Supersymmetric gradient flow in the Wess-Zumino model

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We propose a supersymmetric gradient flow equation in the four-dimensional Wess-Zumino model. The flow is constructed in two ways. One is based on the off-shell component fields and the other is based on the superfield formalism, in which the same result is provided. The obtained flow is supersymmetric because the flow time derivative and the supersymmetry transformation commute with each other. Solving the equation, we find that it has a damping oscillation with the flow time for nonzero mass, which is different from the Yang-Mills flow. The on-shell flow equation is also discussed.

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I. INTRODUCTION

Gradient flow [1,2] has been widely accepted as a new method in lattice field theory and related research areas including supersymmetry (SUSY). In a Yang-Mills flow, any correlation function is ultraviolet (UV) finite at nonzero flow time once four-dimensional Yang-Mills theory is renormalized [3]. The UV finiteness holds even for a QCD flow with an additional renormalization of the time-dependent quarks [4,5]. This property of the flow leads to a lot of interesting applications such as a proper definition of lattice energy momentum tensor [6–14]. The gradient flow approach is also useful in studying the nonlinear sigma model [15–18], nonperturbative renormalization group [19–23], and a theory with anti–de Sitter geometries [24–28]. Other interesting applications are in the references [29–34].

There have been various attempts to apply the gradient flow to SUSY theories so far. In super Yang-Mills (SYM), the most naive approach is to use a non-SUSY flow, which consists of the Yang-Mills flow and an adjoint matter flow [4] although SUSY is broken at a nonzero flow time. From this point of view, a lattice simulation of $\mathcal{N}=1$ SYM has been carried out in [35] and the regularization independent definition of the supercurrent in $\mathcal{N}=1$, 2 SYM has been given in [36,37].

A different approach can be taken, which uses a flow keeping SUSY at a nonzero flow time. Such a SUSY flow has been proposed in the superfield formalism of $\mathcal{N}=1$ SYM [38].[1] The SUSY flow equation is also given for the component fields of the Wess-Zumino gauge in a gauge covariant manner [41]. The obtained flow is supersymmetric in a sense that the flow time derivative and the super transformations commute up to a gauge transformation. The flow equation of supersymmetric O(N) nonlinear sigma model in two dimensions is also studied in [42].

The Wess-Zumino model provides a good testing ground to study the renormalization property of the SUSY theories. The gauge symmetry plays a crucial role to prove the UV finiteness in the Yang-Mills flow. As natural questions, one might ask how SUSY works in the SUSY flows and what kind of influence the nonrenormalization theorem has for the flow theory. Constructing a SUSY flow for the Wess-Zumino model, a deep understanding of the mechanism that leads to the UV finiteness of the SUSY flows could be obtained.

In this paper, we derive a SUSY flow of the four-dimensional Wess-Zumino model, which is referred as

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[1] In the context of the Langevin equation, a flow equation for $\mathcal{N}=1$ SYM was discussed in [39,40].

Wess-Zumino flow in this paper, and derive its formal solution. We give two ways of constructing the Wess-Zumino flow. One way is to use the component fields of the model directly, and the other way is to use the superfield formalism. They give the same result. Solving the Wess-Zumino flow, we find that the solutions behave as damping oscillations with respect to the flow time for nonzero mass, which is different from the Yang-Mills flow.

This paper is organized as follows. In Sec. II, we give the brief review of Wess-Zumino model in four dimensions. In Sec. III, we present two methods of constructing the Wess-Zumino flow equation. We first present the results in Sec. III A. The Wess-Zumino flow is constructed with the component fields in Sec. III B and with the superfield formalism in Sec. III C. The on-shell flow is also discussed in Sec. III D. The formal solutions of the Wess-Zumino flow are given in Sec. IV. We summarize our results in Sec. V. The convention used in this paper is shown in Appendix A.

II. WESS-ZUMINO MODEL

We make a brief review of the Wess-Zumino model, which is the simplest supersymmetric theory made of a complex scalar $A(x)$, Weyl spinors $\psi_\alpha(x)$, $\bar{\psi}_{\dot{\alpha}}(x)$, and a complex auxiliary field $F(x)$.

