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PHYSICAL REVIEW D VOLUME 10, NUMBER 3 1 AUGUST 1974

Phenomenological approach to chiral-symmetry breaking*

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(Received 28 August 1973)

It is shown that all meson-baryon and meson-meson σ terms which can be extracted from experiment can be reconciled with the $(3,\bar{3})$ chiral-symmetry-breaking scheme, $H' = u_0 + cu_8$, provided that c assumes the value $c \sim -1.0$.

I. INTRODUCTION

In a world in which many of the symmetries of nature are approximate, it is important to know how such symmetries are broken. For the case of the algebra of currents, a framework of $SU_3 \times SU_3$ breaking has been given by Gell-Mann[1] and elaborated upon by Gell-Mann, Oakes, and Renner[2] (GMOR) and by Glashow and Weinberg.[3] In the GMOR scheme, the symmetry-breaking part of the Hamiltonian density H' takes the form $u_0 + cu_8$, where u_0 and u_8 transform according to the $(3,\bar{3}) + (\bar{3}, 3)$ representation of $SU_3 \times SU_3$. The parameter c can be determined from the "σ terms" of meson-baryon and meson-meson scattering or from the pseudoscalar mass formula. In the latter case, GMOR assume that all the pseudoscalar mesons are Goldstone bosons which obey a quadratic mass formula. They conclude that $c \approx -1.25$, quite near the $SU_2 \times SU_2$ limit $c = -\sqrt{2}$, which in turn implies that the various pion σ terms should be small. However, $\sigma(\pi N)$ has been estimated[4] to be large, and a recent study of low-energy πN and KN scattering[5,6] has unified previously conflicting analyses[7-9] and finds the large value $\sigma(\pi N)\sim 70$ MeV. As this result is in conflict with the GMOR prediction of 10–20 MeV,[4] either the $(3,\bar{3})$ scheme must be abandoned or the value of c must be changed.

In the present paper we examine the latter alternative and find (Sec. II) that the value $c\sim -1.0$ is consistent with all determinations of the meson-baryon σ terms $\sigma(\pi N)$, $\sigma_0(KN)$, and $\sigma_1(KN)$. The meson-meson σ terms also have a bearing on this question, and in Sec. III we look at $\pi\pi$ scattering, $K_{3\pi}$ decay, K_{l3} decay, $\eta'\to\eta\pi\pi$, and $\eta\to 3\pi$ decays. The most startling conclusion is that $\eta'\to\eta\pi\pi$ and $\eta\to 3\pi$ cannot be explained in the standard GMOR scheme. However, taking $c\sim -1$, corresponding to a large (~ linear) mixing angle for η and η', successfully predicts both of these rates. In Sec. IV we reexamine the pseudoscalar-meson mass formula and attempt to show that the value $c\sim -1.0$ could reasonably arise from moderate vacuum symmetry breaking combined with the kind of corrections which might well be expected from soft-kaon extrapolations. Finally, in Sec. V we point out that the values of the σ terms are such that the approach of assuming other irreducible representations for the symmetry-breaking Hamiltonian is not a viable alternative to the $(3,\bar{3})$ model. The approach of introducing small amounts of other representations in addition to a predominant $(3,\bar{3})$ term cannot be ruled out at this time, but needs to be tested against all the available σ terms in order to be made convincing.

II. BARYON σ TERMS

In the soft-meson limit, the chiral-breaking part of the scattering amplitude for $P^jB\to P^iB$ is $-\sigma^{ij}(PB)/f_p{}^2$, where the σ term is defined as

$$\sigma^{ij}(PB)=\langle B|[F_5^i, i\partial\cdot A^j]|B\rangle \ , \tag{1}$$

with F_5 the axial charge and $\partial\cdot A$ the chiral-breaking divergence of the axial current obeying

$$i\partial\cdot A^i=[F_5^i, H'] \ . \tag{2}$$

Assuming $H'=u_0+cu_8$ along with the $(3,\bar{3})+(\bar{3},3)$ commutation relations

$$\begin{aligned}[][F_5^i, u^j]&=-id^{ijk}v^k,\\ [F_5^i, v^j]&=id^{ijk}u^k \ ,\end{aligned} \tag{3}$$

one computes the πN and KN t-channel $(P\bar{P}\to B\bar{B})$ σ terms from (1) and (2) to be[2,10]

$$\begin{aligned}\sigma(\pi N)&=\sigma_0^t(\pi N)\\ &=\left(\frac{\sqrt{2}+c}{3}\right)(\sqrt{2}\,u_0+u_8)_N \ ,\end{aligned} \tag{4}$$

$$\sigma_0^t(KN)=\left(\frac{\sqrt{2}-\frac{1}{2}c}{3}\right)(\sqrt{2}\,u_0-\tfrac{1}{2}u_8)_N \ , \tag{5}$$

$$\sigma_1^t(KN)=\left(\frac{\sqrt{2}-\frac{1}{2}c}{3}\right)\left(\frac{\sqrt{3}}{2}u_3\right)_N . \tag{6}$$

We shall now show that these $(3,\bar{3})$ predictions can be made consistent with experiment.

A. $\sigma(\pi N)$

The σ term $\sigma(\pi N)$ can be extracted from the data in two independent ways. The on-shell method[4-7] consists of an extrapolation of the isotopic-even forward amplitude $(F_0^t=F^{(+)}=A^{(+)}+\nu B^{(+)})$ to the unphysical point $\nu\equiv(p'+p)\cdot(q'+q)/4m_N=0$, $t\equiv(q-q')^2=2m_\pi{}^2$ (but $q_\pi{}^2=q_\pi'{}^2=m_\pi{}^2$), where σ is related to the background amplitude $\bar{F}$ according to

$$\bar{F}^{(+)}(\nu=0, t=2m_\pi{}^2)=\sigma(\pi N; 2m_\pi{}^2)/f_\pi{}^2+O(m_\pi{}^4) \ . \tag{7}$$

Cheng and Dashen[4] used a broad-area subtraction technique to find $\bar{F}^{(+)}(0, 2m_\pi{}^2)$ and concluded that $\sigma_{\rm CD}(\pi N; 2m_\pi{}^2)\sim 110$ MeV. However, Liu and Termaseren[11] repeated this analysis using the more recent CERN 71 phase shifts and obtained

$$\sigma_{\rm LT}(\pi N; 2m_\pi{}^2)\sim 70\ \text{MeV} \ . \tag{8}$$

An alternative on-shell analysis by Höhler, Jakob, and Strauss[7] combined s- and p-wave near-threshold data with the subthreshold expansion

$$\begin{aligned}\bar{F}^{(+)}(\nu, t)&=f_1^+ +f_2^+t+f_3^+\nu^2+f_4^+\nu^2t\\ &\quad+f_5^+\nu^4+\cdots \ .\end{aligned}$$

