How to extract the dominant part of the Wilson loop average in higher representations

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In previous works, we have proposed a new formulation of Yang-Mills theory on the lattice so that the so-called restricted field obtained from the gauge-covariant decomposition plays the dominant role in quark confinement. This framework improves the Abelian projection in the gauge-independent manner. For quarks in the fundamental representation, we have demonstrated some numerical evidence for the restricted field dominance in the string tension, which means that the string tension extracted from the restricted part of the Wilson loop reproduces the string tension extracted from the original Wilson loop. However, it is known that the restricted field dominance is not observed for the Wilson loop in higher representations if the restricted part of the Wilson loop is extracted by adopting the Abelian projection or the field decomposition naively in the same way as in the fundamental representation. In this paper, therefore, we focus on the confinement of quarks in higher representations. By virtue of the non-Abelian Stokes theorem for the Wilson loop operator, we propose suitable gauge-invariant operators constructed from the restricted field to reproduce the correct behavior of the original Wilson loop averages for higher representations. Moreover, we perform lattice simulations to measure the static potential for quarks in higher representations using the proposed operators. We find that the proposed operators well reproduce the behavior of the original Wilson loop average, namely, the linear part of the static potential with the correct value of the string tension, which overcomes the problem that occurs in naively applying Abelian projection to the Wilson loop operator for higher representations.

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I. INTRODUCTION

The dual superconductor picture is one of the most promising scenarios for quark confinement [1]. According to this picture, magnetic monopoles causing the dual superconductivity are regarded as the dominant degrees of freedom responsible for confinement. However, it is not so easy to verify this hypothesis. Indeed, even the definition of magnetic monopoles in the pure Yang-Mills theory is not obvious. If magnetic charges are naively defined from electric ones by exchanging the role of the magnetic field and the electric one according to the electric-magnetic duality, one needs to introduce singularities to obtain nonvanishing magnetic charges, as represented by the Dirac monopole. For such a configuration, however, the energy becomes divergent.

The most frequently used prescription avoiding this issue in defining monopoles is the *Abelian projection*, which is proposed by 't Hooft [2]. In this method, the "diagonal component" of the Yang-Mills gauge field is identified with the Abelian gauge field and a monopole is defined as the Dirac monopole. The energy density of this monopole can be finite everywhere because the contribution from the singularity of a Dirac monopole can be canceled by that of the off-diagonal components of the gauge field. In this method, however, one needs to fix the gauge because otherwise the "diagonal component" is meaningless.

There is another way to define monopoles, which does not rely on the gauge fixing. This method is called the *field decomposition* that was proposed for the $SU(2)$ Yang-Mills gauge field by Cho [3] and Duan and Ge [4] independently, and later readdressed by Faddeev and Niemi [5], and

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developed by Shabanov [6] and the Chiba University group [7–9]. In this method, as the name suggests, the gauge field is decomposed into two parts. A part called the *restricted field* transforms under the gauge transformation just like the original gauge field, while the other part called the *remaining field* transforms like an adjoint matter. The key ingredient in this decomposition is the Lie-algebra valued field with a unit length that we call the *color field*. The decomposition is constructed in such a way that the field strength of the restricted field is "parallel" to the color field. Then monopoles can be defined by using the gauge-invariant part proportional to the color field in the field strength just as the Abelian field strength in the Abelian projection. The definition of monopoles in this method is equivalent to that in the Abelian projection. By this construction the gauge invariance is manifestly maintained differently from the Abelian projection. The field decomposition was extended to $SU(N)$ $(N \geq 3)$ gauge field in [10–13]. See, e.g., [14] for a review.

While the main advantage of the field decomposition is its gauge covariance, another advantage is that, through a version of the non-Abelian Stokes theorem (NAST) invented originally by Diakonov and Petrov [15,16] and extended in a unified way in [17–23], the restricted field naturally appears in the surface-integral representation of the Wilson loop. By virtue of this method, we understand how monopoles contribute to the Wilson loop at least classically.

It can be numerically examined whether these monopoles actually reproduce the expected infrared behavior of the original Wilson loop average, even if it is impossible to do so analytically. For quarks in the fundamental representation, indeed, such numerical simulations were already performed within the Abelian projection using the maximal Abelian (MA) gauge in $SU(2)$ and $SU(3)$ Yang-Mills theories on the lattice [24–26]. Then it was confirmed that (i) the diagonal part extracted from the original gauge field in the MA gauge reproduces the full string tension calculated from the original Wilson loop average [24,26], which is called the *Abelian dominance*, and that (ii) the monopole part extracted from the diagonal part of the gauge field by applying the Toussaint-DeGrand procedure [27] mostly reproduces the full string tension [25,26], which is called the *monopole dominance*.

However, it should be noted that the MA gauge in the Abelian projection breaks simultaneously the local gauge symmetry and the global color symmetry. This defect should be eliminated to obtain the physical result by giving a procedure to guarantee the gauge invariance. For this purpose, we have developed the lattice version [28–33] of the reformulated Yang-Mills theory written in terms of new variables obtained by the gauge-covariant field decomposition, which enables us to perform the numerical simulations on the lattice in such a way that both the local gauge symmetry and the global color symmetry remain intact, in sharp contrast to the Abelian projection, which breaks both symmetries. In this paper we adopt the gauge-covariant decomposition method to avoid these defects of the Abelian projection, although the conventional treatment equivalent to the Abelian projection and the MA gauge can be reproduced from the gauge-covariant field decomposition method as a special case called the maximal option. Moreover, the MA gauge in the Abelian projection is not the only way to recover the string tension in the fundamental representation. By way of the non-Abelian Stokes theorem [20] for the Wilson loop operator, indeed, it was found that the different type of decomposition called the minimal option is available for $SU(3)$ and $SU(N)$ for $N \geq 4$ [13,29,30]. Even for the minimal option, we have demonstrated the restricted field dominance and monopole dominance in the string tension for quarks in the fundamental representation [31,32]. See [14] for a review. Thus, our method enables one to extract various degrees of freedom to be responsible for quark confinement by combining the option of gauge-covariant field decomposition and the choice of the reduction condition, which is not restricted to the Abelian projection and the MA gauge, respectively. In this paper, indeed, we have adopted three kinds of reduction conditions to examine the contributions from magnetic monopoles of different types.