The action in Euclidean space is given by

$$S = \int d^4x \Big\{ |\partial_\mu A|^2 + i\psi\sigma_\mu\partial_\mu\bar{\psi} + |F|^2 - i(F(mA + gA^2) + \text{H.c.}) + \frac{1}{2}\psi\psi(m + 2gA) + \frac{1}{2}\bar{\psi}\bar{\psi}(m + 2gA)^* \Big\}, \tag{1}$$

where a real and non-negative mass $m \geq 0$ and $g \in \mathbb{C}$, which can be chosen by a phase rotation of the fields without loss of generality. The off-shell SUSY transformation that makes the action (1) invariant is defined as

$$\begin{aligned} \delta_\xi A(x) &= \xi\psi(x), \\ \delta_\xi A^*(x) &= \bar{\xi}\bar{\psi}(x), \\ \delta_\xi \psi(x) &= i\sigma_\mu\bar{\xi}\partial_\mu A(x) + i\xi F(x), \\ \delta_\xi \bar{\psi}(x) &= i\bar{\sigma}_\mu\xi\partial_\mu A^*(x) + i\bar{\xi}F^*(x), \\ \delta_\xi F(x) &= \bar{\xi}\bar{\sigma}_\mu\partial_\mu\psi(x), \\ \delta_\xi F^*(x) &= \xi\sigma_\mu\partial_\mu\bar{\psi}(x), \end{aligned} \tag{2}$$

where ξ_α and $\bar{\xi}_{\dot{\beta}}$ are two anticommuting parameters. The off-shell transformation satisfies

$$[\delta_\xi, \delta_\eta] = -i(\bar{\xi}\bar{\sigma}_\mu\eta + \xi\sigma_\mu\bar{\eta})\partial_\mu, \tag{3}$$

which is a well-known relation derived from the SUSY algebra.

The on-shell action is obtained as

$$S_{\text{on-shell}} = \int d^4x \Big\{ |\partial_\mu A|^2 + |mA + gA^2|^2 + i\psi\sigma_\mu\partial_\mu\bar{\psi} + \frac{1}{2}\psi\psi(m + 2gA) + \frac{1}{2}\bar{\psi}\bar{\psi}(m + 2gA)^* \Big\}, \tag{4}$$

integrating the auxiliary field F of the off-shell one (1). The action (4) is invariant under the on-shell SUSY transformation,

$$\begin{aligned} \delta'_\xi A(x) &= \xi\psi(x), \\ \delta'_\xi A^*(x) &= \bar{\xi}\bar{\psi}(x), \\ \delta'_\xi \psi(x) &= i\sigma_\mu\bar{\xi}\partial_\mu A(x) - \xi(mA^* + g^*A^{*2})(x), \\ \delta'_\xi \bar{\psi}(x) &= i\bar{\sigma}_\mu\xi\partial_\mu A^*(x) - \bar{\xi}(mA + gA^2)(x). \end{aligned} \tag{5}$$

Note that (5) are the first four transformations of (2) replacing $F \to i(mA^* + g^*A^{*2})$ and $F^* \to i(mA + gA^2)$.

The off-shell SUSY theory is also easily defined using the superfield formalism. Suppose that θ_α and $\bar{\theta}_{\dot{\alpha}}$ are two global Grassmann parameters. Superfield is then defined by a function $\mathcal{F}(x, \theta, \bar{\theta})$ whose SUSY transformation is given by

$$\delta_\xi \mathcal{F}(x, \theta, \bar{\theta}) = \frac{1}{\sqrt{2}}(\xi Q + \bar{\xi}\bar{Q})\mathcal{F}(x, \theta, \bar{\theta}), \tag{6}$$

where Q_α and $\bar{Q}_{\dot{\alpha}}$ are differential operators,

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \tag{7}$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha(\sigma_\mu)_{\alpha\dot{\alpha}}\partial_\mu. \tag{8}$$

For later use, we introduce other differential operators,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma_\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \tag{9}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha(\sigma_\mu)_{\alpha\dot{\alpha}}\partial_\mu, \tag{10}$$

which are covariant under SUSY transformation (6) because

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = 2i(\sigma_\mu)_{\alpha\dot{\alpha}}\partial_\mu, \tag{11}$$

and the other commutation relations are 0.