They found $f_1^+\sim -1.6\,m_\pi{}^{-1}$ and $f_2^+\sim -1.13\,m_\pi{}^{-3}$, which, when combined with

$$\bar{F}^{(+)}(0, 2m_\pi{}^2)=f_1^+ +2m_\pi{}^2f_2^+ \tag{9}$$

and (7) implied $\sigma_{\rm HJS}(\pi N; 2m_\pi{}^2)\sim 40$ MeV for the Cabibbo[12] value of $f_\pi\sim 94$ MeV. On the other hand, a reanalysis of this approach using world-average s-wave and p-wave scattering lengths gave[5,6]

$$\begin{aligned}f_1^+&=(-1.40\pm 0.15)m_\pi{}^{-1} \ ,\\ f_2^+&=(1.27\pm 0.15)m_\pi{}^{-3} \ ,\end{aligned} \tag{10}$$

which in turn implied by way of (7) and (9) that[6a]

$$\sigma(\pi N; 2m_\pi{}^2)=(73\pm 21)\ \text{MeV} \ . \tag{11}$$

The off-shell method of determining $\sigma(\pi N)$ corresponds to the Fubini-Furlan[13] lab-frame extrapolation along a parabola to the physical threshold, $\nu=m_\pi$, $t=0$.[9] The algebra of currents then implies the t-channel relations

$$f_\pi{}^2F_0^t(\nu=m_\pi, t=0)=-\sigma(\pi N)+R_0^t(\pi N) \ , \tag{12}$$

$$f_\pi{}^2F_1^t(\nu=m_\pi, t=0)=\tfrac{1}{2}m_\pi+R_1^t(\pi N) \ , \tag{13}$$

where $\sigma(\pi N)$ is evaluated at $t=0$ and R_0^t, R_1^t are off-shell s-wave "rescattering integrals" which are dominated by higher-resonance spin-$\frac{1}{2}^-$ states. Instead of attempting to estimate R_0^t and R_1^t directly [the values in Ref. 9 badly violate the Adler-Weisberger relation (13)], we appeal to two intuitive arguments. The s-channel $I=\frac{1}{2},\frac{3}{2}$ rescattering integrals ought to be dominated by near-threshold $\frac{1}{2}^-$ resonances. Since $N'(1535)$ and $N''(1700)$ have $I=\frac{1}{2}$, $J^P=\frac{1}{2}^-$ and $\Delta(1650)$ has $I=\frac{3}{2}$, $J^P=\frac{1}{2}^-$ and all three resonances couple with approximately the same strength to the πN channel,[14] it is reasonable to assume that

$$R_{1/2}^s/R_{3/2}^s \sim 2-3 \ , \tag{14}$$

consistent with Ref. 9. Since $R_0^t=\frac{1}{3}(R_{1/2}^s+2R_{3/2}^s)$ and $R_1^t=\frac{1}{3}(R_{1/2}^s-R_{3/2}^s)$, (14) then corresponds to

$$R_0^t/R_1^t \sim 4-2.5 \ . \tag{15}$$

Alternatively we can assume the resonance-domination approximation of Ref. 15:

$$R_0^t/R_1^t \approx \frac{m^*-m_N}{m_\pi} \sim 5 \ , \tag{16}$$

corresponding to $m^*\sim 1630$ MeV, the average mass of the three $J^P=\frac{1}{2}^-$ resonances. The various determinations of the isotopic-odd πN scattering lengths have a mean value of $a_{1/2}-a_{3/2}=0.28\,m_\pi^{-1}$ (Ref. 16), or $f_\pi^2F_1^t(m_\pi,0)=84$ MeV (Ref. 17), which implies that $R_1^t=15$ MeV from (13). Taking $R_0^t/R_1^t=4\pm1$ from (15) and (16) then implies that $R_0^t=(60\pm15)$ MeV. Since the mean value of the isotopic-even scattering length is $a_{1/2}+2a_{3/2}=(-0.02\pm0.03)m_\pi^{-1}$ (Ref. 16), or $f_\pi^2F_0^t(m_\pi,0)=(-6\pm9)$ MeV, we conclude from (12) that

$$\sigma(\pi N)\approx(66\pm18)\text{ MeV} \ . \tag{17}$$

We note that the central value of $\sigma(\pi N;2m_\pi^2)$ in (8) or (11) is slightly greater than that of $\sigma(\pi N;0)$ in (17), consistent with a recent estimate[18] of their difference:

$$\sigma(\pi N;2m^2)-\sigma(\pi N)\sim 10\text{ MeV} \ . \tag{18}$$

It is important to realize that the different signs with which σ occurs in (7) and (12) mean that a factor 3.7 increase of $a_{1/2}+2a_{3/2}$ (as used in Ref. 7) will increase $\sigma(\pi N)$ by 17 MeV, and correspondingly decrease $\sigma(\pi N;2m_\pi^2)$ by the same amount, thus conflicting with (18). Given (8), (11), and (17), we conclude that $\sigma(\pi N)\sim 70$ MeV.

In passing we note that the Adler consistency condition[19] requires that $\overline{F}^{(+)}(\nu,t;q^2,q'^2)$ satisfy

$$\overline{F}^{(+)}(0,m_\pi^2;0,m_\pi^2)=0 \ . \tag{19}$$

From the determined values (10), we see that the on-shell analog to (19) is

$$\overline{F}^{(+)}(0,m_\pi^2;m_\pi^2,m_\pi^2)=f_1^+ + m_\pi^2f_2^+=(-0.13\pm0.21)m_\pi^{-1} \tag{20a}$$

$$=\sigma(\pi N)/f_\pi^2+O(m_\pi^2) \ . \tag{20b}$$

Equation (20a) is a measure of the error in the PCAC (partially conserved axial-vector current) extrapolation from $q^2=0$ to $q^2=m_\pi^2$. Equation (20b) would then seem to imply that $\sigma(\pi N)$ is small. However, the model-independent estimate of the $O(m_\pi^2)$ term, which is dominated by the 33 resonance, is large and negative.[6] This is another indication that a large (and positive) $\sigma(\pi N)$ must be correct.

B. $\sigma(KN)$

Unfortunately the on-shell method for $\sigma(KN)$ breaks down because the point $\nu=0$, $t=2m_K^2$ is on the t-channel $\pi\pi$ cut. Therefore we must follow the off-shell (lab-frame) approach of von Hippel and Kim.[9] In analogy with (12) and (13), current algebra leads to the s-channel relations

$$f_K^2F_0^s(\nu=m_K,t=0)=-\sigma_0^s(KN)+R_0^s(KN) \ , \tag{21}$$

$$f_K^2F_1^s(\nu=m_K,t=0)=-\sigma_1^s(KN)-m_K+R_1^s(KN) \ . \tag{22}$$

As in the πN case the rescattering integrals should be dominated by nearby spin-$\frac{1}{2}^-$ s- and u-channel resonances. The s-channel contributions will be small due to the absence of exotic resonances. As for the u channel, the $\Lambda'(1405)$ and $\Sigma''(1750)$ resonances have couplings which approximately cancel when crossed to the s-channel $I=0$ amplitude but add when crossed to the $I=1$ amplitude.[5,10] Thus we conclude that $R_0^s\ll R_1^s$ (Ref. 9 finds $R_0/R_1\sim-0.1$). Now experimentally $a_0^s(KN)\ll a_1^s(KN)$, so it is clear from (21) that $\sigma_0^s(KN)$ is small, which implies

$$\sigma_0^t(KN)\approx 3\sigma_1^t(KN) \ , \tag{23}$$

since[10] $\sigma_0^s=\sigma_0^t-3\sigma_1^t$. Using $(d/f)_{\text{m.s.}}\approx-\frac{1}{3}$ and $(\frac{1}{2}\sqrt{3}\,u_3)_N\approx 0.3(u_8)_N$, the model-independent result (23) implies from (5) and (6) that

