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Rigorous lower bounds on the imaginary parts of the scattering amplitudes and the positions of their zeros

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On the basis of the analyticity and unitarity, rigorous lower bounds on the imaginary parts of the elastic scattering amplitudes are given in terms of the total cross section and the slope parameter. If we assume that the interaction radius R is independent of s and take $R = 1$ fm, the imaginary parts of the elastic scattering amplitudes must not have zeros for $-t \leq 0.45$ $(\mathrm{GeV}/c)^2$ at high energies. A comparison with the CERN ISR data is given.

I. INTRODUCTION

The study of the rigorous bounds based on the analyticity derived from axiomatic field theory and the unitarity of the S matrix has the Froissart bound[1,2] as the first result. The Froissart bound is an upper bound on the imaginary parts of the elastic scattering amplitudes in the forward direction at the high-energy limit. In the physical and unphysical nonforward regions, Martin[2] derived rigorous bounds on the imaginary parts of the elastic scattering amplitudes in terms of the total cross section σ_t. His lower bound is as follows:

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq \frac{P_{L+1}'(1+t/2k^2)+P_L'(1+t/2k^2)+\epsilon(2L+3)P_{L+1}(1+t/2k^2)}{(L+1)^2+\epsilon(2L+3)} \quad \text{for } 0<t<t_0, \tag{1}$$

where $L+1$ is the non-negative integer determined by

$$\frac{k^2}{4\pi}\sigma_t = (L+1)^2+\epsilon(2L+3) \quad (0 \leq \epsilon < 1). \tag{2}$$

Here s, t, and k are the c.m. energy squared, c.m. momentum transfer squared, and c.m. momentum, respectively, and t_0 is the threshold value in the analyticity domain.

Under the assumption that the imaginary parts $a_l(s)$ of the partial-wave amplitudes are monotonically decreasing functions of l, Avni and Savit[3] derived the lower bound on the imaginary parts of the amplitudes in terms of the total cross section and the slope parameter $S=\frac{1}{8}R^2$,

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq 2\frac{J_1(R\sqrt{-t})}{R\sqrt{-t}} \quad \text{for } R\sqrt{-t} \leq 3.83. \tag{3}$$

In this paper, on the basis of the analyticity derived from axiomatic field theory and the unitarity of the S matrix, we derive the rigorous lower bounds on the imaginary parts of the elastic scattering amplitudes in terms of the total cross section and the slope parameter. The bounds hold in the physical and unphysical nonforward regions. For $\pi^{\pm}p$, $K^{\pm}p$, $\bar{p}p$, and pp elastic scatterings, our bounds together with the value $R=1$ fm imply that the imaginary parts of the scattering amplitudes must not have zeros for $-t \leq 0.45$ $(\mathrm{GeV}/c)^2$ at high energies. Furthermore, under an assumption weaker than that of Avni and Savit,[3] it is shown that the same bound holds in a wider region of t. For the elastic scatterings of particles with arbitrary spin, similar bounds are obtained for the imaginary parts of the helicity-nonflip amplitudes. We compare our bound with the CERN ISR data.

Section II is devoted to the derivation of the bounds. In Sec. III a comparison with the ISR data is given.

II. DERIVATION OF THE BOUNDS

We consider the elastic two-body scattering amplitudes of spinless massive particles. In the analyticity domain derived from axiomatic field theory, the imaginary parts of the scattering amplitudes are expanded as follows:

$$\mathrm{Im}F(s,t)=\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(\cos\theta). \tag{4}$$

Here θ is the c.m. scattering angle and $\cos\theta = 1+t/2k^2$ for elastic two-body scatterings. The unitarity of the S matrix gives the constraint

$$0 \leq a_l(s) \leq 1. \tag{5}$$

Now we present the derivation of the bounds in terms of the total cross section σ_t and the slope parameter

$$S \equiv \tfrac{1}{8}R^2 \equiv \frac{d}{dt}\,\mathrm{Im}F(s,t)\Big|_{t=0} \Big/ \mathrm{Im}F(s,0)\,.$$