For quarks in higher representations, however, it is known that, if the Abelian projection is naively applied to the Wilson loop in higher representations, the resulting monopole contribution does not reproduce the string tension extracted from the original Wilson loop average [34]. This is because, in higher representations, the diagonal part of the Wilson loop does not behave in the same way as the original Wilson loop. For example, in the adjoint representation of $SU(2)$, the diagonal part of the Wilson loop average approaches $1/3$ for a large loop, which is obviously different from the behavior of the original Wilson loop. In the language of the field decomposition, this means that in higher representations, the Wilson loop for the restricted field does not behave in the same way as the original Wilson loop. Poulis [35] heuristically found the correct way to extend the Abelian projection approach for the adjoint representation of $SU(2)$. In his approach, the diagonal part of the Wilson loop is further decomposed into the "charged term" and the "neutral term," and then the charged term is used instead of the diagonal part.

In this paper, we propose a systematic prescription to extract the "dominant" part of the Wilson loop average, which can be applied to the Wilson loop operator in an arbitrary representation of an arbitrary compact gauge group. Here the dominant part means that the string tension extracted from this part of the Wilson loop reproduces the string tension extracted from the original Wilson loop. In the prescription, we further extract the "highest weight part" from the diagonal part of the Wilson loop or the Wilson loop for the restricted field. This prescription comes from the NAST. In order to test this proposal, we calculate numerically the dominant part of the Wilson loop for the adjoint representation of the $SU(2)$ group, and adjoint and sextet representations of the $SU(3)$ group. The results support our claim.

This paper is organized as follows. In Sec. II, we briefly review the field decomposition of the gauge field and the NAST for the Wilson loop operator. In Sec. III, we propose an operator suggested from the NAST, which is expected to reproduce the dominant part of the area law falloff of the original Wilson loop average. In Sec. IV, we perform the numerical simulations on the lattice to examine whether the proposed operator exhibits the expected behavior of the Wilson loop average. In Sec. V, we summarize the results obtained in this paper. In the Appendixes A and B we give the derivation of some equations given in Sec. III.

II. FIELD DECOMPOSITION METHOD AND THE NON-ABELIAN STOKES THEOREM

In this section, we give a brief review of the field decomposition, the NAST for the Wilson loop operator and the reduction conditions. First, we introduce the field decomposition in a continuum theory and then in a lattice theory. Here we work in the $SU(N)$ Yang-Mills theory, but the field decomposition can be applied to an arbitrary compact group [23]. Next we introduce the Diakonov-Petrov version of the non-Abelian Stokes theorem [15] for the Wilson loop operator, which is used to see the relationship between the field decomposition and the Wilson loop operator. Finally, we explain the relationship between the field decomposition and the reduction condition, which determines the color fields as a functional of the gauge field. For a more detailed review, see, e.g., [14].

A. Field decomposition

1. Continuum case

In the field decomposition method, we decompose the gauge field $\mathcal{A}_\mu(x)$ into two parts as

$$\mathcal{A}_\mu(x) = \mathcal{V}_\mu(x) + \mathcal{X}_\mu(x). \tag{1}$$

Here the restricted field $\mathcal{V}_\mu(x)$ is required to transform just as the gauge field $\mathcal{A}_\mu$ under the gauge transformation as

$$\mathcal{V}_\mu(x) \to g(x)\mathcal{V}_\mu(x)g^\dagger(x) + ig_{\mathrm{YM}}^{-1}g(x)\partial_\mu g^\dagger(x), \tag{2}$$

where $g(x) \in SU(N)$ and g_{YM} is the Yang-Mills coupling. Hence the remaining field $\mathcal{X}_\mu(x)$ must transform like an adjoint matter field as

$$\mathcal{X}_\mu(x) \to g(x)\mathcal{X}_\mu(x)g^\dagger(x). \tag{3}$$

We wish to regard the restricted field $\mathcal{V}_\mu$ as the dominant part of the gauge field $\mathcal{A}_\mu$ in the IR region. In this paper, we focus on the version of maximal option.

In order to determine the decomposition for the gauge group $SU(N)$, we introduce a set of *color fields* $\boldsymbol{n}^{(k)}(x)$ ($k = 1, \ldots, N-1$) which are expressed using a common $SU(N)$-valued field $\Theta(x)$ as

$$\boldsymbol{n}^{(k)}(x) \coloneqq \Theta(x)H_k\Theta^\dagger(x), \tag{4}$$

where H_k is a Cartan generator. Notice that the color fields are not independent. The transformation property of the color fields under a gauge transformation is given by

$$\boldsymbol{n}^{(k)}(x) \to g(x)\boldsymbol{n}^{(k)}(x)g^\dagger(x). \tag{5}$$

The color fields are determined as functionals of $\mathcal{A}_\mu$ by imposing a condition that we call the *reduction condition* as explicitly given shortly.

The decomposition is constructed such that the field strength of the restricted field, $\mathcal{F}_{\mu\nu}[\mathcal{V}] \coloneqq \partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu - ig[\mathcal{V}_\mu, \mathcal{V}_\nu]$, is expressed by a linear combination of the color fields. This condition can be simply written as

$$\mathcal{D}_\mu[\mathcal{V}]\boldsymbol{n}^{(k)} = 0 \qquad (k = 1, \ldots, N-1), \tag{6}$$

where $\mathcal{D}_\mu[\mathcal{V}] \coloneqq \partial_\mu - ig_{\mathrm{YM}}[\mathcal{V}_\mu, \bullet]$ is the covariant derivative with the restricted field $\mathcal{V}_\mu$. This condition is manifestly gauge covariant. This determines the component of the restricted field orthogonal to the Lie subalgebra spanned by the color fields, but does not determine the component parallel to it. Therefore we need to impose another condition. We wish to identify the restricted field with the dominant part of the original gauge field, and thus it should be as close as possible to the original gauge field in the IR region. For this reason we impose the condition that the component of the restricted field parallel to the color fields is the same as that of the gauge field as

$$\mathrm{tr}(\boldsymbol{n}^{(k)}\mathcal{V}_\mu) = \mathrm{tr}(\boldsymbol{n}^{(k)}\mathcal{A}_\mu) \qquad (k = 1, \ldots, N-1). \tag{7}$$

The two conditions of Eqs. (6) and (7) uniquely determine the decomposition as

$$\mathcal{V}_\mu = \sum_{k=1}^{N-1} 2\mathrm{tr}(\boldsymbol{n}^{(k)}\mathcal{A}_\mu)\boldsymbol{n}^{(k)} - ig_{\mathrm{YM}}^{-1}\sum_{k=1}^{N-1}[\boldsymbol{n}^{(k)}, \partial_\mu\boldsymbol{n}^{(k)}],$$
$$\mathcal{X}_\mu = ig_{\mathrm{YM}}^{-1}\sum_{k=1}^{N-1}[\boldsymbol{n}^{(k)}, \mathcal{D}_\mu[\mathcal{A}]\boldsymbol{n}^{(k)}]. \tag{8}$$

In fact, the resulting decomposed fields satisfy the required transformation properties. As the field strength $\mathcal{F}_{\mu\nu}[\mathcal{V}]$ can be written as the linear combination of the color fields, we can define Abelian-like gauge-invariant field strength as

$$F_{\mu\nu}^{(k)} \coloneqq 2\mathrm{tr}(\boldsymbol{n}^{(k)}\mathcal{F}_{\mu\nu}[\mathcal{V}]), \tag{9}$$

where the normalization of the Cartan generators is given as $\mathrm{tr}(H_k H_l) = \delta_{kl}/2$. Then monopoles are defined in the same manner as the Dirac monopoles for this field strength $F^{(k)}_{\mu\nu}$. The resulting monopoles are gauge invariant by construction.

