Asymptotic behavior of form factors for two- and three-body bound states

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The asymptotic power behavior of the electromagnetic form factors is examined for two- and three-body s-wave bound states, both relativistic and nonrelativistic. In the nonrelativistic case, we consider local and separable two-body potentials and we make use of the Faddeev equations in order to define the three-body bound states. For local potentials which behave as $(|\vec{k}|)^{-1-\theta}$ $(0<\theta)$ for large momentum transfer, we obtain for the asymptotic power behavior of the form factors of the two- and three-body bound states $F_2(\vec{q}^{\,2})\simeq(|\vec{q}|)^{-3-\theta}$ and $F_3(\vec{q}^{\,2})\simeq(|\vec{q}|)^{-6-2\theta}$, respectively. For separable potentials $V=g(|\vec{k}|)g(|\vec{k}'|)$ and $g(|\vec{k}|)\simeq(|\vec{k}|)^{-1/2-\theta}$ we find $F_2(\vec{q}^{\,2})\simeq(|\vec{q}|)^{-2-2\theta}$ $(0<\theta\le\frac{1}{2})$, $F_2(\vec{q}^{\,2})\simeq(|\vec{q}|)^{-2.5-\theta}$ $(\frac{1}{2}\le\theta)$, and $F_3(\vec{q}^{\,2})\simeq(|\vec{q}|)^{-5-2\theta}$, respectively. For the relativistic case, we consider the two- and three-body Bethe-Salpeter equation in the ladder approximation. We treat the spin-zero case only but we believe that our final conclusions will not be affected by the introduction of spin-$\frac{1}{2}$ particles. With an interaction which behaves as $(k^2)^{-\theta}$ $(0<\theta)$ at large momentum transfer, we obtain $F_2(q^2)\simeq(q^2)^{-1-\theta}$ and $F_3(q^2)\simeq(q^2)^{-2-2\theta}$.

I. INTRODUCTION, RESULTS, AND CONCLUSIONS

The evaluation of the electromagnetic hadron form factors has been a constant task for the last five years.[1-6] It soon became clear that the large-momentum-transfer behavior of the form factors provides a powerful means of studying the constituents of the hadrons and their dynamics. It is by now well accepted that the behaviors $F_\pi(q^2)\simeq 1/q^2$ and $F_{1p}(q^2)\simeq 1/(q^2)^2$ are compatible with the experiments.[7] This fact suggests that the pion and the nucleon certainly are of a different nature as far as the electromagnetic interactions are concerned. It seems also to suggest that the pion is less composite than the nucleon because of the faster decrease of the proton form factor. Recently, the previous behaviors have been derived from the minimal quark structure of the pion and the proton;[8,9] so far, however, the three-particle bound state has not been treated relativistically in a convincing way, and this leaves the question open whether the underlying two- and three-particle structure can explain the different behavior of the two form factors.

It is the aim of this paper to investigate the large-q^2 behavior of the form factors of the two- and three-particle s-wave bound states in a systematic way, both in relativistic and nonrelativistic theories. Throughout the paper we consider power behaviors only, neglecting possible logarithmic factors. Here, in a first approach, we restrict ourselves to spinless constituents. We do not believe that the case of spin-$\frac{1}{2}$ constituents makes a real difference in our final conclusions. This case will be discussed elsewhere.[10]

We shall consider the potential scattering case (Sec. II) for two main reasons. First, many features of composite-particle models can be explained by means of the nonrelativistic quark model;[11] moreover, the Bethe-Salpeter equation in the ladder approximation reduces to a nonrelativistic form in the large-momenta limit, as it can be recovered from various (equivalent) three-dimensional equations.[12-15] The second good reason for studying the potential theory is the firm mathematical ground on which the nonrelativistic three-particle theory in the form of the Faddeev equations[16] is based (we do not consider three-particle forces).[17]

For both two- and three-particle cases, we shall assume the two-body local potentials

$$V(|\vec{k}|)\underset{|\vec{k}|\to\infty}{\simeq}(|\vec{k}|)^{-1-\theta},\quad \theta>0$$

and the separable potentials

$$V(\vec{k},\vec{k}')=g(|\vec{k}|)g(|\vec{k}'|)$$

with

$$g(|\vec{k}|)\underset{|\vec{k}|\to\infty}{\simeq}(|\vec{k}|)^{-1/2-\theta},\quad \theta>0\,.$$