The Wess-Zumino model is given by chiral and anti-chiral superfields $\Phi(x, \theta, \bar{\theta})$ and $\bar{\Phi}(x, \theta, \bar{\theta})$, which satisfy

$$\bar{D}_{\dot\alpha}\Phi = D_\alpha\bar\Phi = 0. \tag{12}$$

The θ and $\bar\theta$ expansion of the chiral superfields can easily be written in terms of $y_\mu = x_\mu + i\theta\sigma_\mu\bar\theta$ and $\bar{y}_\mu = x_\mu - i\theta\sigma_\mu\bar\theta$ because, for instance, $\bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}$ in the y coordinate. We thus have

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + i\theta\theta F(y),$$
$$\bar\Phi(\bar{y},\bar\theta) = A^*(\bar{y}) + \sqrt{2}\bar\theta\bar\psi(\bar{y}) + i\bar\theta\bar\theta F^*(\bar{y}). \tag{13}$$

The off-shell SUSY transformation for the component fields (2) are reproduced from the definition (6) with the expansion (13).

The off-shell action (1) can also be expressed as

$$S = -\int d^4x\{\bar\Phi\Phi|_{\theta\theta\bar\theta\bar\theta} + W(\Phi)|_{\theta\theta} + W^*(\bar\Phi)|_{\bar\theta\bar\theta}\}, \tag{14}$$

where

$$W(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3. \tag{15}$$

From the construction presented above, it is obvious that the superfield action (14) is invariant under the off-shell SUSY transformation (6).

III. WESS-ZUMINO FLOW

We construct a supersymmetric flow equation in the Wess-Zumino model. The flow equation is derived in two ways; one is based on the off-shell component fields as shown in Sec. III B and the other is based on the superfield formalism as seen in Sec. III C. We find that they give the same result.

A. 4 + 1-dimensional supersymmetry and supersymmetric flow

We introduce a flow time $t(\geq 0)$ and consider the time-dependent bosonic fields $\phi(t,x)$, $\bar\phi(t,x)$, $G(t,x)$, $\bar{G}(t,x) \in \mathbb{C}$ and spinors $\chi(t,x), \bar\chi(t,x)$. The component fields of the Wess-Zumino model are replaced by those fields as follows:

$$A(x) \to \phi(t,x)$$
$$A^*(x) \to \bar\phi(t,x)$$
$$\psi(x) \to \chi(t,x)$$
$$\bar\psi(x) \to \bar\chi(t,x)$$
$$F(x) \to G(t,x)$$
$$F^*(x) \to \bar{G}(t,x), \tag{16}$$

with boundary conditions,

$$(\phi(t,x), \chi(t,x), G(t,x))|_{t=0} = (A(x), \psi(x), F(x)),$$
$$(\bar\phi(t,x), \bar\chi(t,x), \bar{G}(t,x))|_{t=0} = (A^*(x), \bar\psi(x), F^*(x)). \tag{17}$$

Note that $\bar\phi$ and $\bar{G}$ are no longer the complex conjugates of ϕ and G, respectively, for nonzero flow time.

For the flowed fields, $4+1$-dimensional SUSY transformation can be defined by replacing the fields of (2) according to (16),

$$\delta_\xi\phi = \xi\chi,$$
$$\delta_\xi\bar\phi = \bar\xi\bar\chi,$$
$$\delta_\xi\chi = i\sigma_\mu\bar\xi\partial_\mu\phi + i\xi G,$$
$$\delta_\xi\bar\chi = i\bar\sigma_\mu\xi\partial_\mu\bar\phi + i\bar\xi\bar{G},$$
$$\delta_\xi G = \bar\xi\bar\sigma_\mu\partial_\mu\chi,$$
$$\delta_\xi\bar{G} = \xi\sigma_\mu\partial_\mu\bar\chi, \tag{18}$$

where ξ and $\bar\xi$ are t-independent parameters.