The color fields $\boldsymbol{n}^{(k)}$ are obtained by imposing a reduction condition as we said before. If a reduction condition is given by minimizing a functional

$$R_{\mathrm{MA}}[\mathcal{A}, \{\boldsymbol{n}^{(k)}\}] = \int d^D x \sum_{k=1}^{N-1} \mathrm{tr}(\mathcal{D}_\mu[\mathcal{A}]\boldsymbol{n}^{(k)}(x)\mathcal{D}_\mu[\mathcal{A}]\boldsymbol{n}^{(k)}(x)), \quad (10)$$

the definition of monopoles is equivalent to that for the Abelian projection in the MA gauge.

2. Lattice case

In the lattice version of the field decomposition [28–30], a link variable $U_{x,\mu}$ is decomposed into two variables as

$$U_{x,\mu} = X_{x,\mu} V_{x,\mu}, \qquad X_{x,\mu}, V_{x,\mu} \in SU(N), \quad (11)$$

where $V_{x,\mu}$ gauge transforms just like a link variable as

$$V_{x,\mu} \to g_x V_{x,\mu} g^\dagger_{x+\mu}, \qquad g_x \in SU(N), \quad (12)$$

and $X_{x,\mu}$ transforms like an adjoint matter as

$$X_{x,\mu} \to g_x X_{x,\mu} g^\dagger_x. \quad (13)$$

The decomposition is determined by using the color fields $\boldsymbol{n}^{(k)}_x = \Theta_x H_k \Theta^\dagger_x$ $(k = 1, \ldots, N-1)$ in a similar way to the continuum case. The first condition that determines the decomposition is given by replacing the covariant derivative $\mathcal{D}_\mu[\mathcal{V}]$ in Eq. (6) with the covariant lattice derivative $D_\mu[V]$ as

$$D_\mu[V]\boldsymbol{n}^{(k)}_x := \varepsilon^{-1}(V_{x,\mu}\boldsymbol{n}^{(k)}_{x+\mu} - \boldsymbol{n}^{(k)}_x V_{x,\mu}) = 0, \quad (14)$$

where ε is the lattice spacing. This condition does not determine $V_{x,\mu}$ completely because this equality is maintained if we multiply $V_{x,\mu}$ from the left by $g_x \in SU(N)$, which satisfies $[\boldsymbol{n}^{(k)}_x, g_x] = 0$ for any k. To reproduce the continuum version of the decomposition Eq. (8) in the naive continuum limit, the decomposition is chosen as [30]

$$\begin{aligned} V_{x,\mu} &= \hat{K}_{x,\mu} U_{x,\mu} (\det(\hat{K}_{x,\mu}))^{-1/N}, \\ X_{x,\mu} &= \hat{K}^\dagger_{x,\mu} (\det(\hat{K}_{x,\mu}))^{1/N}, \\ \hat{K}_{x,\mu} &:= \left(\sqrt{K_{x,\mu} K^\dagger_{x,\mu}}\right)^{-1} K_{x,\mu}, \\ K_{x,\mu} &:= \boldsymbol{1} + 2N \sum_{k=1}^{N-1} \boldsymbol{n}^{(k)}_x U_{x,\mu} \boldsymbol{n}^{(k)}_{x+\mu} U^\dagger_{x,\mu}. \end{aligned} \quad (15)$$

The color fields are determined by minimizing a reduction functional as in the continuum case. The lattice version of Eq. (10) is given by replacing the covariant derivative with the covariant lattice derivative as

$$R_{\mathrm{MA}}[U, \{\boldsymbol{n}^{(k)}\}] := \sum_{x,\mu} \sum_{k=1}^{N-1} \mathrm{tr}[(D_\mu[U]\boldsymbol{n}^{(k)}_x)^\dagger D_\mu[U]\boldsymbol{n}^{(k)}_x]. \quad (16)$$

B. Non-Abelian Stokes theorem

The Wilson loop operator in a representation R is defined by

$$W_R[\mathcal{V}; C] := \frac{1}{D_R} \mathrm{tr}_R \mathcal{P} \exp\left(i g_{\mathrm{YM}} \oint_C \mathcal{A}\right), \quad (17)$$

where D_R is the dimension of R, tr_R denotes the trace in R, and $\mathcal{P}$ denotes the path ordering. We can relate the decomposed field variables to a Wilson loop operator through a version of the NAST that was proposed by Diakonov and Petrov [15]. In this version of the NAST, a Wilson loop operator in a representation R is rewritten into the surface integral form by introducing a functional integral on the surface S surrounded by the loop C as

$$\begin{aligned} W_R[\mathcal{A}; C] &= \int D\Omega \exp\left(i g_{\mathrm{YM}} \int_{S:\partial S = C} \sum_{k=1}^{N-1} \Lambda_k F^{(k)}\right), \\ F^{(k)} &:= \frac{1}{2} F^{(k)}_{\mu\nu} dx^\mu \wedge dx^\nu, \quad D\Omega := \prod_{x \in S} d\Omega(x), \end{aligned} \quad (18)$$

where $D\Omega$ is the product of the Haar measure $d\Omega(x)$ over the surface S with the loop C as the boundary, Λ_k is the kth component of the highest weight of the representation R, the color fields are defined by $\boldsymbol{n}^{(k)} = \Omega H_k \Omega^\dagger$, and $F^{(k)}_{\mu\nu}$ is the Abelian-like field strength defined by Eq. (9). Thus we can relate the restricted field to the Wilson loop operator in the manifestly gauge-invariant way.