$$(u_0/u_8)_N\approx 1 \tag{24}$$

in the $(3,\overline{3})$ model, independent of c.[5,10a] Combining (24) with (4), $\sigma(\pi N)\sim 70$ MeV, and using $(cu_8)_N=-209$ MeV from the baryon mass splittings, we obtain the value

$$c\approx-1.0 \ . \tag{25}$$

Assuming the von Hippel-Kim estimates of $R_0^s(KN)=29$ MeV and $R_1^s=288$ MeV further substantiates the value of $c\sim-1.0$. Taking

$$f_K = 119 \text{ MeV } [f_K/f_\pi = 1.27,\ f_+(0) = 1] ,$$

the phenomenological s-wave scattering lengths[20] $a_0^s(KN) = 0.02$ F and $a_1^s(KN) = -0.29$ F correspond, respectively, to $f_K{}^2 F_0^s(m_K, 0) = 28$ MeV and $f_K{}^2 F_1^s(m_K, 0) = -400$ MeV. Therefore from (21) and (22) we find ($\sigma_1^s = \sigma_0^t + \sigma_1^t$)

$$\sigma_0^s(KN) \approx 0 , \tag{26}$$

$$\sigma_1^s(KN) \approx 190 \text{ MeV} . \tag{27}$$

Equation (26) is equivalent to our model-independent result (23) and therefore (24), and combining the latter with (5), (6), and (27) leads to $c \sim -0.8$. Put another way, $\sigma(\pi N)$ is directly related to $\sigma_0(KN)$ and $\sigma_1(KN)$ in the $(3,\bar{3})$ model as

$$\sigma(\pi N) = \left[1 - \frac{3\Delta m_N}{5[\sigma_0^s(KN) - \sigma_1^s(KN)]}\right] \times [2\sigma_1^s(KN) - \sigma_0^s(KN)] , \tag{28}$$

where Δm_N is the SU_3 nucleon mass suppression, -209 MeV. Using (26) and (27), (26) predicts $\sigma(\pi N) \approx 130$ MeV. However, $f_+(0)$ is expected to be slightly less than 1, and the value $f_+(0) = 0.92$, which follows from the K_{e3} relation $f_+(0) \sin\theta_V = 0.21$, would lead to $\sigma(\pi N) \approx 70$ MeV. One would need an unreasonably low value of $f_+(0)$ to reconcile the $(3,\bar{3})$ relation (28) with a small value of $\sigma(\pi N)$, such as 20 MeV, which would follow from (24) along with the GMOR value $c = -1.25$.

The presence of the Y_0^* and $\Lambda\pi$ cut at the $\bar{K}N$ threshold makes any estimate of $\sigma_0(\bar{K}N)$ and $\sigma_1(\bar{K}N)$ far too model-dependent to be trusted. Likewise the unknown $I=2$ $\pi\Sigma$ scattering length and rescattering integral make any determination of $\sigma_{0,1}(\pi\Sigma)$ unreliable. However, an estimate of $\sigma_2^t(\pi\Sigma)$ may be reasonable because the isospin crossing matrix gives

$$M_2^t(\pi\Sigma) = \tfrac{1}{3}M_0^s(\pi\Sigma) - \tfrac{1}{2}M_1^s(\pi\Sigma) + \tfrac{1}{6}M_2^s(\pi\Sigma) ,$$

and therefore one might roughly neglect the unknown amplitude $M_2^s(\pi\Sigma)$. The von Hippel-Kim estimates of $R_{0,1}^s(\pi\Sigma)$ and $F_{0,1}^s(\pi\Sigma)$ at threshold then lead to $\sigma_2^t(\pi\Sigma) \approx 0$. This result is not inconsistent with the $(3,\bar{3})$ model, which demands that only $\sigma_0^t(\pi\Sigma)$ be nonvanishing.

III. MESON σ TERMS

A. $\sigma(\pi\pi)$

The meson σ terms are defined in an analogous fashion to (1):

$$\sigma^{\pi\pi}(P'P) \equiv \langle P' | [F_5^\pi, i\partial\cdot A^\pi] | P\rangle \tag{29a}$$

$$= \left(\frac{\sqrt{2}+c}{3}\right)(\sqrt{2}\,u_0 + u_8)_{P'P} \tag{29b}$$

in the $(3,\bar{3})$ model, where we will only have occasion to consider soft-pion limits. Letting $P' = P = \pi$ become soft it then follows that

$$\sigma(\pi\pi) \equiv \langle \pi | \sigma^{\pi\pi} | \pi\rangle = m_\pi{}^2 , \tag{30}$$

a result which is also implied by the Weinberg form[21] of the $\pi\pi$ low-energy amplitude,

$$f_\pi{}^2 M_{\pi\pi} = [(q_a + q_b)^2 - m_\pi{}^2]\,\delta_{ab}\,\delta_{cd} + [(q_a + q_c)^2 - m_\pi{}^2]\,\delta_{ac}\,\delta_{bd} + [(q_a + q_d)^2 - m_\pi{}^2]\,\delta_{ad}\,\delta_{bc} , \tag{31}$$

and follows from current algebra and the $(3,\bar{3})$ assumption that the $I=2$ cross-channel scattering length, $a_2^t(\pi\pi)$, vanishes. This in turn is roughly equivalent to[21] $a_0^s(\pi\pi) \approx (7m_\pi/32\pi f_\pi{}^2) \approx 0.15 m_\pi{}^{-1}$ and $a_2^s(\pi\pi) \approx -(m_\pi/16\pi f_\pi{}^2) \approx -0.04 m_\pi{}^{-1}$. The latest results from K_{l4} decay yield[22] $a_0^s(\pi\pi) = (0.17 \pm 0.13) \times m_\pi{}^{-1}$. A recent analysis of $\pi N \to \pi\pi N$ and $\pi N \to \pi\pi\Delta$ also points to the Weinberg values for $a_0^s(\pi\pi)$ and $a_2^s(\pi\pi)$.[23] Thus once again the $(3,\bar{3})$ model has been given experimental support.

B. Kaon σ terms

The σ term $\sigma(KK) \equiv \langle K | \sigma^{\pi\pi} | K\rangle$ can, in principle, be extracted from the nonleptonic decays $K \to 2\pi$ and $K \to 3\pi$. While the analysis[24] is extremely involved, it has been shown that the $K_{3\pi}$ $\Delta I = \frac{3}{2}$ Dalitz-plot slope parameters imply a value of

$$\sigma(KK) \approx 2m_\pi{}^2 . \tag{32}$$

Comparing (32) with the GMOR prediction

$$\sigma(KK) = \tfrac{1}{2} m_\pi{}^2 , \tag{33}$$

we can infer that $\sigma(KK)$, being proportional to $(\sqrt{2}+c)$ by Eq. (29b), corresponds to $c \sim -0.9$, rather than the GMOR value of -1.25. A somewhat model-dependent analysis of $K\pi$ phase shifts[25] also yields a value of $\sigma(KK)$ substantially greater than (33).