For convenience we set

$$\mathrm{Im}F_L(s,t) \equiv \sum_{l=0}^{L}(2l+1)a_l(s)P_l(\cos\theta)\,, \tag{6}$$

$$\mathrm{Im}F_L(s,0) = \sum_{l=0}^{L}(2l+1)a_l(s)$$

$$\equiv (2C_L+1)\alpha_L\,, \tag{7}$$

and

$$\mathrm{Im}F_L'(s,0) = \frac{1}{4k^2}\sum_{l=0}^{L}(2l+1)\,(l^2+l)a_l(s)$$

$$\equiv \frac{1}{4k^2}(2C_L+1)\,(C_L{}^2+C_L)\alpha_L\,. \tag{8}$$

Using the formula

$$P_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{1-z}{2}\right)^n \prod_{m=0}^{n-1}(\nu^2+\nu-m^2-m)\,, \tag{9}$$

we find that $P_\nu(z)$ is a convex function of the argument $\nu^2+\nu$ in the $(1-z)$ region such that $(1-z)\,(\nu^2+\nu)$ is sufficiently small:

$$G(\textstyle\sum_i \lambda_i(\nu_i{}^2+\nu_i)) \leqslant \sum_i \lambda_i G(\nu_i{}^2+\nu_i) \tag{10}$$

for

$$G(\nu^2+\nu) \equiv P_\nu(z),\ \textstyle\sum_i\lambda_i = 1,\ \text{and}\ \lambda_i \geqslant 0\,. \tag{11}$$

The existence of such a region is shown in the Appendix. Thus the following bound is obtained:

$$G(C_L{}^2+C_L) \leqslant \sum_{l=0}^{L} G(l^2+l)\frac{(2l+1)a_l(s)}{(2C_L+1)\alpha}\,, \tag{12}$$

i.e.,

$$\frac{\mathrm{Im}F_L(s,t)}{\mathrm{Im}F_L(s,0)} \geqslant P_{C_L}(\cos\theta)\,, \tag{13}$$

where use is made of Eqs. (7) and (8), and

$$C_L{}^2+C_L = \frac{\sum_{l=0}^{L}(l^2+l)\,(2l+1)a_l(s)}{\sum_{l=0}^{L}(2l+1)a_l(s)}\,. \tag{14}$$

Hence C_L is a monotonically increasing function of L. As is shown in the Appendix, $P_\nu(z)$ is a monotonically decreasing function of ν in the positive $1-z$ region such that $(1-z)\,(\nu^2+\nu)$ is sufficiently small. Then

$$\frac{\mathrm{Im}F_L(s,t)}{\mathrm{Im}F_L(s,0)} \geqslant P_{C_\infty}(\cos\theta) \tag{15}$$

for any L. Therefore we obtain the lower bound

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geqslant P_C(\cos\theta) \tag{16}$$

for sufficiently small positive $1-\cos\theta$. Here $C=C_\infty$ is given by

$$C = [\tfrac{1}{2}(kR)^2+\tfrac{1}{4}]^{1/2}-\tfrac{1}{2}\,, \tag{17}$$

which is obtained by using the definition

$$S = \tfrac{1}{8}R^2 = \frac{d}{dt}\,\mathrm{Im}F(s,t)\Big|_{t=0}\Big/ \mathrm{Im}F(s,0)\,.$$

By using the relation

$$P_\nu(\cos\theta) = J_0\left(\frac{\nu+\frac{1}{2}}{k}\sqrt{-t}\right) + \frac{-t}{4k^2}\left[\tfrac{1}{3}J_0\left(\frac{\nu+\frac{1}{2}}{k}\sqrt{-t}\right) - \frac{1}{6}\,\frac{J_1((\nu+\frac{1}{2})\sqrt{-t}/k)}{(\nu+\frac{1}{2})\sqrt{-t}/k}\right] + O((-t/4k^2)^2) \tag{18}$$

at high energies, the lower bound

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geqslant J_0(R\sqrt{-t}/\sqrt{2}) \quad \text{as } k\to\infty \text{ for } 0 \leqslant R\sqrt{-t}/\sqrt{2} \leqslant 3.83 \tag{19}$$

is obtained in the same way as Eq. (16). Here we have used the fact that $J_0(x)$ has the smallest value at $x=3.83$ and is a convex function of x^2 in the interval $0 \leqslant x^2 \leqslant (3.83)^2$.