The simplified version of the derivation is as follows. See, e.g., [14,19,20] for a more detailed derivation of Eq. (18) along the following line. First, we divide the loop into small pieces and represent the Wilson loop operator as the product of the parallel transporter for each piece. Next we insert between parallel transporters the completeness relation

$$\mathbf{1} = \int d\Omega \Omega |\Lambda\rangle\langle\Lambda|\Omega^\dagger, \tag{19}$$

where $d\Omega$ is the Haar measure and $|\Lambda\rangle$ is the highest weight state of the representation R, and we rewrite the trace by using the equality

$$\mathrm{tr}\mathcal{O} = \int d\Omega \langle\Lambda|\Omega^\dagger \mathcal{O} \Omega|\Lambda\rangle. \tag{20}$$

Then, by taking the limit where the length of each piece of the loop goes to zero, we obtain

$$W_R[\mathcal{A}; C] = \int \prod_{x\in C} d\Omega(x) \exp\left(ig_{\mathrm{YM}} \oint_C \langle\Lambda|\mathcal{A}^\Omega|\Lambda\rangle \right),$$
$$\mathcal{A}^\Omega(x) \coloneqq \Omega^\dagger(x)\mathcal{A}(x)\Omega(x) + ig_{\mathrm{YM}}^{-1}\Omega^\dagger(x) d\Omega(x). \tag{21}$$

In this expression, the path ordering disappears, and therefore we can use the usual Stokes theorem as

$$W_R[\mathcal{A}; C] = \int \prod_{x\in S} d\Omega(x) \exp\left(ig_{\mathrm{YM}} \int_{S:\partial S = C} F^\Omega \right),$$
$$F^\Omega(x) \coloneqq d\langle\Lambda|\mathcal{A}^\Omega|\Lambda\rangle. \tag{22}$$

We can show that $F^\Omega_{\mu\nu}$ is written as the linear combination of the Abelian-like field strengths Eq. (9) as [22]

$$F^\Omega_{\mu\nu} = \sum_{k=1}^{N-1} \Lambda_k F^{(k)}_{\mu\nu}, \qquad \boldsymbol{n}^{(k)}(x) \coloneqq \Omega(x) H_k \Omega^\dagger(x), \tag{23}$$

where the color fields $\boldsymbol{n}^{(k)}$ are defined by using the integration variable $\Omega(x)$ instead of $\Theta(x)$.

Clearly, the NAST can be applied not only to the fundamental representation but also to any representation, suggesting the correct way for extracting the dominant part of the Wilson loop in higher representations as we explain in the next section.

C. The relationship between the NAST and the reduction condition

Here we consider the relation between the reduction condition and the NAST. In the NAST Eq. (18), we observe that the field strength $F^{(k)}_{\mu\nu}$ is defined in terms of the integration variable $\Omega(x)$. At this stage, $\Omega(x)$ is distinct from $\Theta(x)$ used to define the field decomposition. Therefore, there is no clear relationship between the Wilson loop operator and the field decomposition defined by using the color field $\boldsymbol{n}^{(k)}(x)$ constructed from $\Theta(x)$. Instead of performing the integration over the measure $D\Omega$, the color fields defined using $\Omega(x)$ in Eq. (18) are replaced by the color fields defined using $\Theta(x)$ determined by solving the reduction condition. The validity of this replacement should be checked by numerical calculations. In the fundamental representation, if we use Θ determined by minimizing Eq. (10), the integrand of Eq. (18) with $\Omega = \Theta$ is equal to the "Abelian Wilson loop" obtained by taking the Abelian projection in the MA gauge. This gauge is chosen so as to maximize the Abelian part of the gauge field. In this case the validity of this replacement has already been checked by the Abelian dominance in the previous studies. In higher representations, we follow the same strategy as the fundamental representation, and the validity will be checked by the numerical calculations in this paper.

The reduction condition is not determined uniquely. To see the dependence on the reduction condition, in the present study for the $SU(3)$ Yang-Mills theory, we performed numerical simulations under the two additional reduction conditions that are defined by minimizing the functionals

$$R_{n3}[U, \{\boldsymbol{n}^{(k)}\}] = \sum_{x,\mu} \mathrm{tr}[(D_\mu[U]\boldsymbol{n}^3_x)^\dagger D_\mu[U]\boldsymbol{n}^3_x], \tag{24}$$

$$R_{n8}[U, \{\boldsymbol{n}^{(k)}\}] = \sum_{x,\mu} \mathrm{tr}[(D_\mu[U]\boldsymbol{n}^8_x)^\dagger D_\mu[U]\boldsymbol{n}^8_x], \tag{25}$$

where $\boldsymbol{n}^3_x \coloneqq \Theta_x T^3 \Theta^\dagger_x$ and $\boldsymbol{n}^8_x \coloneqq \Theta_x T^8 \Theta^\dagger_x$. Note that the reduction functional Eq. (25) does not determine $\boldsymbol{n}^3_x$ and therefore does not determine the decomposition Eq. (15) completely. However, as we explain in the next section, a specific part of Eq. (35) of the Wilson loop for the restricted field is determined.

III. WILSON LOOPS IN HIGHER REPRESENTATIONS

In the preceding numerical simulations [25,26] by using the Abelian projection and [28,31–33,36] by using the field decomposition, it was shown that the area law of the average of a Wilson loop in the fundamental representation is reproduced by the monopole contribution. However, this might be an accidental agreement restricted to the fundamental representation. Therefore, we should check the other quantities. The Wilson loops in higher representations are appropriate for this purpose because they have a clear physical meaning. However, it is known that if we apply the Abelian projection naively to higher representations, the monopole contributions in the Abelian part do not reproduce the correct behavior [34]. For example, in the adjoint representation of $SU(2)$, the Abelian Wilson loop average approaches $1/3$ as the loop size increases according to the numerical simulation [35]. In this case, we cannot extract the static potential $V(R)$ from the exponential falloff behavior $e^{-V(L)T}$ of the Wilson loop average defined for the rectangular loop with length T and width L, since

$e^{-V(L)T} \to 0$ as $T \to \infty$. In the spin-3/2 representation, the string tension extracted from the Abelian Wilson loop has the same value as that for the fundamental representation [34], which is different from the correct behavior. Thus we need to find a more appropriate way to extract the monopole contributions in the Abelian part.

As we mentioned before, the NAST suggests how we extract the dominant part of the Wilson loop average, which means that by using an appropriate operator $\tilde{W}_R[\mathcal{V};C]$ suggested by the NAST, we can reproduce the full string tension extracted using the original Wilson loop $W_R[\mathcal{A};C]$. In the language of the field decomposition, the diagonal part of the Wilson loop is equivalent to the "restricted Wilson loop" $W_R[\mathcal{V};C]$, the Wilson loop for the restricted field $\mathcal{V}$. Therefore, the average of $W_R[\mathcal{V};C]$ does not reproduce the string tension extracted from the original Wilson loop $W_R[\mathcal{A};C]$. On the other hand, the NAST Eq. (18) suggests the distinct operator $\tilde{W}_R[\mathcal{V};C]$ as the dominant part of the Wilson loop in higher representations.