The kaon leptonic decays K_{l3} might also be expected to shed some light upon chiral-symmetry breaking. In terms of the $\langle \pi | V_\mu | K\rangle$ form factors $f_\pm(t; q_\pi{}^2)$, where $t = (q_K - q_\pi)^2$, the soft-pion Callan-Treiman relation states that[26]

$$f_+(m_K{}^2; 0) + f_-(m_K{}^2; 0) = f_K/f_\pi . \tag{34}$$

This is to be compared with the on-shell analog of the Callan-Treiman relation

$$f_+(m_K{}^2; m_\pi{}^2) + f_-(m_K{}^2; m_\pi{}^2) = (f_K/f_\pi) - (2\sigma_K/m_K{}^2 f_\pi) + O(m_\pi{}^2) , \tag{35}$$

where σ_K is the K_{l3} analog of a chiral-breaking "σ term",

$$\sigma_K \equiv \langle 0 | [F^K, i\partial\cdot A^\pi] | K\rangle \ , \tag{36}$$

and F^K is the kaon vector charge. In the $(3,\bar{3})$ model, (36) implies[27]

$$2\sigma_K/m_K{}^2 f_\pi = \frac{f_K}{f_\pi}\left(\frac{\sqrt{2}+c}{\sqrt{2}-\frac{1}{2}c}\right) \ . \tag{37}$$

If $c \sim -1.25$, then (37) is of order $m_\pi{}^2/m_K{}^2$ and negligible in (35) and pion PCAC as measured by (34) and (35) would seem to be well satisfied if the right-hand side of (35) indeed turns out to be 1.2 ($\lambda_0 \sim 0.02$, Ref. 28). However, if $c \sim -1.0$, then $2\sigma_K/m_K{}^2 f_\pi \sim 3 m_\pi{}^2/m_K{}^2 \sim 0.25$ seems to imply $\lambda_0 \sim 0$ and a 25% correction to pion PCAC in (35). Such conclusions are based upon the assumption that the $O(m_\pi{}^2)$ background in (35) is small, and we simply point out that the analogous assumption for πN scattering, Eq. (20b) is *not* valid. Unlike the case of πN scattering, there is no value of t for K_{l3} decay which would make the background correction to (35) of $O(m_\pi{}^4)$. Therefore, in order to extract the exact value of σ_K or c from (35), one must make a model-dependent estimate of the $O(m_\pi{}^2)$ background.[28a]

C. $\sigma(\eta'\eta)$

The meson σ term most sensitive to variations in c is $\sigma(\eta\eta')$, occurring in the strong decay $\eta' \to \eta\pi\pi$. The on-shell Ward identity for this process is

$$f_\pi{}^2 M(\eta' \to \eta\pi\pi) + q_\pi'^\mu \overline{M}_{\mu\nu} q_\pi^\nu = -\sigma(\eta\eta') = -\langle \eta | [F_5^\pi, i\partial\cdot A^\pi] | \eta'\rangle \ . \tag{38}$$

It will prove most convenient first to take the soft limit $q_\pi \to 0$, $q_\pi'^2 = 0$ in (38) and then correct for the slope dependence of the on-shell decay amplitude[29]:

$$M(\eta' \to \eta\pi\pi) = [\sigma(\eta\eta')/f_\pi{}^2] + b q_\pi' \cdot q_\pi \ . \tag{39}$$

The experimental parametrization of the decay amplitude is $A(1+\alpha y)$ where[29]

$$A = \frac{\sigma(\eta\eta')}{f_\pi{}^2(1+2\alpha)} \tag{40}$$

from (39), leading to the decay rate[30]

$$\Gamma(\eta' \to \eta\pi\pi) = 3(1+0.24\alpha + 0.27\alpha^2)|A|^2 \text{ keV} \ . \tag{41}$$

The combined world data for the slope parameter are[31] $\alpha = -0.08 \pm 0.03$, leading to

$$\Gamma(\eta' \to \eta\pi\pi) \approx 4.2\left(\frac{\sigma(\eta'\eta)}{f_\pi{}^2}\right)^2 \text{ keV} \ . \tag{42}$$

If η were pure octet and η' pure singlet, $\sigma(\eta\eta')$ would vanish and the decay $\eta' \to \eta\pi\pi$ would be forbidden by (40). However, it is by now clear that η-η' mixing effects are important, and so we write

$$|\eta\rangle = \cos\theta\,|\eta_8\rangle - \sin\theta\,|\eta_1\rangle \ ,$$
$$|\eta'\rangle = \sin\theta\,|\eta_8\rangle + \cos\theta\,|\eta_1\rangle \ , \tag{43}$$

where $\theta \approx -11^\circ\,(-24^\circ)$ for a quadratic (linear) pseudoscalar mass formula. Then $\sigma(\eta\eta')$ can be written as

$$\sigma(\eta'\eta) = \sigma(\eta\eta') = -\cos\theta\sin\theta\,\overline{\sigma}(\eta'\eta) \ . \tag{44}$$

The rate $\Gamma(\eta' \to \eta\pi\pi)$ can be expressed in terms of the branching ratios for $\eta' \to \eta\pi\pi$ (68%), $\eta' \to 2\gamma$ (1.9%), the width of $\pi^0 \to 2\gamma$ (7.8 eV),[14] the mixing angle θ, and the rate $\Gamma(\eta \to \gamma\gamma)$ as

$$\Gamma(\eta' \to \eta\pi\pi) \approx 36\left(\frac{X\cos\theta - 1}{\sin\theta}\right)^2 \text{ keV} \ , \tag{45}$$

where[32]

$$X \equiv \left|\frac{3 m_\pi{}^3 \Gamma(\eta \to \gamma\gamma)}{m_\eta{}^3 \Gamma(\pi \to \gamma\gamma)}\right|^{1/2} \ . \tag{46}$$

Now comparing (42) and (45), we obtain $\overline{\sigma}(\eta\eta')$:

$$\overline{\sigma}(\eta'\eta) \approx 1.3 m_\pi{}^2\left(\frac{X\cos\theta - 1}{\sin^2\theta\cos\theta}\right) \ . \tag{47}$$

It is clear from the $\sin^2\theta$ dependence of (47) that the quadratic mixing angle will make (47) at least four times larger than will the linear mixing angle. The (old) measured width of $\Gamma(\eta \to \gamma\gamma) = 1.0 \pm 0.22$ keV (Ref. 33) gives $X \approx 2.38$, leading to

$$\overline{\sigma}(\eta'\eta)_{\text{old}} \approx (11 m_\pi{}^2)_L, \ (52 m_\pi{}^2)_Q \ , \tag{48}$$

corresponding to $\Gamma_{\text{old}}(\eta' \to \eta\pi\pi) \sim 0.3$ MeV, 1.9 MeV for linear and quadratic mixing, respectively. A new measurement[34] of $\Gamma(\eta \to \gamma\gamma)$ is (0.324 ± 0.046) keV, giving $X = 1.37$, which in turn demands

$$\overline{\sigma}(\eta'\eta)_{\text{new}} \approx (2 m_\pi{}^2)_L \ , \ (14 m_\pi{}^2)_Q \ , \tag{49}$$

corresponding to $\Gamma_{\text{new}}(\eta' \to \eta\pi\pi) \sim 15$ keV, 140 keV, respectively.