It is shown that a supersymmetric flow equation is given by

$$\partial_t\phi = \Box\phi + im\bar{G} + g^*(2i\bar\phi\bar{G} - \bar\chi\bar\chi), \tag{19}$$

$$\partial_t\bar\phi = \Box\bar\phi + imG + g(2i\phi G - \chi\chi), \tag{20}$$

$$\partial_t\chi = \Box\chi + i\sigma_\mu\partial_\mu(m\bar\chi + 2g^*\bar\phi\bar\chi), \tag{21}$$

$$\partial_t\bar\chi = \Box\bar\chi + i\bar\sigma_\mu\partial_\mu(m\chi + 2g\phi\chi), \tag{22}$$

$$\partial_t G = \Box G - i\Box(m\bar\phi + g^*\bar\phi^2), \tag{23}$$

$$\partial_t\bar{G} = \Box\bar{G} - i\Box(m\phi + g\phi^2), \tag{24}$$

where $\Box = \sum_\mu\partial_\mu\partial_\mu$.

The Wess-Zumino flow tells us that each component field does not flow independently but mixes with other fields to keep SUSY. The flowed fields G and $\bar{G}$ are no longer auxiliary fields because derivative terms are in (23) and (24). It is possible to show that

$$[\partial_t, \delta_\xi] = 0, \tag{25}$$

which means that SUSY is kept at nonzero flow time. As we see in the next two sections, (25) can also be easily confirmed from the construction of the Wess-Zumino flow equation.

B. Derivation of the Wess-Zumino flow in component fields

We begin with considering a derivative of S with respect to $A(x)$. Since $\delta S/\delta A(x)$ has $\Box A^*(x)$, a gradient flow for $\phi(t,x)$ as a diffusion equation $\partial_t\phi \simeq \Box\phi$ should be defined by

$$\partial_t \phi(t,x) = -\frac{\delta S}{\delta A^*(x)}\bigg|_{\text{fields}\to\text{flowed fields}}, \tag{26}$$

where $X|_{\text{fields}\to\text{flowed fields}}$ means that the field variables in X are replaced according to (16). The first flow equation (19) is obtained from (26). Similarly, (20) is derived from a gradient flow equation as (26) by replacing $\partial_t\phi$ and δA^* by $\partial_t\bar{\phi}$ and δA, respectively.

Suppose that (25) holds for ϕ. Then the lhs of (19) becomes

$$\delta_\xi \partial_t \phi(t,x) = \partial_t \delta_\xi \phi(t,x) = \xi \partial_t \chi(t,x), \tag{27}$$

while the SUSY transformation of the rhs of (19) is

$$\delta_\xi(\text{rhs of}(19)) = \xi(\Box\chi + i\sigma_\mu\partial_\mu(m\bar{\chi} + 2g^*\bar{\phi}\bar{\chi})). \tag{28}$$

Since (27) coincides with (28), we obtain (21). We can also find (22) assuming (25) for $\bar{\phi}$ as well.

The flow equations for G and $\bar{G}$ are derived in the same manner. If (25) holds for χ and $\bar{\chi}$, we immediately find (23) and (24) by performing the SUSY transformation of the flow equations for χ and $\bar{\chi}$.

Once the flow equations are given for the scalar fields, we found that those for the other fields can be constructed by repeating the SUSY transformation (18). Since we then assumed (25) for $\phi, \bar{\phi}, \chi$ and $\bar{\chi}$, it is obvious that the obtained flows satisfy (25) for them. So all we have to do is check whether (25) holds for G and $\bar{G}$ or not.

It is enough to consider two cases: (a) δ_ξ for $\bar{\xi} = 0$ (and $\xi \neq 0$) and (b) δ_ξ for $\xi = 0$ (and $\bar{\xi} \neq 0$) since a general δ_ξ is the summation of (a) and (b). We now have $\delta_\xi\bar{\phi} = \delta_\xi G = 0$ for $\bar{\xi} = 0$. So it can be immediately shown that $[\partial_t, \delta_\xi]G = 0$ for $\bar{\xi} = 0$. Moreover, one can show that

$$[\partial_t, \delta_\xi]\eta G - [\partial_t, \delta_\eta]\xi G = 0, \tag{29}$$

from the SUSY transformation of χ using $[\partial_t, \delta_\xi] = 0$ for ϕ, χ and $[\partial_t, [\delta_\xi, \delta_\eta]] = 0$. From (29), we also confirm that $[\partial_t, \delta_\xi]G = 0$ for $\xi = 0$. We thus obtain $[\partial_t, \delta_\xi]G = 0$ for a general δ_ξ. Repeating the same argument, (25) is also true for $\bar{G}$.