We now give the explicit expressions for the operators suggested by the NAST, $\tilde{W}_R[\mathcal{V};C]$, and the restricted Wilson loop operator $W_R[\mathcal{V};C]$ to see the difference between the two operators. The restricted Wilson loop operator $W_R[\mathcal{V};C]$ is rewritten as

$$\begin{aligned} W_R[\mathcal{V};C] &\coloneqq \frac{1}{D_R}\mathrm{tr}_R P\exp\left(ig_{\mathrm{YM}}\oint_C \mathcal{V}\right) \\ &= \frac{1}{D_R}\sum_{\boldsymbol{\mu}\in\Delta_R} d_\mu \exp\left(ig_{\mathrm{YM}}\oint_C \langle\mu|\mathcal{A}^\Theta|\mu\rangle\right), \\ \mathcal{A}^\Theta(x) &\coloneqq \Theta^\dagger(x)\mathcal{A}(x)\Theta(x) + ig_{\mathrm{YM}}^{-1}\Theta^\dagger(x)d\Theta(x), \end{aligned} \tag{26}$$

where D_R is the dimension of the representation R, Δ_R is the set of all weights of R, d_μ is the multiplicity of a weight $\boldsymbol{\mu}$, and $|\mu\rangle$ is a normalized state corresponding to $\boldsymbol{\mu}$. Note that this operator of Eq. (26) is gauge invariant just as the original Wilson loop. The derivation of Eq. (26) is given in Appendix A.

For example, in the adjoint representation of $SU(2)$, the Wilson loop for the restricted field is written as

$$\begin{aligned} W_{J=1}[\mathcal{V};C] &= \frac{1}{3}(e^{i\phi} + e^{-i\phi} + 1), \\ \phi &\coloneqq g_{\mathrm{YM}}\oint 2\mathrm{tr}(\mathcal{A}^\Theta T^3). \end{aligned} \tag{27}$$

In [35], it was confirmed that the average of this operator approaches 1/3 as the loop size increases. This behavior is clearly different from the original Wilson loop.

In the adjoint representation [1, 1] and the sextet representation [0, 2] of $SU(3)$, the weight diagram is given in Figs. 1(a) and 1(b), respectively. Then the Wilson loop for the restricted field is written as

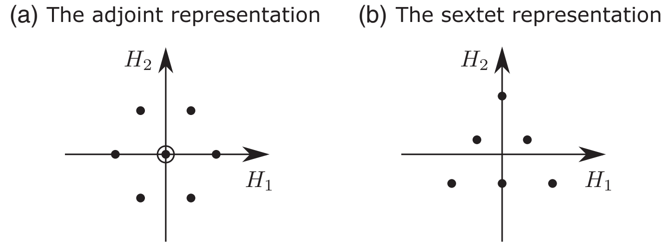

FIG. 1. The weight diagram of (a) the adjoint representation [1, 1] and (b) the sextet representation [0, 2] of $SU(3)$. A single dot represents a weight $\boldsymbol{\mu}$ with multiplicity one, $d_\mu = 1$, and a circled dot represents a weight $\boldsymbol{\mu}$ with multiplicity two, $d_\mu = 2$.

$$\begin{aligned} W_{[1,1]}[\mathcal{V};C] &= \frac{1}{8}\Big(e^{i\frac{\phi_3+\sqrt{3}\phi_8}{2}} + e^{-i\frac{\phi_3+\sqrt{3}\phi_8}{2}} + e^{i\frac{-\phi_3+\sqrt{3}\phi_8}{2}} \\ &\quad + e^{-i\frac{-\phi_3+\sqrt{3}\phi_8}{2}} + e^{i\phi_3} + e^{-i\phi_3} + 2\Big), \\ W_{[0,2]}[\mathcal{V};C] &= \frac{1}{6}\Big(e^{i\frac{2}{\sqrt{3}}\phi_8} + e^{i\frac{3\phi_3+\sqrt{3}\phi_8}{3}} + e^{i\frac{-3\phi_3+\sqrt{3}\phi_8}{3}} \\ &\quad + e^{i\frac{3\phi_3+\sqrt{3}\phi_8}{6}} + e^{i\frac{-3\phi_3+\sqrt{3}\phi_8}{6}} + e^{-i\frac{1}{\sqrt{3}}\phi_8}\Big), \\ \phi_3 &\coloneqq g_{\mathrm{YM}}\oint_C 2\mathrm{tr}(\mathcal{A}^\Theta T^3), \\ \phi_8 &\coloneqq g_{\mathrm{YM}}\oint_C 2\mathrm{tr}(\mathcal{A}^\Theta T^8). \end{aligned} \tag{28}$$

On the other hand, the operator $\tilde{W}_R[\mathcal{V};C]$ suggested by the NAST is the integrand of the NAST using the color fields satisfying the reduction condition, i.e., the integrand of Eq. (21) with $\Omega(x) = \Theta(x)$. We include the contribution of the weights that are equivalent to the highest weight under the action of the Weyl group. Let the set of such weights be Δ_R^h. Thus we propose the operator

$$\tilde{W}_R[\mathcal{V};C] = \frac{1}{D_R^h}\sum_{\boldsymbol{\Lambda}\in\Delta_R^h}\exp\left(ig_{\mathrm{YM}}\oint_C\langle\Lambda|\mathcal{A}^\Theta|\Lambda\rangle\right), \tag{29}$$

where D_R^h is the number of elements in Δ_R^h. We call this operator *the highest weight part* of the Abelian Wilson loop. Note that this operator of Eq. (29) is gauge invariant because $\Theta(x)$ transforms as $\Theta(x) \to g(x)\Theta(x)$ under the gauge transformation. In the fundamental representation, the highest weight part of the Abelian Wilson loop, Eq. (29), is the same as the Abelian Wilson loop because all weights of the fundamental representation are equivalent to the highest weight under the action of the Weyl group.