We wish to give, numerically, the domain of k and t in which our lower bounds (16) and (19) hold. For any fixed z, we can draw the curve of the function $P_\nu(z)$ for the argument $\nu^2+\nu$ in Fig. 1. Each Legendre function $P_\nu(z)$ in the expansion (4) of the scattering amplitudes lies on that curve. For each given $\{a_0(s), a_1(s), \ldots, a_l(s), \ldots\}$, the point with the coordinate

$$\left(\frac{\sum_{l=0}^{\infty}(2l+1)a_l(s)\,(l^2+l)}{\sum_{l=0}^{\infty}(2l+1)a_l(s)}\,,\ \frac{\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(z)}{\sum_{l=0}^{\infty}(2l+1)a_l(s)}\right) = \left(C^2+C\,,\ \frac{\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(z)}{\sum_{l=0}^{\infty}(2l+1)a_l(s)}\right) \tag{20}$$

lies on a segment between certain two points on that curve. It is known from the numerical calculation that $P_\nu(z)$ is a convex function of $\nu^2+\nu$ in the interval $0 \leqslant \nu \leqslant \nu_0(z)$, where $\nu_0(z)$ is the value of ν at which $P_\nu(z)$ takes its smallest value. Thus the inequality (16) holds for $C \leqslant \nu_0(\cos\theta)$. Now we can give the domain of k and t in which our lower bound (16) holds for the interaction radius, for example, $R=1$ fm. This is shown in Fig. 2. The domain at the high-energy limit is

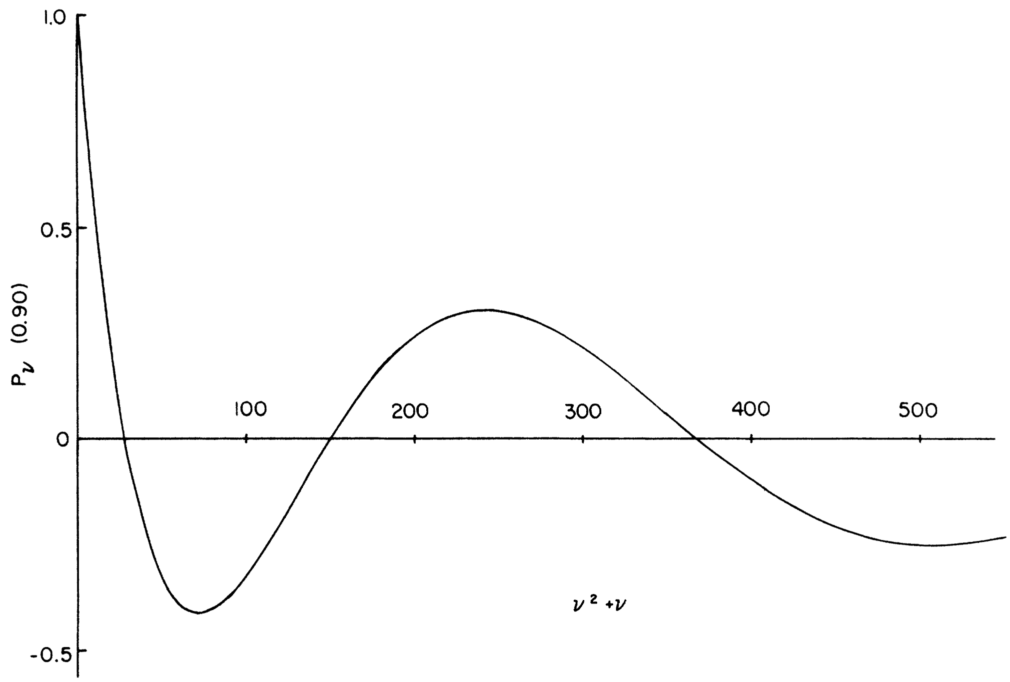

FIG. 1. The curve of the Legendre function $P_\nu(z)$ of the argument $\nu^2+\nu$ with $z=0.90$. The functional $P_\nu(0.90)$ takes its smallest value at $\nu=\nu_0=7.989$ and is convex in the interval $0\le\nu\le\nu_0$. That $P_\nu(z)$ takes its smallest value at certain $\nu=\nu_0(z)$ and is a convex function of $\nu^2+\nu$ in the interval $0\le\nu\le\nu_0(z)$ was confirmed by numerical calculations for $-0.94\le z\le 0.99$ and $0.11\le\nu^2+\nu\le 3600$.