In order to estimate the size of $\overline{\sigma}(\eta'\eta)$ theoretically, we further specialize to the $(3,\bar{3})$ model, where (29b) implies after some algebra,[29,35]

$$\overline{\sigma}(\eta'\eta) = -\frac{\sqrt{2}+c}{3c}\{(1-\sqrt{2}\,c)[m_1{}^2 - \sqrt{2}m_8{}^2(\sqrt{2}-c)^{-1}] + \sqrt{2}\,c m_0{}^2\} \ . \tag{50}$$

Here m_0 is the "bare" chiral singlet mass $(H_0)_{\eta_1\eta_1} = m_0{}^2$, and m_1, m_8 are the singlet, octet masses obeying $m_1{}^2 + m_8{}^2 = m_\eta{}^2 + m_{\eta'}{}^2$ with $m_1{}^2 \approx (44\, m_\pi{}^2)_L$, $(47\, m_\pi{}^2)_Q$, and $m_8{}^2 \approx (20\, m_\pi{}^2)_L$, $(17\, m_\pi{}^2)_Q$ for linear and quadratic mixing, respectively. The value $c \approx -1.25$ is associated with quadratic mixing and $c \approx -0.9$ with linear mixing (see Sec. IV). Then

(50) leads to

$$\bar{\sigma}(\eta'\eta)_Q \approx 0.08(59m_\pi^2 - m_0^2)\,, \tag{51a}$$

$$\bar{\sigma}(\eta'\eta)_L \approx 0.24(57m_\pi^2 - m_0^2)\,. \tag{51b}$$

Since the pseudoscalar octet mass breaking is upward to 600 MeV, it is reasonable to assume that m_0 is less than m_1 by a corresponding amount; viz., $350 \lesssim m_0 \lesssim 950$ MeV, or $6m_\pi^2 \lesssim m_0^2 \lesssim 47m_\pi^2$. Using this estimate, Eqs. (51) then predict

$$4m_\pi^2 \gtrsim \bar{\sigma}(\eta'\eta)_Q \gtrsim m_\pi^2\,, \tag{52a}$$

$$12m_\pi^2 \gtrsim \bar{\sigma}(\eta'\eta)_L \gtrsim 2m_\pi^2\,. \tag{52b}$$

Comparing the range of theory, (52), with the range of experiment, (48) and (49), we see that the *quadratic prediction for* $\bar{\sigma}(\eta'\eta)$ *is 3 to 13 times smaller than experiment.*[36] On the other hand, the linear prediction for $\bar{\sigma}(\eta'\eta)$ is precisely within the experimental range. While this argument is certainly no proof that linear mixing and $c \approx -0.9$ are correct, it does illustrate the sensitive dependence of $\bar{\sigma}(\eta'\eta)$ upon c and θ, and the likelihood that $-c$ is less than 1.25.

D. $\sigma(\eta\eta)$

Finally we estimate the size of the meson σ terms which describe the electromagnetic decays $\eta \to 3\pi$. Following the current-algebra analysis of Bardeen *et al.*,[37] but including all *three* pseudoscalar meson poles $\eta \xrightarrow{E} \pi_P \to \pi\pi\pi$, $\eta \to \pi\pi\eta_P \xrightarrow{E} \pi\pi\pi$, and $\eta \to \pi\pi\eta'_P \xrightarrow{E} \pi\pi\pi$, it is easy to show that for

$$\langle \pi_a \pi_b \pi_c | M | \eta\rangle = \delta_{3a}\delta_{bc}M_a + \delta_{3b}\delta_{ac}M_b + \delta_{3c}\delta_{ab}M_c\,, \tag{53}$$

the pole model gives[38,39]

$$M_{a,b,c} = \frac{\langle\pi|H_E|\eta\rangle}{m_\eta^2 - m_\pi^2}\frac{1}{f_\pi^2 m_\pi^2}[\sigma(\pi\pi) - \sigma(\eta\eta) - \chi\sigma(\eta'\eta)] \times (s_{a,b,c} - m_\pi^2)\,, \tag{54}$$

where

$$\chi = \frac{m_\eta^2 - m_\pi^2}{m_{\eta'}^2 - m_\pi^2}\frac{\langle\pi|H_E|\eta'\rangle}{\langle\pi|H_E|\eta\rangle}\,. \tag{55}$$

Here we have used $s_{a,b,c} = (p_\eta - q_{a,b,c})^2$ along with the (mixing) electromagnetic Hamiltonian[39a]

$$\langle\pi|H_E|\eta\rangle = -\frac{1}{\sqrt{3}}(\cos\theta - S_E\sin\theta) \times (m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2 - m_{\pi^0}^2)\,, \tag{56}$$

where the singlet-octet ratio S_E is taken to be of order one,[35] making χ in (54) roughly $\frac{1}{3}$. As noted in Ref. 37, the Weinberg[21,38] slope structure $s - m_\pi^2$ in (54) implies the branching ratio $\Gamma(\eta \to \pi^0\pi^0\pi^0)/\Gamma(\eta \to \pi^+\pi^-\pi^0) \approx \frac{3}{2}$ along with the Dalitz-plot slope parameter $\alpha = 0.49$, both of which are in reasonable agreement with experiment.

To find the $\eta \to 3\pi^0$ rate, we note that (54) depends upon the same $\sigma(\eta'\eta)$ σ term involved in $\eta' \to \eta\pi\pi$ decay. In a manner similar to (50), we can obtain $\sigma(\eta\eta)$ in the $(3, \bar{3})$ model as[35]

$$\sigma(\eta\eta) = -\frac{\sqrt{2} + c}{3c}\{\cos^2\theta(\sqrt{2} - 2c)(\sqrt{2} - c)^{-1}m_8^2 + \sin^2\theta[(1 - \sqrt{2}c)m_1^2 + \sqrt{2}cm_0^2] - m_\eta^2\} \tag{57}$$

leading to

$$\sigma(\eta\eta)_Q \approx 0.0026\,(212m_\pi^2 - m_0^2)\,, \tag{58a}$$

$$\sigma(\eta\eta)_L \approx 0.04\,(113m_\pi^2 - m_0^2)\,. \tag{58b}$$

Using the previously deduced constraints on m_0, $6m_\pi^2 \lesssim m_0^2 \lesssim 47\,m_\pi^2$, Eqs. (58) imply

$$0.5m_\pi^2 \gtrsim \sigma(\eta\eta)_Q \gtrsim 0.4m_\pi^2\,, \tag{59a}$$

$$4m_\pi^2 \gtrsim \sigma(\eta\eta)_L \gtrsim 2.5m_\pi^2\,, \tag{59b}$$

where it is clear that (59a) corresponds to the GMOR prediction

$$\sigma(\eta_8\eta_8) = \tfrac{1}{3}m_\pi^2\,. \tag{60}$$

Since the pion pole term in (54) gives a rate[30] of $\Gamma(\eta \to 3\pi^0) \approx 107$ eV with $\sigma(\pi\pi) = m_\pi^2$, inclusion of the η and η' pole terms in (54) leads to the predicted widths

$$10 \lesssim \Gamma(\eta \to 3\pi^0)_Q \lesssim 40 \text{ eV}\,, \tag{61a}$$

$$2000 \gtrsim \Gamma(\eta \to 3\pi^0)_L \gtrsim 300 \text{ eV}\,. \tag{61b}$$

The experimental rate is obtained from the known[14] branching ratios of $B(\eta \to 3\pi^0) = 0.33 \pm 0.03$, $B(\eta \to \gamma\gamma) = 0.38 \pm 0.01$ combined with the Primakoff measurements of $\eta \to \gamma\gamma$ (Refs. 33 and 34). This yields

$$\Gamma(\eta \to 3\pi^0)_{\text{old}} = 870 \pm 200 \text{ eV}, \quad \Gamma(\eta \to 3\pi^0)_{\text{new}} = 280 \pm 40 \text{ eV}\,. \tag{62}$$

It is again clear that the GMOR quadratic values of $\Gamma(\eta \to 3\pi^0)$, (61a), due to the σ terms (59a) or (60) are much smaller than the experimental widths in (62). Note too that the inequalities in (61) are opposite to those in (52a), which means that the most favorable (but still inadequate) value of m_0 in (52a), namely $m_0 \sim 350$ MeV is the least favorable in (61a). On the other hand, the linear values in (61b) not only cover the range of experiment, but the *correlation* with the $\eta' \to \eta\pi\pi$ rate will ultimately lead to a unique value of m_0 which could satisfy both (48) and (49), and (62).