Our choice of the potentials is determined by simple reasons. For the local potentials, the limiting behavior $(|\vec{k}|)^{-1}$ is characteristic of the singular potential $(-\lambda/r^2)$ which produces the unpleasant feature of a wave-function falloff depending on the coupling constant.[18,19] On the other hand, an even more singular potential gives rise to the exponential decrease of both the wave function and the form factor,[20] and this does not seem to be the physical case. As far as the separable potential is concerned, the choice $\theta > 0$ is imposed by the very existence of scattering processes. The use of nonlocal potentials is suggested both by the existence of tensor forces in the spin-$\frac{1}{2}$ case and by the structure of the relativistic potential as recovered in the three-dimensional version of the Bethe-Salpeter equation.[10,12-15]

Our results are as follows. For the two-body and three-body bound-state form factors we obtain

$$F_2(\vec{q}^2) \simeq (|\vec{q}|)^{-3-\theta}$$

and

$$F_3(\vec{q}^2) \simeq (|\vec{q}|)^{-6-2\theta}$$

with local potentials, whereas we obtain

$$F_2(\vec{q}^2) \simeq (|\vec{q}|)^{-2-2\theta} \quad (0 < \theta \leq \tfrac{1}{2}),$$

$$F_2(\vec{q}^2) \simeq (|\vec{q}|)^{-2.5-\theta} \quad (\tfrac{1}{2} \leq \theta),$$

and

$$F_3(\vec{q}^2) \simeq (|\vec{q}|)^{-5-2\theta}$$

with separable potentials.

In the framework of relativistic theories, we consider (Sec. III) the s-wave bound states of two and three particles described by the two-body Bethe-Salpeter equation in the ladder approximation (Sec. III A) and by the relativistic Faddeev equations (Sec. III B).[12-15,21] We shall assume a two-body interaction of the form

$$V(k) \underset{k^2 \to \infty}{\simeq} (k^2)^{-\theta}, \quad \theta > 0.$$

Our interactions correspond to the $\lambda\varphi^3$ theory for $\theta = 1$ and to the $\lambda\varphi^4$ theory for the limiting case $\theta = 0$. For the latter case it has been proved[2,4,19] that the high-momentum-transfer behavior of the two-body wave functions and form factors depends on the coupling constant, as in the singular $(-\lambda/r^2)$ potential.

Our results for the asymptotic behavior of the two- and three-body form factors are $F_2(q^2) \simeq (q^2)^{-1-\theta}$, $F_3(q^2) \simeq (q^2)^{-2-2\theta}$.

Since the $\lambda\varphi^4$ theory leads to that strange dependence on the coupling constant, we define the physical form factors as given by our superrenormalizable interaction in the limit $\theta \to 0$; the asymptotic behavior of our "pion" and "nucleon" form factors turns out to be $(q^2)^{-1}$ and $(q^2)^{-2}$, respectively.

The spin-$\frac{1}{2}$ constituents, which are more interesting for the physical situation, present some technical difficulties: Apart from the complicated spin structure of the three-body wave function, there appears a delicate region of integration, so that one has to be more careful than in the spin-zero case. However, we do not agree with Ref. 4, note 25, where it is claimed that the consistency argument, widely applied in our paper, does not work for superrenormalizable interactions.

Finally, it is worthwhile to remark that our results are in agreement with the predictions given in Refs. 8 and 9. Furthermore, our wave functions turn out to be integrable, as was assumed in Ref. 8 as a crucial hypothesis.