For example, in the adjoint representation of $SU(2)$ the proposed operator is written as

$$\tilde{W}_{J=1}[\mathcal{V};C] = \frac{1}{2}(e^{i\phi} + e^{-i\phi}). \tag{30}$$

In [35], Poulis heuristically found that this operator reproduces the full adjoint string tension without giving the theoretical justification. In the adjoint representation [1, 1] and the sextet representation [0, 2] of $SU(3)$ it can be written as

$$\tilde{W}_{[1,1]}[\mathcal{V};C] = \frac{1}{6}(e^{i\frac{\phi_3+\sqrt{3}\phi_8}{2}} + e^{-i\frac{\phi_3+\sqrt{3}\phi_8}{2}} + e^{i\frac{-\phi_3+\sqrt{3}\phi_8}{2}} + e^{-i\frac{-\phi_3+\sqrt{3}\phi_8}{2}} + e^{i\phi_3} + e^{-i\phi_3}),$$

$$\tilde{W}_{[0,2]}[\mathcal{V};C] = \frac{1}{3}(e^{i\frac{2}{\sqrt{3}}\phi_8} + e^{i\frac{3\phi_3+\sqrt{3}\phi_8}{3}} + e^{i\frac{-3\phi_3+\sqrt{3}\phi_8}{3}}). \quad (31)$$

For $SU(2)$, the proposed operator of Eq. (29) for the spin-J representation can be written as

$$\tilde{W}_J[V;C] = \frac{1}{2}\mathrm{tr}((V_C)^{2J}), \quad (32)$$

by using the untraced restricted Wilson loop V_C in the fundamental representation defined as

$$V_C \coloneqq \prod_{\langle x,\mu\rangle\in C} V_{x,\mu}. \quad (33)$$

For $SU(3)$, the proposed operator for the representation with the Dynkin index $[m,n]$ can be written as

$$\tilde{W}_{[m,n]}[V;C] = \frac{1}{6}(\mathrm{tr}((V_C)^m)\mathrm{tr}((V_C^\dagger)^n) - \mathrm{tr}((V_C)^m(V_C^\dagger)^n)), \quad (34)$$

where $(V_C)^0 = \boldsymbol{1}$. The derivation of Eqs. (32) and (34) is given in Appendix B. Note that Eqs. (32) and (34) are gauge invariant because of the gauge-transformation property of $V_{x,\mu}$, Eq. (12). Indeed, Eq. (32) for $J = 1/2$ in $SU(2)$ and Eq. (34) for $[m,n] = [1,0]$ in $SU(3)$ are the same as the ordinary Abelian Wilson loop in the fundamental representation.

Finally, we consider what part of the Abelian Wilson loop is determined by the reduction condition Eq. (25). The color field $\boldsymbol{n}_x^8$ does not change under a transformation $\Theta_x \to \Theta_x g_x$, $g_x \in U(2)$, where $U(2)$ is generated by T^1, T^2, T^3, T^8. Under this transformation ϕ_8 does not change but ϕ_3 changes. Thus the part of the Abelian Wilson loop that is determined by Eq. (25) is written as

$$e^{i\frac{n}{\sqrt{3}}\phi_8}, \qquad n \in \mathbf{Z}. \quad (35)$$

This part is contained in the highest weight part of the Abelian Wilson loop only for representations $[m,0]$ and $[0,n]$. Therefore, in the numerical simulation, we have not calculated the highest weight part of the Abelian Wilson loop in the adjoint representation [1, 1] for the reduction condition Eq. (25).

IV. NUMERICAL RESULT

In order to support our claim that the dominant part of the Wilson loops in higher representation is given by the highest weight part of the Abelian Wilson loop, Eq. (29), we examine numerically whether the string tension extracted from Eqs. (32) and (34) reproduce the full string tension. In this paper we investigate the Wilson loop in the adjoint representation of $SU(2)$ and in the adjoint representation [1, 1] and the sextet representation [0, 2] of $SU(3)$.

We set up the gauge configurations for the standard Wilson action at $\beta = 2.5$ on the 24^4 lattice for $SU(2)$ and at $\beta = 6.2$ on the 24^4 lattice for $SU(3)$. For the $SU(2)$ case, we prepare 500 configurations every 100 sweeps after 3000 thermalization by using the heat bath method. For the $SU(3)$ case, we prepare 1500 configurations every 50 sweeps after 1000 thermalization by using the pseudo–heat bath method with the overrelaxation algorithm (20 steps per sweep). In the measurement of the Wilson loop average we apply the HYP smearing [37] for the $SU(2)$ case and the APE smearing technique [38] for the $SU(3)$ case to reduce noises and the exciting modes. In the $SU(3)$ case, the number of the smearing is determined so that the ground state overlap is enhanced [39]. We have calculated the Wilson loop average $W(L,T)$ for a rectangular loop with length T and width L to derive the potential $V(L,T)$ through the formula

$$V(L,T) = -\log\frac{W(L,T+1)}{W(L,T)}. \quad (36)$$

In the case of $SU(2)$, we investigate the Wilson loop in the adjoint representation **3** ($J = 1$). The restricted link variable $V_{x,\mu}$ is obtained by using Eq. (15) for the color field $\boldsymbol{n}_x$ which minimizes the reduction functional of Eq. (16) ($N = 2$). Figure 2 shows that the static potentials from the proposed operator of Eq. (32) for $J = 1$ and the full Wilson loop in the adjoint representation are in good agreement. The string tensions σ_{full} and σ_{rest} for the full Wilson loop and the proposed operator that are extracted by fitting the data with the Cornel potential are

$$\sigma_{\mathrm{full}} = 0.1021(234), \qquad \sigma_{\mathrm{rest}} = 0.0968(159),$$
$$\sigma_{\mathrm{rest}}/\sigma_{\mathrm{full}} \simeq 0.95. \quad (37)$$

Note that in the fundamental representation **2** ($J = 1/2$), we obtain the perfect Abelian dominance in the string tension in [33].

In the case of $SU(3)$, we investigate the Wilson loop in the fundamental representation $[0,1] = \mathbf{3}$, the adjoint representation $[1,1] = \mathbf{8}$, and the sextet representation $[0,2] = \mathbf{6}$. For each representation, we measure the Wilson loop average for possible reduction functionals, Eqs. (16), (24), and (25). Figure 3 shows the static potentials from the proposed operator of Eq. (34) for $[m,n] = [0,1], [1,1], [0,2]$ and the

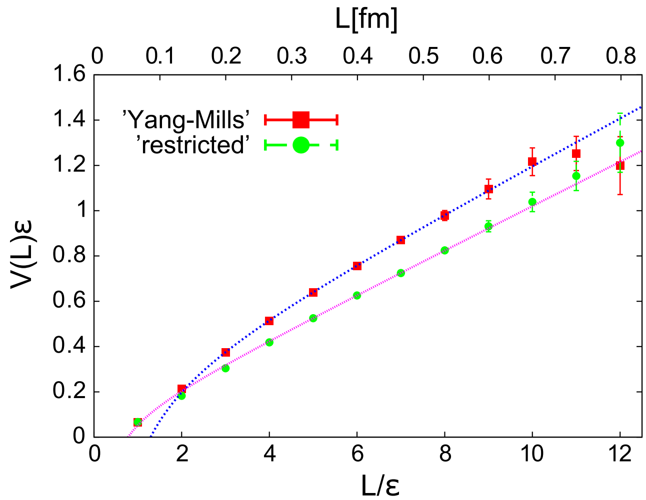

FIG. 2. The static potential $V(L, T = 6)$ between the sources in the adjoint representation of $SU(2)$ using Eq. (32) for $J = 1$ and for comparison the full Wilson loop average in the adjoint representation. The result is consistent with that of [35,40] where the same quantity is calculated by the Abelian projection method. The curves are obtained by fitting the data with the Cornel potential. The fit range is $1 \leq L/\varepsilon \leq 8$.

full Wilson loop in the fundamental, adjoint and sextet representations. Table I shows the string tensions that are extracted by fitting the data with the linear potential. Note that the data for the adjoint representation [1, 1] under the reduction condition n8 is not available, since the highest weight part of the Abelian Wilson loop in the adjoint representation [1, 1] is not determined by the reduction condition n8 of Eq. (25), as explained in the final part of the previous section. The string tensions extracted from the proposed operator reproduce nearly equal to or more than 90% of the full string tension for any of the reduction conditions Eqs. (16), (24), and (25). These results indicate that the proposed operators actually give the dominant part of the Wilson loop average.