IV. THE MESON MASS FORMULA

A. Soft-meson approach

According to the GMOR scheme, the octet of pseudoscalar mesons are Goldstone bosons obeying $\langle P|H_0|P\rangle=0$ and $(H=H_0+H')$

$$\langle P_i|H'|P_i\rangle \sim m_i^2 \,. \tag{63}$$

Requiring the consistency of all single soft-meson reductions of the above formula, GMOR were led to the approximate equality of the meson decay constants, $f_\pi \approx f_K \approx f_\eta$, and the quadratic mass formula

$$m_\pi^2+3m_{\eta_8}^2=4m_K^2 \tag{64}$$

corresponding to

$$c_Q=-\sqrt{2}\left(\frac{m_K^2-m_\pi^2}{m_K^2+\frac{1}{2}m_\pi^2}\right)=-1.25\,. \tag{65}$$

At the double soft-meson reduction level, the formula

$$-m_i^2=f_i^{-2}(d_{0ii}+cd_{8ii})[(\tfrac{2}{3})^{1/2}\langle 0|u_0|0\rangle+d_{8ii}\langle 0|u_8|0\rangle]$$
$$+\tfrac{2}{3}f_8^{-2}c\delta_{i8}\langle 0|u_8|0\rangle \tag{66}$$

then implies SU(3) invariance of the vacuum, $\langle 0|u_8|0\rangle=0$.

At first glance, this *derivation* of the quadratic meson mass formula with $c=-1.25$ seems airtight, with little freedom to vary c. One would have to make $(u_8)_0/(u_0)_0\sim-0.8$ to lower c to ~-1.0, clearly unacceptable. One might try to infer directly a linear mass formula from (66), with[40] $\langle 0|u_8|0\rangle=0$ and

$$c_L=(-\sqrt{2})\left(\frac{m_K-m_\pi}{m_K+\frac{1}{2}m_\pi}\right)=-0.90\,. \tag{67}$$

However, this would seem to require $f_K/f_\pi=(m_\pi/m_K)^{1/2}=0.53$ which is definitely unacceptable.[40]

It is usually argued that (65) reflects the validity of the chiral $SU_2\times SU_2$ soft-*pion* PCAC limit, $\sqrt{2}+c=0$. And yet in the derivation of (63)–(66) GMOR assumed soft-*kaon* (and η) PCAC as well. Why should $c\sim-\sqrt{2}$ allow the extrapolation in kaon momentum to have no corrections?

B. Hard-meson approach

An alternative approach to deriving c comes up against the kaon extrapolation problem in another form. Following Glashow and Weinberg,[3] we take the matrix element of (2) between the vacuum and a single pseudoscalar meson state, and then, using (3), obtain

$$-f_i m_i^2=[(\tfrac{2}{3})^{1/2}+cd_{ii8}]\langle 0|v_i|P_i\rangle$$
$$+(\tfrac{2}{3})^{1/2}c\langle 0|v_0|P_8\rangle\delta_{i8}\,, \tag{68}$$

which in turn implies[41]

$$c=-\sqrt{2}\left(\frac{f_K m_K^2\zeta-f_\pi m_\pi^2}{f_K m_K^2\zeta+\frac{1}{2}f_\pi m_\pi^2}\right), \tag{69}$$

where

$$\zeta\equiv\frac{\langle 0|v_\pi|\pi\rangle}{\langle 0|v_K|K\rangle}\,. \tag{70}$$

If SU(3) were a good symmetry for $\langle 0|v_i|P_i\rangle$, one could take $\zeta=1$. However, since there exists no Ademollo-Gatto–type theorem for ζ [i.e. $\zeta=1+O(\epsilon^2)$], and in view of the large differences between the only invariants m_π^2 and $m_K^2\approx 13\,m_\pi^2$, we see no fundamental reason[41a] why $\zeta=1$ (and $c=-1.29$).

The above derivation of c in terms of ζ is independent of any soft limits, but now to probe the value of ζ we appeal to the soft-pion and -kaon values

$$\langle 0|v_\pi|\pi\rangle=\alpha_\pi\langle 0|v_\pi|\pi(q=0)\rangle$$
$$=\frac{\alpha_\pi}{\sqrt{3}f_\pi}(\sqrt{2}\langle 0|u_0|0\rangle+\langle 0|u_8|0\rangle)\,, \tag{71a}$$
$$\langle 0|v_K|K\rangle=\alpha_K\langle 0|v_K|K(q=0)\rangle$$
$$=\frac{\alpha_K}{\sqrt{3}f_K}(\sqrt{2}\langle 0|u_0|0\rangle-\tfrac{1}{2}\langle 0|u_8|0\rangle)\,, \tag{71b}$$

where $\alpha_{\pi,K}$ measure the corrections due to the soft-kaon and -pion limits. Then (70) and (71) lead to

$$\zeta=\frac{\alpha_\pi f_K}{\alpha_K f_\pi}\left(\frac{1+b}{1-\frac{1}{2}b}\right)\,, \tag{72}$$

where b measures the SU_3 breaking of the vacuum

$$b\equiv\frac{\langle 0|u_8|0\rangle}{\sqrt{2}\langle 0|u_0|0\rangle}\,. \tag{73}$$

We shall show below that soft-kaon corrections to meson systems tend to underestimate the on-shell values, indicating that $\alpha_\pi/\alpha_K<1$. Furthermore, the existence of the scalar κ meson would imply that $b<0$ [see point C (iii) below].

Thus it would seem that $\zeta<1$. A value of $\zeta\sim\frac{1}{3}$ could arise from the not unreasonable kaon PCAC corrections of $\alpha_\pi/\alpha_K\sim0.6$ along with $b\sim-0.3$. Then (72) leads to $\zeta\sim\frac{1}{3}$, which in turn implies $c\sim-1.0$ from (69). This value of c is consistent with our phenomenological estimates of $\sigma(\pi N)$, $\sigma_0(KN)$, $\sigma_1(KN)$, $\sigma(KK)$, $\sigma(\eta'\eta)$, and $\sigma(\eta\eta)$ in Secs. II and III along with an $\eta'\eta$ mixing angle which is somewhat larger than that implied by the quadratic mass formula ($\theta^\circ \sim -11^\circ$) and possibly near the linear value of $\theta \sim -24^\circ$.