II. POTENTIAL SCATTERING

In the framework of a potential-scattering theory we shall discuss the asymptotic behavior of the two- and three-body bound-state form factors at large momentum transfer; we shall consider s-wave bound states only. Furthermore, in order to simplify things, we shall always assume that only one particle is charged. Let us start with the two-body case. Here the charge form factor reads

$$F_2(\vec{q}^2) = \int d\vec{q}\,'\, \psi^*(\vec{q}\,')\psi(\vec{q}\,' - \vec{q}), \tag{1}$$

where the wave function ψ satisfies the homogeneous Schrödinger equation

$$\psi(\vec{q}) = \frac{1}{q^2 - E} \int d\vec{k}\, V(\vec{q} - \vec{k})\psi(\vec{k}). \tag{2}$$

If we now consider a central potential which behaves at large $|\vec{k}|$ as

$$V(\vec{k}) \simeq \frac{1}{|\vec{k}|^{1+\theta}}, \quad \theta > 0 \tag{3}$$

we get the following behavior for ψ and F_2:

$$\begin{aligned} \psi(\vec{q}) &\underset{|\vec{q}| \to \infty}{\simeq} \frac{1}{|\vec{q}|^{3+\theta}}, \\ F_2(\vec{q}^2) &\underset{|\vec{q}| \to \infty}{\simeq} \frac{1}{|\vec{q}|^{3+\theta}}. \end{aligned} \tag{4}$$

In the limiting case $\theta = 0$ which corresponds to the potential $(-\lambda/r^2)$ the form factor behaves like $(|\vec{q}|)^{-2-2(1/4-\lambda)^{1/2}}$ $(0 < \lambda < \frac{1}{4})$ with an explicit dependence on the coupling constant λ (a similar phenomenon occurs in the Bethe-Salpeter equation[2,4,19]). With an even more singular potential, the wave function (and hence the form factor) becomes exponentially decreasing.[20] In conclusion, with a central potential, the desired $1/q^2$ behavior of the "pion" form factor is achieved only with the singular potential $(-\lambda/r^2)$ and only in the particular limit $\lambda \to \frac{1}{4}$.

For the three-body case, we consider the Faddeev equations[16] with minor changes in the notations. Let $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$ be the three-momenta of the three particles and let us introduce the new variables

$$\vec{Q}=\vec{p}_1+\vec{p}_2+\vec{p}_2\,,$$
$$\vec{k}_1=\frac{m_3\vec{p}_2-m_2\vec{p}_3}{m_2+m_3}\,, \tag{5}$$
$$\vec{q}_1=\frac{(m_2+m_3)\vec{p}_1-m_1(\vec{p}_2+\vec{p}_3)}{m_1+m_2+m_3}$$

and their cyclic permutations $\vec{k}_2, \vec{q}_2$ and $\vec{k}_3, \vec{q}_3$. $\vec{Q}$ is the total momentum, $\vec{k}_1$ is the relative momentum between the particles 2 and 3, and $\vec{q}_1$ is the relative momentum of the particle 1 with respect to the cluster 2-3. These variables are the most suitable ones for our purposes and any pair $\{\vec{k}_i, \vec{q}_i\}$ can be used for the description of the system. From now on we shall assume equal masses and $m=1$. For practical purposes, we write down some relations between the different variables:

$$\vec{p}_1=\tfrac{1}{3}\vec{Q}+\vec{q}_1\,,\quad \vec{p}_2=\tfrac{1}{3}\vec{Q}-\tfrac{1}{2}\vec{q}_1+\vec{k}_1\,,\quad \vec{p}_3=\tfrac{1}{3}\vec{Q}-\tfrac{1}{2}\vec{q}_1-\vec{k}_1\,,$$
$$\vec{q}_2=-\tfrac{1}{2}\vec{q}_1+\vec{k}_1\,,\quad \vec{k}_2=-\tfrac{3}{4}\vec{q}_1-\tfrac{1}{2}\vec{k}_1\,,$$
$$\vec{q}_3=-\tfrac{1}{2}\vec{q}_1-\vec{k}_1\,,\quad \vec{k}_3=\tfrac{3}{4}\vec{q}_1-\tfrac{1}{2}\vec{k}_1\,, \tag{6}$$
$$\vec{q}_1+\vec{q}_2+\vec{q}_3=0\,.$$

We assume that the particle 1 only is charged; then the form factor reads

$$F_3(\vec{q}^{\,2})=\int\int d\vec{k}_1'\,d\vec{q}_1'\,\psi^*_{\vec{Q}}(\vec{k}_1',\vec{q}_1')\psi_{\vec{Q}+\vec{q}}(\vec{k}_1',\vec{q}_1'-\vec{q})\,, \tag{7}$$

where $\psi=\psi^1+\psi^2+\psi^3$ and ψ^i are the Faddeev components satisfying the equation

$$\begin{bmatrix}\psi^1\\ \psi^2\\ \psi^3\end{bmatrix}=-G_0(E)\begin{bmatrix}0 & T_1(E) & T_1(E)\\ T_2(E) & 0 & T_2(E)\\ T_3(E) & T_3(E) & 0\end{bmatrix}\begin{bmatrix}\psi^1\\ \psi^2\\ \psi^3\end{bmatrix}\,. \tag{8}$$