C. Derivation of Wess-Zumino flow in superfield formalism

The flowed superfields are given by replacing

$$\Phi(z) \to \Psi(t,z), \qquad \bar{\Phi}(z) \to \bar{\Psi}(t,z), \tag{30}$$

with $z = (x, \theta, \bar{\theta})$. Suppose that

$$\Psi(t,z)|_{t=0} = \Phi(z), \qquad \bar{\Psi}(t,z)|_{t=0} = \bar{\Phi}(z) \tag{31}$$

as an initial condition and the SUSY transformation of $\Psi(t,z)$ and $\bar{\Psi}(t,z)$ is defined by (6).

The gradient flow should be given such that $\Phi(t,z)$ and $\bar{\Phi}(t,z)$ are chiral and antichiral superfields satisfying (12). The field variation of the chiral superfield is defined as

$$\frac{\delta}{\delta\bar{\Phi}(\bar{y},\theta)}\bar{\Phi}(\bar{y}',\theta') = \delta^4(\bar{y}-\bar{y}')\delta^2(\theta-\theta'). \tag{32}$$

It can be shown that

$$\frac{\delta S}{\delta\bar{\Phi}(x,\theta,\bar{\theta})} = \frac{1}{4}DD\Phi(x,\theta,\bar{\theta}) - \frac{\partial W^*(\bar{\Phi}(x,\theta,\bar{\theta}))}{\partial\bar{\Phi}(x,\theta,\bar{\theta})}. \tag{33}$$

Although it is natural to use $\delta S/\delta\bar{\Phi}$ for a gradient flow for Φ, such a derivative does not satisfy the supersymmetric chiral condition (12) for Φ.

It is possible to keep the condition (12) multiplying $\delta S/\delta\bar{\Phi}$ by $\bar{D}\bar{D}$. Thus a proper flow equation is

$$\partial_t\Psi(t,z) = \frac{1}{4}\bar{D}\bar{D}\frac{\delta S}{\delta\bar{\Phi}(z)}\bigg|_{\Phi(z),\bar{\Phi}(z)\to\Psi(t,z),\bar{\Psi}(t,z)}, \tag{34}$$

and similarly

$$\partial_t\bar{\Psi}(t,z) = \frac{1}{4}DD\frac{\delta S}{\delta\Phi(z)}\bigg|_{\Phi(z),\bar{\Phi}(z)\to\Psi(t,z),\bar{\Psi}(t,z)}. \tag{35}$$

Since $\bar{D}\bar{D}DD = 16\Box$, we have

$$\partial_t\Psi = \Box\Psi - \frac{1}{4}\bar{D}\bar{D}W'^*(\bar{\Psi}),$$
$$\partial_t\bar{\Psi} = \Box\bar{\Psi} - \frac{1}{4}DDW'(\Psi), \tag{36}$$

where $W'(x) = \partial W(x)/\partial x$. The supersymmetric chiral condition (12) is actually kept for any nonzero flow time because, noticing $D^3 = \bar{D}^3 = 0$ and $[D, \partial_t] = [\bar{D}, \partial_t] = 0$,

$$\partial_t(\bar{D}_{\dot{\alpha}}\Psi(t,x)) = \partial_t(D_\alpha\bar{\Psi}(t,x)) = 0, \tag{37}$$

with $\bar{D}_{\dot{\alpha}}\Psi(t=0,x) = D_\alpha\bar{\Psi}(t=0,x) = 0$.

The definitions of the gradient flow (34) and (35) are consistent with the SUSY transformation given by (6) because $[Q, \partial_t] = [\bar{Q}, \partial_t] = 0$. So the commutation relation (25) is manifestly satisfied.

Since the flowed superfields obey the supersymmetric chiral condition (12), they can also be expanded as

$$\Psi(t,y,\theta) = \phi(t,y) + \sqrt{2}\theta\chi(t,y) + i\theta\theta G(t,y),$$
$$\bar{\Psi}(t,\bar{y},\bar{\theta}) = \bar{\phi}(t,\bar{y}) + \sqrt{2}\bar{\theta}\bar{\chi}(t,\bar{y}) + i\bar{\theta}\bar{\theta}\bar{G}(t,\bar{y}). \tag{38}$$

Substituting these expansions into (36), we find that the same flow equations as (19)–(24) are obtained.

