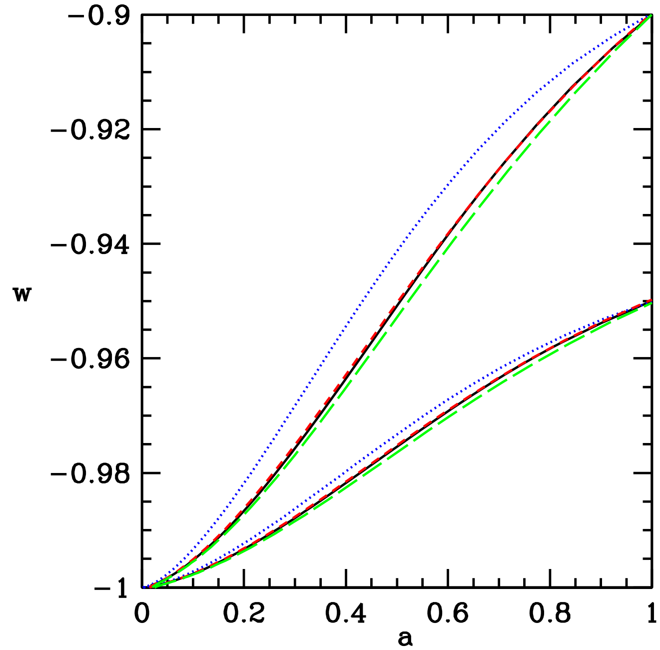

FIG. 2. Evolution of w as a function of a, normalized to $a = 1$ at the present, with $\Omega_{\phi 0} = 0.7$ and $w_0 = -0.9$ and -0.95, for models with $F_X \approx 0$. Solid (black) curve is our analytic approximation (Eq. (34)). Dotted (blue) curve is for $V(\phi) = V_0/\phi$, short-dashed (red) curve is $V(\phi) = e^{-\lambda\phi}$, and long-dashed (green) curve is $V(\phi) = e^{-\phi^2/\sigma^2}$. We take $F(X) = F_0 + F_2(X - X_m)^2$ for all three cases.

IV. DISCUSSION

Now we can compare the behavior of k-essence models with $F_X \approx 0$ to those of Ref. [22] with $X \approx 0$. The $F_X \approx 0$ models yield a new form for the evolution of $w(a)$, distinct from previous behaviors that have been derived for other models, while $X \approx 0$ models correspond to behavior that is identical to the results for quintessence evolution given in Refs. [17,18]. On the other hand, our results for $F_X \approx 0$ models are applicable to a much more restricted set of scalar field potentials than is the case for $X \approx 0$; namely, our results apply only to potentials for which $(V'/V)'/(V'/V)\delta\phi \ll 1$. This is the reason that Chiba *et al.* [22] found a variety of possible behaviors for $w(a)$ with $F_X \approx 0$ (see Fig. 4 of Ref. [22]); the potentials examined in that paper did not satisfy our (very restrictive) conditions on $V(\phi)$.

Now consider the observational constraints on our model. We will compare with the recent results of Alam *et al.* [44], derived from baryon acoustic oscillation measurements from the Sloan Digital Sky Survey III, cosmic microwave background observations from Planck, and type Ia supernovae data. Alam *et al.* express their constraints on w in terms of the Chevallier-Linder-Polarski (CPL) parametrization [45,46],

$$w = w_0 + (1 - a)w_a, \qquad (35)$$

where w_a and w_0 are constants, with w_0 being the present-day value of w. The most stringent bounds on w_a and w_0 in Alam *et al.* correspond to a narrow ellipse in the w_0, w_a plane. In this two-parameter model, neither w_0 nor w_a is individually strongly constrained, but a linear combination of the two is tightly bounded. The reason for this characteristic narrow elliptical bounded region in $w_0 - w_a$ space is the existence of a pivot redshift z_p, at which the errors on w are minimized [47]. In particular, Alam *et al.* [44] find a pivot redshift of $z_p = 0.37$, at which $w(z_p) = -1.05 \pm 0.05$.

We can exploit the fact that our model and the other models discussed in this paper are well-fit by the CPL parametrization for $a_p < a < 1$, and each model gives a unique prediction for $w(a_p)$ as a function of w_0. Hence, much stronger constraints can be placed on these models than on a generic dark energy model; in particular, we can derive a tight upper bound on the present-day value of w. First note that the pivot redshift z_p is related to a_p through $a_p = 1/(1 + z_p)$, so we have $a_p = 0.73$. Then our k-essence model with $F_X \approx 0$ must satisfy the $2 - \sigma$ upper bound $w(a = 0.73) < -0.95$. We can then simply read off the allowed value of w_0 from Fig. 1; namely, $w_0 < -0.93$.

It is clear from this argument that the models that allow the largest values of w_0 are those for which w increases most rapidly from $a = 0.73$ to $a = 1$. Hence, our new k-essence model with $F_X \approx 0$ is the most strongly constrained of those displayed in Fig. 1. In comparison, the quintessence model with a nearly flat potential [16] yields the constraint $w_0 < -0.91$, while the least strongly constrained model is the k-essence model with $X \approx 0$ (or equivalently, the quintessence model with non-negligible curvature in the potential) with $K = 2$, for which $w_0 < -0.87$. Larger values of K are even less strongly constrained [17]. (See Ref. [48] for another approach to observational constraints on thawing models.)

Note further that the k-essence models considered here with $F_X \approx 0$ make a very different prediction for the dark energy sound speed than do the previously examined models with $X \approx 0$. From Eq. (18), we see that our models give $c_s^2 \approx 0$, while the $X \approx 0$ models have $c_s^2 \approx 1$. Current observations are unable to significantly constrain c_s for dark energy (see, e.g., Refs. [49–52]), so these two extreme cases are not currently distinguishable, but future experiments such as Euclid [53] may provide useful constraints on the sound speed of dark energy.

In summary, we have derived a new generic thawing evolution of k-essence with w near -1; this is essentially a special case of the $F_X \approx 0$ solutions previously derived in Ref. [22], but for which additional constraints on the potential $V(\phi)$ yield a single set of evolutionary behaviors for $w(a)$. It is interesting that w in this model evolves away from -1 more rapidly than in any of the other models considered here, which allows us to place tighter constraints on this model than on any of the others. In contrast, the k-essence models with $X \approx 0$ examined in Ref. [22] require fewer conditions on the potential $V(\phi)$ and are less tightly constrained by observations. Our $F_X \approx 0$ models also provide a simple case for which $w \approx -1$, but the dark energy sound speed is close to zero.

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