Here $G_0(E)=1/(H_0-E)$, where H_0 is the free three-body Hamiltonian and E (the mass of the three-body ground state) is below any threshold. $T_i(E)$ is the two-body scattering operator between particles j and k $(i\neq j\neq k)$.

In order to evaluate the asymptotic behaviors, we now introduce the "vertex function" φ:

$$\varphi=\varphi^1+\varphi^2+\varphi^3\,,\quad \psi^i=G_0\varphi^i\,, \tag{9}$$

and we consider the once-iterated Faddeev equations:

$$\begin{aligned}\varphi^1&=T_1G_0T_2G_0\varphi^1+T_1G_0T_3G_0\varphi^1\\&\quad+T_1G_0T_2G_0\varphi^3+T_1G_0T_3G_0\varphi^2\,,\\ \varphi^2&=T_2G_0T_1G_0\varphi^2+T_2G_0T_3G_0\varphi^2\\&\quad+T_2G_0T_1G_0\varphi^3+T_2G_0T_3G_0\varphi^1\,,\\ \varphi^3&=T_3G_0T_1G_0\varphi^3+T_3G_0T_2G_0\varphi^3\\&\quad+T_3G_0T_1G_0\varphi^2+T_3G_0T_2G_0\varphi^1\,.\end{aligned} \tag{10}$$

The first term of the first equation reads explicitly

$$\begin{aligned}\varphi^1_{\vec{Q}}(\vec{k}_1,\vec{q}_1)=\int\int d\vec{k}_2'\,d\vec{q}_2'\,t_1(\vec{k}_1,\vec{q}_2'+\tfrac{1}{2}\vec{q}_1;E-\tfrac{3}{4}\vec{q}_1^{\,2}-\tfrac{1}{6}\vec{Q}^2)\frac{1}{\vec{q}_1^{\,2}+\vec{q}_1\cdot\vec{q}_2'+\vec{q}_2'^{\,2}-E+\tfrac{1}{6}\vec{Q}^2}\\ \times t_2(-\vec{q}_1-\tfrac{1}{2}\vec{q}_2',\vec{k}_2';E-\tfrac{3}{4}\vec{q}_1^{\,2}-\tfrac{1}{6}\vec{Q}^2)\frac{1}{\vec{q}_2'^{\,2}+\vec{q}_2'\cdot\vec{q}_1'+\vec{q}_1'^{\,2}-E+\tfrac{1}{6}\vec{Q}^2}\varphi^1_{\vec{Q}}(\vec{q}_1',\vec{k}_1')+\cdots\,,\end{aligned} \tag{11}$$

where $\{\vec{k}_1',\vec{q}_1'\}$ and $\{\vec{k}_2',\vec{q}_2'\}$ are related by Eq. (6). The high-$\vec{q}^{\,2}$ behavior of the form factor (7) is given once we know the behavior of ψ (or φ) for large $|\vec{q}|$ *and* $|\vec{k}|$ (we always suppose that the low momenta do not create any trouble). In this region the t matrix behaves as the potential up to logarithms, so that, for the potential (3),

$$t(\vec{k},\vec{k}')\simeq\frac{1}{|\vec{k}-\vec{k}'|^{1+\theta}}\;;$$

by means of a simple consistency argument we find that the only behavior consistent with Eq. (11) is given by

$$\varphi^1_{\vec{Q}}(\vec{q}_1,\vec{k}_1)\underset{\substack{|\vec{k}_1|\to\infty\\|\vec{q}_1|\to\infty}}{\simeq}\frac{1}{\vec{q}_1^{\,2}}\left\{\frac{1}{|-\tfrac{1}{2}\vec{q}_1+\vec{k}_1|^{1+\theta}}\frac{1}{|\vec{q}_1|^{1+\theta}}+\frac{1}{|\tfrac{1}{2}\vec{q}_1+\vec{k}_1|^{1+\theta}}\frac{1}{|\vec{q}_1|^{1+\theta}}\right\}\,. \tag{12}$$