D. The on-shell flow

The relation (25) is shown to be satisfied for the off-shell supersymmetric gradient flow. We mention an on-shell case in which the auxiliary field is integrated out.

We consider an on-shell flow by replacing G and $\bar{G}$ of (19)–(22) as

$$G = i(m\bar{\phi} + g^*\bar{\phi}^2), \bar{G} = i(m\phi + g\phi^2), \qquad (39)$$

which are the equations of motion of F and F^* at $t = 0$. Here we do not consider the flow equation of G and $\bar{G}$. An on-shell SUSY transformation δ'_ξ for the flowed fields is given by (19) with the replacement (22).

The commutation relation between the flow derivative and the on-shell SUSY transformation does not vanish in general but is proportional to $\delta S/\delta h$ for $h = \psi, A, \bar{\psi}, A^*$. For instance,

$$[\partial_t, \delta'_\xi]\phi = W''^*(\bar{\phi})\xi \frac{\delta S}{\delta \psi}\bigg|_{\text{fields}\to\text{flowed fields}}. \qquad (40)$$

One can easily show that the commutators for other fields do not also vanish but satisfy the similar relations.

IV. FORMAL SOLUTION OF WESS-ZUMINO FLOW

The flowed chiral and antichiral superfields are directly coupled even in the linear part of the Wess-Zumino flow,

$$\partial_t \begin{pmatrix} \Psi_0 \\ \bar{\Psi}_0 \end{pmatrix} = \begin{pmatrix} \Box & -\frac{m}{4}\bar{D}\bar{D} \\ -\frac{m}{4}DD & \Box \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \bar{\Psi}_0 \end{pmatrix}, \qquad (41)$$

where the suffix 0 means they are solutions to the linear part of the flow equation.

To solve the formal solution of the Wess-Zumino flow, let us move on to a basis that diagonalizes the matrix of (41) as

$$\begin{pmatrix} \Pi_+ \\ \Pi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i}{4}\frac{DD}{\sqrt{-\Box}} & 1 \\ -\frac{i}{4}\frac{DD}{\sqrt{-\Box}} & 1 \end{pmatrix} \begin{pmatrix} \Psi \\ \bar{\Psi} \end{pmatrix}. \qquad (42)$$

Then the Wess-Zumino flow equation is given in terms of Π_+ and Π_-,

$$\partial_t \Pi_\pm = \left(\Box \pm im\sqrt{-\Box}\right)\Pi_\pm + R_\pm, \qquad (43)$$

where

$$R_\pm = \pm \frac{ig^*\sqrt{-\Box}}{\sqrt{2}}\bar{\Psi}^2 - \frac{g}{4\sqrt{2}}DD\Psi^2. \qquad (44)$$

Note that the initial conditions for $\Pi_\pm$ are derived from those of Ψ and $\bar{\Psi}$ via (42).

A formal solution of (43) is given by

$$\Pi_\pm(t,x) = \int d^4y \bigg\{ K_t^\pm(x-y)\Pi_\pm(0,y) + \int_0^t ds K_{t-s}^\pm(x-y)R_\pm(s,y) \bigg\}, \qquad (45)$$

where θ, $\bar{\theta}$ are abbreviated for $\Pi_\pm(t,x,\theta,\bar{\theta})$ and $R_\pm(t,x,\theta,\bar{\theta})$. Here $K_t^\pm(x)$ is a heat kernel defined by

$$K_t^\pm(x) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} e^{-t(p^2 \mp im\sqrt{p^2})}. \qquad (46)$$

Note that (46) coincides with the normal one for $m = 0$, and it still works as a damping factor for $m \neq 0$. We can actually show that (45) satisfies (43) because $K_t^\pm(x)$ provides a solution to the free part of (43).