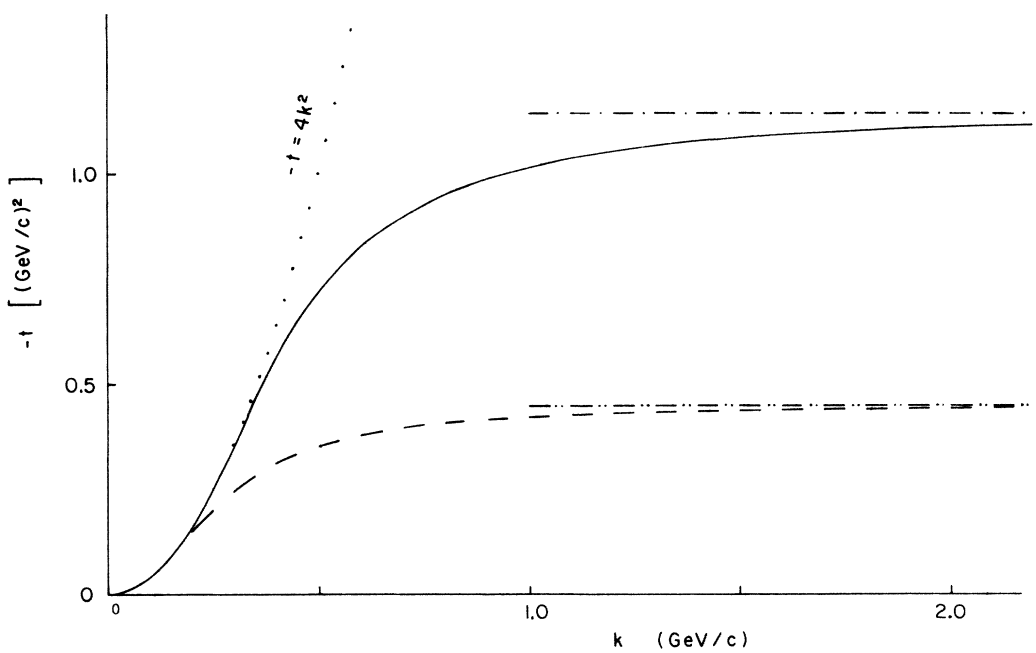

FIG. 2. The domain of k and t in which our lower bound (16) holds. The slope parameter is taken to be $R=1$ fm. The bound (16) holds in the domain between the k axis and the solid line (——). The dashed line (---) refers to the zeros of the right-hand side of Eq. (16). The dash-dot (-•-•) and dash-dot-dot (-••-••) lines are asymptotes of the solid and dashed lines, respectively.

$0 \leq -t \leq 1.14$ $(\mathrm{GeV}/c)^2$. This is nothing but the domain in which the lower bound (19) holds.

For the unphysical t, the lower bounds

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq P_C\left(1+\frac{t}{2k^2}\right) \text{ for } 0<t<t_0\,, \qquad (21)$$

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq I_0[R(\tfrac{1}{2}t)^{1/2}] \text{ as } k\to\infty \text{ for } 0<t<t_0 \qquad (22)$$

are obtained in a similar way, where C is given by Eq. (17) and t_0 is the threshold value in the analyticity domain. Here one uses the fact that $\mathrm{Im}F_L(s,t)/\mathrm{Im}F_L(s,0)$ is monotonically increasing with L for $t>0$. It should be noted that the lower bounds (16), (19), (21), and (22) are obtained only from the analyticity and unitarity and are valid even for finite energies.