We can obtain this result starting with a definite ansatz on the asymptotic behavior of the vertex function [for example the estimate (7.36) of Ref. 16]. Because we are faced with a Euclidean metric, we can apply the Weinberg theorem[22] and the asymptotic behaviors are simply given by a power counting. We find inconsistency, unless the ansatz is precisely the one given in formula (12). The behavior of φ^2 and φ^3 is easily found, so that from (9) and (12) we recover the following behavior for the wave function:

$$\psi_{\vec{Q}}(\vec{q}_1,\vec{k}_1)=\frac{1}{\vec{k}_1^{\,2}+\frac{3}{4}\vec{q}_1^{\,2}+\frac{1}{6}\vec{Q}^2-E}\,\varphi_{\vec{Q}}(\vec{q}_1,\vec{k}_1)$$
$$\simeq\frac{1}{\vec{k}_1^{\,2}+\vec{q}_1^{\,2}}\left\{\frac{1}{|q_1|^{3+\theta}}\left[\frac{1}{|-\frac{1}{2}\vec{q}_1+\vec{k}_1|^{1+\theta}}+\frac{1}{|\frac{1}{2}\vec{q}_1+\vec{k}_1|^{1+\theta}}\right]\right.$$
$$\left.+\frac{1}{|-\frac{1}{2}\vec{q}_1+\vec{k}_1|^{3+\theta}}\left[\frac{1}{|q_1|^{1+\theta}}+\frac{1}{|\frac{1}{2}\vec{q}_1+\vec{k}_1|^{1+\theta}}\right]+\frac{1}{|\frac{1}{2}\vec{q}_1+\vec{k}_1|^{3+\theta}}\left[\frac{1}{|q_1|^{1+\theta}}+\frac{1}{|-\frac{1}{2}\vec{q}_1+\vec{k}_1|^{1+\theta}}\right]\right\}. \tag{13}$$

The three terms which appear in Eq. (13) are easily understood. Equation (13), in fact, turns out to be symmetric in $\vec{q}_1,\vec{q}_2,\vec{q}_3$ and, consequently, in $\vec{p}_1,\vec{p}_2,\vec{p}_3$. By counting the powers in Eq. (13) and by observing that no dangerous region of integration exists (we could express everything as a function of $\vec{p}_1,\vec{p}_2,\vec{p}_3$), we see that the wave function ψ is integrable and

$$\psi_{\vec{Q}+\vec{q}}(\vec{k}_1,\vec{q}_1-\vec{q})\underset{\substack{|\vec{q}|\to\infty\\ \vec{k}_1,\vec{q}_1\ \text{fixed}}}{\simeq}\frac{1}{|\vec{q}|^{6+2\theta}}\,. \tag{14}$$

From (7), (13), and (14), making use of the Weinberg theorem,[22] we finally get

$$F_3(\vec{q}^{\,2})\underset{|\vec{q}|\to\infty}{\simeq}\frac{1}{|\vec{q}|^{6+2\theta}}\,,\quad 0<\theta \tag{15}$$

which has to be compared with $F_2(\vec{q}^{\,2})\simeq 1/|\vec{q}|^{3+\theta}$ given in Eq. (4). The asymptotic behavior of the form factors, therefore, does depend on the number of the constituents (at least for 2 and 3). The slowest decrease we can achieve is $|\vec{q}|^{-3}$ and $|\vec{q}|^{-6}$ in the limiting case $\theta\to 0$.

It is interesting to remark that the three-body result is not affected by the existence or nonexistence of bound states in the two-body subchannels; in fact, for large momentum transfer, the two-body t matrix is dominated by the scattering part and not by the discrete spectrum (cf. Ref. 16, Theorem 4.2).