We can also give the formal of (36) as

$$\begin{aligned} \Psi_t(p) &= C_t(p)\Phi(p) - S_t(p)\frac{\bar{D}\bar{D}}{4\sqrt{p^2}}\bar{\Phi}(p) \\ &\quad - \int_0^t ds \bigg(g^* C_{t-s}(p)\frac{\bar{D}\bar{D}}{4}(\bar{\Psi}_s \star \bar{\Psi}_s)(p) + g S_{t-s}(p) \\ &\quad \times \sqrt{p^2}(\Psi_s \star \Psi_s)(p) \bigg), \\ \bar{\Psi}_t(p) &= C_t(p)\bar{\Phi}(p) - S_t(p)\frac{DD}{4\sqrt{p^2}}\Phi(p) \\ &\quad - \int_0^t ds \bigg(g C_{t-s}(p)\frac{DD}{4}(\Psi_s \star \Psi_s)(p) + g^* S_{t-s}(p) \\ &\quad \times \sqrt{p^2}(\bar{\Psi}_s \star \bar{\Psi}_s)(p) \bigg), \end{aligned} \qquad (47)$$

where we again abbreviate θ, $\bar{\theta}$ of $\Psi_t(p,\theta,\bar{\theta})$ and $\Phi(p,\theta,\bar{\theta})$. Here D and $\bar{D}$ are the momentum representation of (10), and $C_t(p)$ and $S_t(p)$ are defined by

$$C_t(p) \equiv e^{-tp^2}\cos\left(tm\sqrt{p^2}\right), \qquad (48)$$

$$S_t(p) \equiv e^{-tp^2}\sin\left(tm\sqrt{p^2}\right), \qquad (49)$$

which come from (46) in the momentum space as $K_t^\pm(p) = C_t(p) \pm iS_t(p)$. The star symbol means the convolution integral in the momentum space,

$$(A\star B)(p) \equiv \int \frac{d^4q}{(2\pi)^4} A(q)B(p-q), \qquad (50)$$

for any functions A and B. Note that $(A\star B)(p) = (B\star A)(p)$.

We finally find the formal solutions for the component fields inserting (13) and (38) into (47),

$$\phi_t(p) = C_t(p)A(p) + \frac{i}{\sqrt{p^2}}S_t(p)F^*(p) - g\sqrt{p^2}\int_0^t dsS_{t-s}(p)(\phi_s\star\phi_s)(p)$$
$$+ g^*\int_0^t dsC_{t-s}(p)\{2i(\bar{\phi}_s\star\bar{G}_s)(p) - (\bar{\chi}_s\star\bar{\chi}_s)(p)\}, \tag{51}$$

$$\bar{\phi}_t(p) = C_t(p)A^*(p) + \frac{i}{\sqrt{p^2}}S_t(p)F(p) - g^*\sqrt{p^2}\int_0^t dsS_{t-s}(p)(\bar{\phi}_s\star\bar{\phi}_s)(p)$$
$$+ g\int_0^t dsC_{t-s}(p)\{2i(\phi_s\star G_s)(p) - (\chi_s\star\chi_s)(p)\}, \tag{52}$$

$$\chi_t(p) = C_t(p)\psi(p) - \frac{\sigma_\mu p_\mu}{\sqrt{p^2}}S_t(p)\bar{\psi}(p) - 2g^*\sigma_\mu p_\mu\int_0^t dsC_{t-s}(p)(\bar{\phi}_s\star\bar{\chi}_s)(p)$$
$$- 2g\sqrt{p^2}\int_0^t dsS_{t-s}(p)(\phi_s\star\chi_s)(p), \tag{53}$$

$$\bar{\chi}_t(p) = C_t(p)\bar{\psi}(p) - \frac{\bar{\sigma}_\mu p_\mu}{\sqrt{p^2}}S_t(p)\psi(p) - 2g\bar{\sigma}_\mu p_\mu\int_0^t dsC_{t-s}(p)(\phi_s\star\chi_s)(p)$$
$$- 2g^*\sqrt{p^2}\int_0^t dsS_{t-s}(p)(\bar{\phi}_s\star\bar{\chi}_s)(p), \tag{54}$$

$$G_t(p) = C_t(p)F(p) + i\sqrt{p^2}S_t(p)A^*(p) + ig^*p^2\int_0^t dsC_{t-s}(p)(\bar{\phi}_s\star\bar{\phi}_s)(p)$$
$$- g\sqrt{p^2}\int_0^t dsS_{t-s}(p)\{2(\phi_s\star G_s)(p) + i(\chi_s\star\chi_s)(p)\}, \tag{55}$$

$$\bar{G}_t(p) = C_t(p)F^*(p) + i\sqrt{p^2}S_t(p)A(p) + igp^2\int_0^t dsC_{t-s}(p)(\phi_s\star\phi_s)(p)$$
$$- g^*\sqrt{p^2}\int_0^t dsS_{t-s}(p)\{2(\bar{\phi}_s\star\bar{G}_s)(p) + i(\bar{\chi}_s\star\bar{\chi}_s)(p)\}. \tag{56}$$

Note that the terms with $1/\sqrt{p^2}$ are well-defined because they appear with $S_t(p)$ and $1/\sqrt{p^2}S_t(p)|_{p=0} = 0$.