C. Theoretical models

(i) Theoretical models encompass a wide range of values for ζ. By analyzing the process $\pi S_K \to K^*\pi'$, where S_K and π' are massless 0^+ and 0^- spurions representing ∂V_K and v_π, respectively, Sebastian[41] finds $\zeta \approx m_\pi^2/m_K^2 \approx \frac{1}{13}$. This rather extreme result, which, however, does correspond to the value obtained in the quite different weak PCAC framework of Brandt and Preparata,[27] may result from dynamical pole-dominance assumptions and a truncated form used for a vector transition amplitude.

(ii) By consideration of positivity conditions on two-point spectral functions, Prasad[42] obtains allowed regions for c and b, which, with certain SU(3) assumptions, exclude the GMOR values. The values $c \sim -1.0$, $b \sim -0.3$ indicated above by the analysis of σ terms do, however, fall on the boundary of one of the allowed regions.

(iii) The scalar κ meson with mass m_s and strength f_s can be related to b by pole-dominating the 2-point functions, leading to[43]

$$\frac{m_s^2 f_s^2}{m_\pi^2 f_\pi^2} \approx \frac{9}{4}\left(\frac{c}{\sqrt{2}+c}\right)\left(\frac{b}{1+b}\right) . \tag{74}$$

With $c \sim -1.0$, $b \sim -0.3$, and $m_s \sim 1$ GeV, (74) implies $f_s^2/f_\pi^2 \sim 0.05$. Then $f_K/f_\pi f_+(0) = 1.27$ and the Glashow-Weinberg formula[3]

$$f_+(0) = (f_K^2 + f_\pi^2 - f_s^2)/2 f_K f_\pi \tag{75}$$

lead to the reasonable solution $f_+(0) \approx 1.0$. While we cannot claim to derive an exact value of b from (74), we can say that b should be negative and that a breaking of the vacuum of the order of $b \sim -0.3$ is not so large as to be inconsistent with (75) and $f_+(0) = 1 + O(\epsilon^2)$.

(iv) In the chiral perturbation theory of Langacker and Pagels,[44] the leading $\epsilon \ln\epsilon$ corrections to the chiral limit can be calculated. The extension of SU(3) × SU(3) (which the authors acknowledge to be marginal) implies

$$\frac{f_K}{f_\pi} - 1 = \delta + O(\epsilon) , \tag{76}$$

$$\frac{1}{\zeta} - 1 = \tfrac{1}{9}\,\delta + O(\epsilon') . \tag{77}$$

Here the δ term, of order $\epsilon \ln\epsilon$, is explicitly

$$\delta = \frac{3(m_K^2 - m_\pi^2)}{64\pi^2 f_\pi^2} \ln\frac{\Lambda}{4\mu^2} , \tag{78}$$

where μ^2 is the average meson squared mass, $\mu^2 \approx 9 m_\pi^2$, and Λ is a cutoff which may reasonably be taken as around $4 m_N^2$, in which case δ has the value of 0.20. If the $O(\epsilon)$ terms in (76) and (77) could be neglected compared with δ, then these equations would indeed imply $\zeta \approx 1$. However, in the first place $O(\epsilon \ln\epsilon)$ terms dominate over $O(\epsilon)$ terms only when ϵ is small enough for $\ln\epsilon$ to be large, whereas in practice the logarithm in (78) is only 1.65. Second, the exact form for the left-hand side of (77) should be $\ln(1/\zeta)$, which makes the difference of ζ from 1 much more sensitive to the unknown and possibly dominant $O(\epsilon)$ terms. Finally, (77) is based on the SU(3) × SU(3) extension[45] of the Weinberg $\pi\pi$ amplitude, (31), which involves the rather dubious procedure of extending the region of convergence of the power-series expansion in s, t, u beyond the 2π cut by virtue of soft-K and η limits. Furthermore, since Ref. 45 demonstrates that the SU(3)×SU(3) extension of (31) depends explicitly upon $c = -1.25$, it would seem that (77) simply reflects a consistency pattern within the GMOR scheme. In fact we have seen that $\sigma(KK)$ and $\sigma(\eta\eta)$ are consistent only with the amplitude structure $s - m_\pi^2$, which is not equivalent to the SU(3) × SU(3) extension of the Weinberg amplitude.[24,38,39] Indeed, this extension would force the $K \to 2\pi$ amplitude to vanish identically.[24,46,47]

D. Phenomenological analogies

In keeping with the phenomenological spirit of our analysis of chiral-symmetry breaking, we offer a number of examples consistent with an SU_3 breaking of $\zeta \sim \frac{1}{3}$ or with large kaon PCAC corrections:

(i) At present, all two-body meson decays, two-body high-energy charge-exchange reactions and all three-body meson decays are consistent with *either* the quadratic or linear pseudoscalar mass formula.[48]

(ii) The Cabibbo–Gell-Mann[46] theorem states that the $K \to 2\pi$ λ_6 weak matrix elements vanish in the strict SU_3 limit. At first glance, this might seem surprising since SU_3 is known to work well for nonleptonic hyperon decays, whereas the $K_s \to \pi\pi$ lifetime of 10^{-10} sec corresponds to the largest nonleptonic amplitude. A more detailed analysis,[49] however, reveals that the Cabibbo–Gell-Mann argument, which is based upon SU_3 Bose statistics, breaks down if one maintains the physical masses in the invariant amplitude $A(m_K^2, m_\pi^2, m_\pi^2)$. The $K \to 2\pi$ "paradox" is then avoided if one breaks SU_3 for these dynamical amplitudes, $A(m_K^2, m_\pi^2, m_\pi^2) \neq A(m_\pi^2, m_K^2, m_\pi^2)$. One need not destroy SU_3 completely in the sense that the SU_3 Clebsch-Gordan coefficients are still preserved. This is precisely the situation in (68)–(70).

(iii) The best process to isolate the soft-kaon PCAC corrections involving only the pseudoscalar mesons is K_{l3} decay, where the soft-kaon Callan-Treiman relation analogous to (34) is

$$f_+(t=m_\pi^2;k^2=0)-f_-(t=m_\pi^2;k^2=0)=f_\pi/f_K\approx 0.8/f_+(0)\ . \quad (79)$$

On the kaon mass shell, the combination f_+-f_- occurring in (79) can be written as

$$f_+(t=m_\pi^2;k^2=m_K^2)-f_-(t=m_\pi^2;k^2=m_K^2)\approx(1-\xi)f_+(0)\ , \quad (80)$$

where $\xi=f_-(0)/f_+(0)$ and $\lambda_\pm$ are small enough to neglect the extrapolation from $t=0$ to $t=m_\pi^2$. The 1970 compilation of Chounet and Gaillard[28] gave $\xi=-0.85\pm0.20$. More recent experiments find $\xi=0.01\pm0.04$ (Donaldson *et al.*, Ref. 28) and $\xi=-0.6\pm0.2$ (Ref. 50). If ξ should indeed turn out to be negative, then the corrections due to soft-kaon PCAC as measured by (79) and (80) could be large and in the direction of underestimating the on-shell amplitude [i.e., the right-hand side of (79) is less than the right-hand side of (80)]. As we have seen, this corresponds to $\alpha_\pi/\alpha_K<0$, which in turn lowers ζ and therefore implies $-c<1.25$.

(iv) As noted above, the $SU_3\times SU_3$ meson-meson scattering amplitudes are not in agreement with the GMOR extension[44,45] of the Weinberg formula for $\pi\pi$ scattering If they were, then the $K\to\pi\pi$ amplitude would be constrained to vanish.[24,46,47] Furthermore, the $\eta\to3\pi$ rate would become vanishingly small.[39] Finally, the $\eta'\to\eta\pi\pi$ slope and rate would not agree with experiment.