TABLE I. The string tensions in the lattice unit in the $SU(3)$ case: the string tensions obtained under reduction conditions MA Eq. (16), n3 Eq. (24) and n8 Eq. (25), in comparison with the full string tension. The second line of each cell indicates the ratio of the string tensions which are extracted from the proposed operator and the full Wilson loop for each reduction condition. Note that the data in the slot [1, 1]-n8 is not available, because the highest weight part of the Abelian Wilson loop in the adjoint representation [1, 1] is not determined by the reduction condition n8 Eq. (25).

	Full	MA	n3	n8
[0, 1]	0.02776(2)	0.02458(1) 89%	0.02884(3) 104%	0.02544(3) 91%
[1, 1]	0.0576(1)	0.0522(1) 91%	0.062(1) 108%	…
[0, 2]	0.0647(1)	0.05691(9) 91%	0.0635(2) 98%	0.0641(6) 99%

V. CONCLUSION

In this paper, we have proposed a solution for the problem that the correct string tension extracted from the Wilson loop in higher representations cannot be reproduced if the restricted part of the Wilson loop is naively extracted by adopting the Abelian projection or the field decomposition in the same way as in the fundamental representation. We have given a prescription to construct the gauge invariant operator of Eq. (29) suitable for this purpose. We have performed numerical simulations to show that this prescription works well in the adjoint representation **3** for the $SU(2)$ color group, and the adjoint representation $[1, 1] = \mathbf{8}$ and the sextet representation $[0, 2] = \mathbf{6}$ for the $SU(3)$ color group. In comparison, we have investigated the Wilson loop for the restricted field in the fundamental representation of $SU(3)$ by using the reduction conditions Eqs. (16), (24), and (25). It should be compared to the result

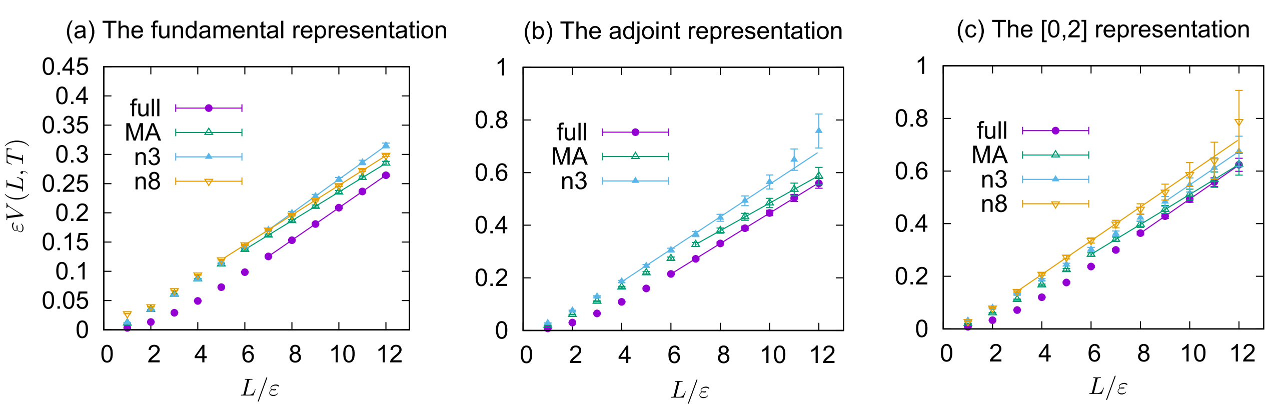

FIG. 3. The static potential $\langle V(L, T = 8)\rangle$ between the sources in (a) the fundamental [0, 1], (b) the adjoint [1, 1], and (c) the sextet [0, 2] representations of $SU(3)$ calculated using Eq. (34), in comparison with the full Wilson loop average. The legends, MA, n3, and n8 represents the measurements by using the corresponding reduction conditions Eqs. (16), (24), and (25), respectively. The straight lines are obtained by fitting the data with the linear potential. The fit range is indicated by the plotting range of the lines.

of [31] calculated by using the minimal option, which is a different option of the field decomposition where $V_{x,\mu}$ and $X_{x,\mu}$ are determined by using only $\boldsymbol{n}_x^8$.

Further studies are needed in order to establish the magnetic monopole dominance in the Wilson loop average for higher representations, supplementary to the fundamental representation for which the magnetic monopole dominance was established. In addition, we should investigate on a lattice with a larger physical spatial size because it was stated in [41] that for the sufficiently large spatial size, the Abelian part of the string tension perfectly reproduced the full string tension in the fundamental representation of $SU(3)$. It should also be checked whether the string breaking occurs for the highest weight part of the Abelian Wilson loop in the adjoint representation of $SU(3)$, similar to the $SU(2)$ case [40].

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APPENDIX A: THE DERIVATION OF EQ. (26)

The following derivation can be applied to an arbitrary compact gauge group. The two conditions that determine the decomposition, Eqs. (6) and (7), are common to all compact gauge groups.