One of the interesting points is that the solutions have a damping oscillation with the flow time for nonzero mass, C_t and S_t. This behavior is different from the solution of the Yang-Mills flow whose damping factor is e^{-tp^2}. In the case of $m = 0$, we have much simpler solutions because $C_t(p) = e^{-tp^2}$ and $S_t(p) = 0$.

V. SUMMARY

We have constructed a supersymmetric gradient flow equation in the four-dimensional Wess-Zumino model. The Wess-Zumino flow equation is given in two ways. One is based on the off-shell component fields in which the flow for the scalar field is given by the gradient of the action. The flow equations for the other fields are derived from it by repeating the SUSY transformation. The other way is based on the superfield formalism. The gradient flow for the chiral superfield is determined from the gradient of the action with respect to the superfield with keeping the supersymmetric chiral condition. We found that the resultant equations are the same.

The obtained flow is supersymmetric in a sense that the flow time derivative and the SUSY transformation commute with each other for nonzero flow time. On the other hand, the commutator does not vanish for the on-shell flow. The flowed components fields G and $\bar{G}$ are not auxiliary but dynamical fields because the derivative terms are provided by their flows. We have obtained the formal solution of the Wess-Zumino flow equation and find that it behaves as a damping oscillation with respect to the flow time for nonzero mass, which is different from the Yang-Mills flow.

Since we have constructed the SUSY flow for the Wess-Zumino model, we achieved the first step toward the further understanding of the mechanism that leads to the UV finiteness of SUSY gradient flows. It is interesting whether the Wess-Zumino flow shows the UV finiteness at one loop order or not. In order to show that, further studies are now in progress.

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APPENDIX: CONVENTION

The Lorentz index μ runs $\mu = 0, 1, 2, 3$. All boundary fields are defined on $\mathbb{R}^4$. The fermions ψ_α and $\bar{\psi}^{\dot{\beta}}$ transform as spinors of $SO(4) \simeq SU(2)_L \times SU(2)_R$. The spinor indices α, β take the values 1,2. We basically follow [43] as the convention of spinors, but we perform the Wick rotation $t \to -it$ from [43]. Then the auxiliary field F is also replaced as $F \to iF$. Useful identities of a Euclidean (Wick rotated) version of [43] are summarized in [41].

The antisymmetric tensors $\epsilon_{\alpha\beta}, \epsilon^{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}, \epsilon^{\dot{\alpha}\dot{\beta}}$ are defined as $\epsilon_{21} = \epsilon^{12} = \epsilon_{\dot{2}\dot{1}} = \epsilon^{\dot{1}\dot{2}} = 1$. Spinors with upper and lower indices are defined as

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}. \tag{A1}$$

Then Lorentz scalars made of two spinors are given by

$$\psi\chi \equiv \psi^\alpha\chi_\alpha, \qquad \bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \tag{A2}$$

Note that $\bar{\psi}_{\dot{\alpha}}$ is not a complex conjugate of ψ_α in Euclidean space.

The four-dimensional sigma matrices $(\sigma_\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}$ are defined by

$$\sigma_0 = \bar{\sigma}_0 = -i\mathbf{1}, \qquad \sigma_i = -\bar{\sigma}_i = \sigma^i, \tag{A3}$$

where σ^i for $i = 1, 2, 3$ are the standard Pauli matrices. We often abbreviate the spinor index such as (A2) throughout this paper. For instance, $\psi\sigma_\mu\bar{\psi}$ in the action (1) means $\psi^\alpha(\sigma_\mu)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}$. The index structure of σ_μ and $\bar{\sigma}_\mu$ can be specified as $(\sigma_\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}$. They are related to each other as

$$(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma_\mu)_{\beta\dot{\beta}}. \tag{A4}$$

See [41] for the other useful formulas after the Wick rotation.

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