From the phenomenological analysis for $\pi^{\pm}p$, $K^{\pm}p$, $\bar{p}p$, and pp elastic scatterings at high energies, the imaginary parts $a_l(s)$ of the partial-wave amplitudes appear to be monotonically decreasing functions of l. Now let us assume the following inequality:

$$a_{l-1}(s)-a_{l+1}(s) \geq 0 \text{ for any positive integer } l\,. \qquad (23)$$

By using the formula $(2l+1)P_l(z)=P_{l+1}'(z)-P_{l-1}'(z)$, the imaginary parts (4) of the scattering amplitudes are rewritten as follows:

$$\mathrm{Im}F(s,t)=\sum_{l=1}^{\infty}[a_{l-1}(s)-a_{l+1}(s)]\,P_l'(\cos\theta)\,. \qquad (24)$$

The function $2P_\nu'(z)/(\nu^2+\nu)$ is convex for the argument $\nu^2+\nu-2$ in the $(1-z)$ region such that $(1-z)(\nu^2+\nu-2)$ is sufficiently small. From analyticity, unitarity, and Eq. (23), we obtain the lower bounds,

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq 2\frac{P_{C'}'(\cos\theta)}{(C'^2+C')} \text{ for a certain small-angle region,} \qquad (25)$$

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq 2\frac{J_1(R\sqrt{-t})}{R\sqrt{-t}} \text{ as } k\to\infty \text{ for } 0\leq R\sqrt{-t}\leq 5.14\,, \qquad (26)$$

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq 2\frac{P_{C'}'(1+t/2k^2)}{(C'^2+C')} \text{ for } 0<t<t_0\,, \qquad (27)$$

$$\frac{\mathrm{Im}F(s,t)}{\mathrm{Im}F(s,0)} \geq 2\frac{I_1(R\sqrt{t})}{R\sqrt{t}} \text{ as } k\to\infty \text{ for } 0<t<t_0\,, \qquad (28)$$

where

$$C'=[(kR)^2+\tfrac{9}{4}]^{1/2}-\tfrac{1}{2}\,. \qquad (29)$$

The derivation of the bounds (25), (26), (27), and (28) is quite similar to that of the bound (16). In this way, under an assumption (23) weaker than that of Avni and Savit,[3] we obtained the same bound which holds in a region of t wider than theirs.

III. COMPARISON WITH CERN ISR DATA

Our lower bounds are the constraints on the imaginary parts of the elastic scattering amplitudes. For elastic differential cross sections, our bound (19) gives

$$\frac{\frac{d\sigma}{dt}}{\text{(optical point)}} \geq \{J_0[R(-\tfrac{1}{2}t)^{1/2}]\}^2 \text{ as } k\to\infty\,. \qquad (30)$$

Here (optical point) is $(1/16\pi)\sigma_t^2$. The same bound can be obtained also for the scatterings of particles with arbitrary spin. The domain of t in which the bound (30) holds is $0\leq -t\leq 0.45/R^2$ $(\mathrm{GeV}/c)^2$. Here R is in units of fm.

For the pp scatterings at CERN ISR (Intersecting Storage Rings) energies, the ratios of the real to imaginary parts of the helicity-nonflip amplitudes are experimentally[4] consistent with zero at $t=0$. Also the helicity-flip amplitudes are kinematically zero at $t=0$. Hence we may safely assume that the helicity-flip amplitudes and the real parts of the helicity-nonflip amplitudes are negligibly small in the neighborhood of $t=0$. Under this assumption, the ISR data for elastic differential cross sections tell us that $R=1.4$ fm at $s=2800$ GeV^2 (i.e., $k=27$ GeV/c). Then we can compare the lower bound $\{J_0[R(-\frac{1}{2}t)^{1/2}]\}^2$ with the experimental values[5] of the elastic differential cross sections for pp scatterings. This comparison is given in Fig. 3, from which it is seen that our lower bound (30) is consistent with the ISR data.

IV. SUMMARY

For the elastic scatterings of spinless particles, we obtained the lower bounds (16), (19), (21), and (22) on the imaginary parts of the scattering amplitudes in the physical and unphysical regions based on the analyticity derived from axiomatic field theory and the unitarity of the S matrix. These bounds are given in terms of the total cross section and the slope parameter. It should be noticed that our bounds are obtained based on the analyticity and unitarity only and that they are valid for any finite energies. Further assuming that the imaginary parts of the partial-wave amplitudes are decreasing functions