By gauge transforming $\mathcal{V}_\mu$ by Θ in Eq. (6) and using Eq. (4), we obtain

$$[\mathcal{V}_\mu^\Theta, H_k] = 0, \qquad k = 1, \ldots, r, \tag{A1}$$

where $\mathcal{V}_\mu^\Theta \coloneqq \Theta^\dagger V_\mu \Theta + i g_{\mathrm{YM}}^{-1} \Theta^\dagger \partial_\mu \Theta$. This means that $\mathcal{V}_\mu^\Theta$ belongs to the Cartan subalgebra, and thus it is commutable with itself, $[\mathcal{V}_\mu^\Theta(x), \mathcal{V}_\nu^\Theta(y)] = 0$. Therefore by transforming $\mathcal{V}_\mu$ by Θ in Eq. (26), we obtain

$$\begin{aligned} &\frac{1}{D_R} \mathrm{tr}_R \mathcal{P} \exp\left(i g_{\mathrm{YM}} \oint \mathcal{V} \right) \\ &\quad = \frac{1}{D_R} \mathrm{tr}_R \exp\left(i g_{\mathrm{YM}} \oint \mathcal{V}^\Theta \right), \end{aligned} \tag{A2}$$

where we can omit the path ordering because $\mathcal{V}_\mu^\Theta$ is commutable. The trace of an element $\exp(i\phi_k H_k)$ of the Cartan subgroup in R is calculated as

$$\begin{aligned} \mathrm{tr}_R \exp(\phi_k H_k) &= \sum_{\mu \in \Delta_R} d_\mu \langle \mu | \exp(i\phi_k H_k) | \mu \rangle \\ &= \sum_{\mu \in \Delta_R} d_\mu \exp(i\phi_k \mu_k) \\ &= \sum_{\mu \in \Delta_R} d_\mu \exp(i \langle \mu | \phi_k H_k | \mu \rangle), \end{aligned} \tag{A3}$$

where we have used $H_k|\mu\rangle = \mu_k|\mu\rangle$. Therefore, by performing the trace in Eq. (A2), we obtain

$$(A2) = \frac{1}{D_R} \sum_{\mu \in \Delta_R} d_\mu \exp\left(i g_{\mathrm{YM}} \oint \langle \mu | \mathcal{V}^\Theta | \mu \rangle \right). \tag{A4}$$

Now Eq. (7) implies

$$\begin{aligned} &\mathrm{tr}(\mathcal{V}_\mu^\Theta H_k) = \mathrm{tr}(\mathcal{A}_\mu^\Theta H_k) \\ &\quad \Rightarrow \mathcal{V}_\mu^\Theta = \kappa \mathrm{tr}(\mathcal{A}_\mu^\Theta H_k) H_k, \end{aligned} \tag{A5}$$

where κ is the normalization of the trace and the second line follows from the fact that $\mathcal{V}_\mu^\Theta$ belongs to the Cartan subalgebra. Because for an element E of the Lie algebra,

$$\langle \mu | \mathrm{tr}(E H_k) H_k | \mu \rangle = \langle \mu | E | \mu \rangle, \tag{A6}$$

we obtain

$$(A4) = \frac{1}{D_R} \sum_{\mu \in \Delta_R} d_\mu \exp\left(i g_{\mathrm{YM}} \oint \langle \mu | \mathcal{A}^\Theta | \mu \rangle \right). \tag{A7}$$

This completes the derivation of Eq. (26).

APPENDIX B: THE DERIVATION OF EQS. (32) and (34)

First we show Eq. (32) in $SU(2)$ Yang-Mills theory. The Wilson loop for the restricted field can be written by using Abelian link variables $u_{x,\mu}$ that are defined by

$$u_{x,\mu} \coloneqq \Theta_x^\dagger V_{x,\mu} \Theta_{x+\mu}. \tag{B1}$$

Here it should be noted that an Abelian link variable $u_{x,\mu}$ belongs to the Cartan subgroup $U(1)^{N-1}$ because of Eq. (14). The (normalized) trace of the product of the Abelian link variables along a closed loop C is equal to the Wilson loop for the restricted link variables $V_{x,\mu}$ as

$$\begin{aligned} W_R[V; C] &\coloneqq \frac{1}{D_R} \mathrm{tr}_R \prod_{\langle x,\mu \rangle \in C} V_{x,\mu} \\ &= \frac{1}{D_R} \mathrm{tr}_R \prod_{\langle x,\mu \rangle \in C} u_{x,\mu}, \end{aligned} \tag{B2}$$

where D_R is the dimension of a representation R and tr_R denotes the trace in R. Now we define the untraced Abelian Wilson loop w_C, which belongs to $U(1)$, by using Eq. (B1) as

$$w_C \coloneqq \prod_{l \in C} u_l = \Theta_x^\dagger V_C \Theta_x, \tag{B3}$$

where x is the starting point of C. Let us parametrize the untraced Abelian Wilson loop as

$$\Theta_x^\dagger V_C \Theta_x = \mathrm{diag}(e^{i\theta}, e^{-i\theta}). \tag{B4}$$

Then the proposed operator of Eq. (29) in the spin-J representation is written as

$$\tilde{W}_J[V; C] = \frac{1}{2}(e^{i2J\theta} + e^{-i2J\theta}). \tag{B5}$$

Therefore we obtain

$$\frac{1}{2}\mathrm{tr}((V_C)^{2J}) = \frac{1}{2}\mathrm{tr}(\Theta_x^\dagger V_C \Theta_x)^{2J} = \tilde{W}_J[V; C]. \tag{B6}$$

This completes the derivation of Eq. (32).

In the $SU(3)$ Yang-Mills theory, let us parametrize the untraced Abelian Wilson loop as

$$w_C \coloneqq \Theta_x^\dagger V_C \Theta_x = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}), \tag{B7}$$

where $\theta_1 + \theta_2 + \theta_3 = 0 \bmod 2\pi$. Then the proposed operator of Eq. (29) in the $[m, n]$ representation is written as

$$\begin{aligned}\tilde{W}_{[m,n]}[V; C] = \frac{1}{6}(&e^{i(m\theta_1 - n\theta_3)} + e^{i(m\theta_3 - n\theta_1)} \\ &+ e^{i(m\theta_3 - n\theta_2)} + e^{i(m\theta_2 - n\theta_3)} \\ &+ e^{i(m\theta_2 - n\theta_1)} + e^{i(m\theta_1 - n\theta_2)}).\end{aligned} \tag{B8}$$

Therefore we obtain

$$\begin{aligned}\mathrm{tr}((V_C)^m)\mathrm{tr}((V_C^\dagger)^n) &= \mathrm{tr}((w_C)^m)\mathrm{tr}((w_C^\dagger)^n) \\ &= (e^{im\theta_1} + e^{im\theta_2} + e^{im\theta_3}) \\ &\quad \times (e^{-in\theta_1} + e^{-in\theta_2} + e^{-in\theta_3}) \\ &= 6\tilde{W}_{[m,n]}[V; C] \\ &\quad + e^{i(m-n)\theta_1} + e^{i(m-n)\theta_2} + e^{i(m-n)\theta_3} \\ &= 6\tilde{W}_{[m,n]}[V; C] + \mathrm{tr}((w_C)^m (w_C^\dagger)^n) \\ &= 6\tilde{W}_{[m,n]}[V; C] + \mathrm{tr}((V_C)^m (V_C^\dagger)^n).\end{aligned} \tag{B9}$$

This completes the derivation of Eq. (34).

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