V. DISCUSSION

If our conjecture that $\zeta\sim\frac{1}{3}$ and $c\sim-1.0$ is correct, then the $SU_2\times SU_2$ limit is still a reasonable symmetry of the Hamiltonian, though the $SU_3\times SU_3$ limit is somewhat obscured by soft-kaon and -η corrections. However, in our analysis of σ terms, only pions are allowed to become soft without corrections. In the case of $\sigma(KN)$, the soft-kaon extrapolations are corrected for by the rescattering integrals in the von Hippel-Kim approach. The soft-kaon correction to $\langle 0|v_K|K\rangle$ is accounted for by α_K.

On the other hand, if for some reason one could conclusively show that $\zeta=1$, so that $c=-1.25$, then our analysis indicates that the pure $(3,\overline{3})$-breaking model might have to be abandoned. It is important to note, however, that alternative chiral-breaking schemes lead to far more drastic predictions than does the $(3,\overline{3})$ model. For example, the reasonable $SU_3\times SU_3$ octet baryon mass of 1 GeV implies[51] $\sigma(\pi N)\sim-150$ MeV in the $(6,\overline{6})$ model and $\sigma(\pi N)\sim-200$ MeV in the $(8,\overline{8})$ model.[52] Furthermore, the $I=0$ $\pi\pi$ scattering lengths are predicted to be $a_0=-0.36\ m_\pi^{-1}$ in the $(8,\overline{8})$ model[53] and $a_0=-0.08\ m_\pi^{-1}$ in the $(6,\overline{6})$ model,[54] compared with the Weinberg $(3,\overline{3})$ value, $a_0=0.15\ m_\pi^{-1}$. Finally, it has been shown[55] that the $(8,\overline{8})$ model implies an extremely large value $\Gamma_{\eta'\to\eta\pi\pi}\gtrsim1.2$ MeV, which is now ruled out by experiment,[32] $\Gamma_{\eta'\to\eta\pi\pi}<0.54$ MeV. Suffice it to say that experiment can only admit of very small admixtures of $(6,\overline{6})$ or $(8,\overline{8})$ terms in the symmetry-breaking Hamiltonian. Such admixtures have been considered,[56] but have yet to be confronted in a systematic way with the entire range of experimental data.

ACKNOWLEDGMENTS

We would like to thank L. R. Thebaud, P. C. McNamee, and G. Kalbfleisch for helpful discussions. One of us (MDS) is indebted to N. P. Samios and K.-W. Lai for their hospitality at Brookhaven National Laboratory and the other (HFJ) to S. D. Drell for his hospitality at SLAC.

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*Work supported by the U. S. Atomic Energy Commission.

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$$M(\pi N)=\delta_{ij}M_0^t+i\epsilon_{ijk}\tau_kM_1^t=\tfrac{1}{3}(\delta_{ij}+i\epsilon_{ijk}\tau_k)M_{1/2}^s+\tfrac{1}{3}(2\delta_{ij}-i\epsilon_{ijk}\tau_k)M_{3/2}^s$$

and

$$\begin{aligned} M(KN) &= M_0^t + \vec{\tau}^K\cdot\vec{\tau}^N M_1^t \\ &= \tfrac{1}{4}(1-\vec{\tau}^K\cdot\vec{\tau}^N)M_0^s + \tfrac{1}{4}(3+\vec{\tau}^K\cdot\vec{\tau}^N)M_1^s \\ &= \tfrac{1}{4}(1+\vec{\tau}^K\cdot\vec{\tau}^N)M_0^u + \tfrac{1}{4}(3-\vec{\tau}^K\cdot\vec{\tau}^N)M_1^u . \end{aligned}$$

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$$f_\pi^2 F_1^t(m_\pi, 0) = 4\pi(1+m_\pi/m_N)\tfrac{1}{3}(a_{1/2}-a_{3/2}),$$
$$f_\pi^2 F_0^t(m_\pi, 0) = 4\pi(1+m_\pi/m_N)\tfrac{1}{3}(a_{1/2}+2a_{3/2}).$$

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[36] Reference 29 also found $\sigma(\eta'\eta)$ and $\Gamma(\eta'\to\eta\pi\pi)$ to be anomalously small for the GMOR quadratic mixing scheme, but did not state that the phenomenology of the $\gamma\gamma$ decays simultaneously requires $\sigma(\eta'\eta)$ and $\Gamma(\eta'\to\eta\pi\pi)$ to be *large* for the quadratic mixing angle.

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PHYSICAL REVIEW D VOLUME 10, NUMBER 3 1 AUGUST 1974

Distribution functions for collisions at finite high energy in the fragmentation model*

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(Received 19 January 1973)

The distribution functions for collisions at finite high energy in the fragmentation model are calculated and plotted for different energies with the number of production particles taken to be finite. The results show that while the characteristic behaviors of these distribution functions in the x variable (relative longitudinal momentum in the c.m. system) are still in qualitative agreement with experimental data, those in the y variable (rapidity) change drastically at small values of y.

In a recent letter,[1] Quigg, Wang, and Yang (QWY) speculated about the distributions of the particles produced by collisions at very high energy within the fragmentation model.[2] They constructed an interesting simple model in which there is only one kind of particle and assumed

$$\sigma(l)=\frac{K}{l(l-1)},\quad l\geq 2,\quad \sigma(1)=0 \tag{1}$$

where l is the number of charged particles observed in the right hemisphere in the center-of-mass system. They found that the one- and two-particle distribution functions are

$$\rho_1(x_1)=\frac{K}{x_1},\quad \sigma_T=K\ , \tag{2}$$

$$\rho_2(x_1,x_2)=\frac{2K}{(x_1+x_2)^3}+K\delta(1-x_1-x_2)\qquad \text{for } x_1>0 \text{ and } x_2>0\ , \tag{3}$$

and

$$m_R(x_1)=\frac{1}{x_1}\ ,\ \text{with } x=\frac{p^*_\parallel}{(p^*)_{\text{incoming}}}\ . \tag{4}$$

$m_R(x_1)$ is the average number of additional charged particles in the right hemisphere for all events where a charged particle with x_1 is known to be emitted.

The above formulas were derived within the assumption that the incoming energy is becoming infinite, and consequently the number of charged particles produced in the fragmentation will presumably also become infinite. However, in practice, experiments to determine ρ_1 and ρ_2 are done with energy which, though high, is still far from being able to be considered as infinite.[3-7] Although Quigg, Wang, and Yang stressed that their model is only intended to speculate on some qualitative behaviors of the distribution functions at infinite energy, i.e., strictly speaking, their model is valid at the point $x=0$ only, experimental data obtained for these distributions are available at values of x other than zero. These data at finite energies seem to indicate that there will exist a singularity at $x=0$ for the x distribution when the energy of the experiment is increased to infinity. The experimental correlation function

$$Z(x_1,x_2)=\frac{\sigma_T\,\rho_2(x_1,x_2)}{\rho_1(x_1)\rho_1(x_2)}$$

also seems to develop a positive spike at $x_1=0+$ and $x_2=0+$. Other authors[8] have even tested the statement by QWY that the dependence of the two-body distribution functions on x_1 and x_2 is mainly