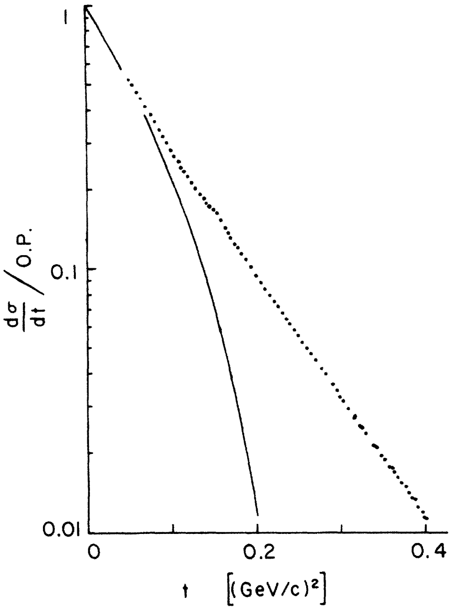

FIG. 3. Comparison of our lower bound with the CERN ISR data on the pp elastic differential cross section at 2800 GeV^2 (i.e., $k=27$ GeV/c). The solid line is for our lower bound $\{J_0[R(-\frac{1}{2}t)^{1/2}]\}^2$. O.P. means "optical point."

of l, we have derived the lower bounds (25), (26), (27), and (28). Similar bounds are obtained also for the helicity-nonflip amplitudes for the scatterings of particles with arbitrary spin.

The bound (19) implies that the imaginary parts of the scattering amplitudes must not have zeros for $0 \leq -t \leq 0.45/R^2$ (GeV/c)^2 at the high-energy limit. Similarly the bound (26) implies the nonexistence of zeros for $0 \leq -t \leq 0.55/R^2$ (GeV/c)^2. Here R is in units of fm. In particular, for $\pi^{\pm}p$, $K^{\pm}p$, $\bar{p}p$, and pp elastic scatterings, the interaction radius R is about 1 fm at high energies.

In Sec. III, our bound (30) was compared with the ISR data and found to be consistent with them.

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APPENDIX

We present a proof that the Legendre function $P_\nu(z)$ is convex with respect to the argument $\nu^2+\nu$ in a region of sufficiently small positive $1-z$. The function is written for any positive number ν as follows:

$$P_\nu(z) = F\left(-\nu,\ \nu+1,\ 1;\ \frac{1-z}{2}\right) \tag{A1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{1-z}{2}\right)^n \times \prod_{m=1}^{n}\left[\nu^2+\nu-(m-1)^2-(m-1)\right] \tag{A2}$$

$$= \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n \prod_{m=1}^{n}\left[\frac{\nu^2+\nu-(m-1)^2-(m-1)}{m^2}\right]. \tag{A3}$$

The second derivative with respect to $\nu^2+\nu$ is

$$\left(\frac{\partial}{\partial(\nu^2+\nu)}\right)^2 P_\nu(z) = \sum_{n=2}^{\infty}\left(\frac{z-1}{2}\right)^n \sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1}{(ij)^2} \times \prod_{\substack{m=1\\ m\neq i,j}}^{n}\left[\frac{\nu^2+\nu-(m-1)^2-(m-1)}{m^2}\right]. \tag{A4}$$

Then

$$(\nu^2+\nu)^2\left|\left(\frac{\partial}{\partial(\nu^2+\nu)}\right)^2 P_\nu(z) - \frac{1}{(2!)^2}\left(\frac{1-z}{2}\right)^2\right|$$
$$\leq \sum_{n=3}^{[\nu]}\left\{\left(\frac{1-z}{2}\right)(\nu^2+\nu)\right\}^n \frac{1}{(n!)^2}\,\tfrac{1}{2}n(n-1) + \sum_{n=[\nu]+1(>3)}^{\infty}\left\{\left(\frac{1-z}{2}\right)(\nu^2+\nu)\right\}^n \tfrac{1}{2}n(n-1)\frac{(\nu^2+\nu)^{[\nu]+2-n}}{([\nu]!)^2}$$
$$= O\left(\left\{\left(\frac{1-z}{2}\right)(\nu^2+\nu)\right\}^3\right), \tag{A5}$$

where $[\nu]$ is the Gauss symbol, i.e., the largest integer which is not larger than the value of ν.

The same method applies to positive $z-1$. Hence, there is a region of z such that $P_\nu(z)$ is a convex function of the argument $\nu^2+\nu$.

In a similar way we can show that in a region of sufficiently small positive $1-z$ the Legendre function $P_\nu(z)$ decreases monotonically as $\nu^2+\nu$ increases.

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