In the second part of this section we shall discuss the case of separable potentials for reasons given above. Let us assume a separable contribution to the potential:

$$\tilde{V}(\vec{k},\vec{k}')=\lambda g(|\vec{k}|)g(|\vec{k}'|)\,. \tag{16}$$

Furthermore, we shall assume that this part of the potential is dominating at short distances so that we can consider an interaction entirely described by the potential (16). The related t matrix is given by the simple expression[23]

$$t(\vec{k},\vec{k}';E)=g(|\vec{k}|)t(E)g(|\vec{k}|)\,, \tag{17}$$

where

$$t(E)=\left(1+4\pi\lambda\int_0^\infty d|\vec{q}|\frac{\vec{q}^{\,2}|g(|\vec{q}|)|^2}{\vec{q}^{\,2}-E+i\epsilon}\right)^{-1}\,. \tag{18}$$

In order that $t(E)$ may exist, we have to assume

$$g(|\vec{q}|)\underset{|\vec{q}|\to\infty}{\simeq}\frac{1}{|\vec{q}|^{1/2+\theta}}\,,\quad\theta>0\,. \tag{19}$$

Then, in the two-particle case, we immediately obtain from (1) and (2)

$$\begin{aligned}
&\psi_{\vec{Q}}(\vec{q})\underset{|\vec{q}|\to\infty}{\simeq}\frac{1}{|\vec{q}|^{2.5+\theta}}\,,\\
&F_2(\vec{q}^{\,2})\underset{|\vec{q}|\to\infty}{\simeq}\frac{1}{|\vec{q}|^{2+2\theta}}\,,\quad 0<\theta\le\tfrac{1}{2}\,,\\
&F_2(\vec{q}^{\,2})\underset{|\vec{q}|\to\infty}{\simeq}\frac{1}{|\vec{q}|^{2.5+\theta}}\,,\quad \tfrac{1}{2}\le\theta\,.
\end{aligned} \tag{20}$$

If we insert the t matrix (17) in Eq. (8), we obtain the following simple structure of the three-body bound-state wave function:

$$\psi^i_{\vec{Q}}(\vec{k}_i,\vec{q}_i)=\frac{g(|\vec{k}_i|)t(E-\frac{3}{4}\vec{q}_i^{\,2}-\frac{1}{6}\vec{Q}^2)\tilde{g}^i(|\vec{q}_i|)}{\vec{k}_1^{\,2}+\frac{3}{4}\vec{q}_1^{\,2}+\frac{1}{6}\vec{Q}^2-E}\,, \tag{21}$$

where the functions $\tilde{g}^i$ satisfy the (noniterated) coupled equations

$$\tilde{g}^i(|\vec{q}_i|)=\int d\vec{p}\,\frac{g(|\vec{p}+\frac{1}{2}\vec{q}_i|)t(E-\frac{3}{4}\vec{p}^{\,2}-\frac{1}{6}\vec{Q}^2)g(|\frac{1}{2}\vec{p}+\vec{q}_i|)}{\vec{q}_i^{\,2}+\vec{q}_i\cdot\vec{p}+\vec{p}^{\,2}-E+\frac{1}{6}\vec{Q}^2}\,[\tilde{g}^j(|\vec{p}|)+\tilde{g}^k(|\vec{p}|)]\,,\quad i\ne j\ne k=1,2,3\,. \tag{22}$$

From the assumption (19) and the structure (18) of the t matrix, it directly follows that the only behavior compatible with Eq. (22) is

$$\tilde{g}^i(\vec{q}_i)\simeq\frac{1}{|\vec{q}_i|^{3+2\theta}}\,. \tag{23}$$

This leads to the asymptotic behavior of the wave function

$$\psi_{\vec{Q}}(\vec{k}_1,\vec{q}_1)\underset{\substack{|\vec{k}_1|\to\infty\\ |\vec{q}_1|\to\infty}}{\simeq}\frac{1}{\vec{k}_1^{\,2}+\vec{q}_1^{\,2}}\sum_{i=1}^{3}\frac{1}{|\vec{k}_i|^{1/2+\theta}}\,\frac{1}{|\vec{q}_i|^{3+2\theta}}\,. \tag{24}$$

This wave function is not integrable, but the same analysis we have applied in the local case still works and we obtain

$$F_3(\vec{q})^2 \simeq \frac{1}{|\vec{q}|^{5+2\theta}} . \tag{25}$$

Therefore, the $|\vec{q}|^{-2}$ behavior of the two-body form factor is achieved with the potentials (16) in the limit $\theta \to 0$ without any dependence on the coupling constant. With the same limiting potential the three-body form factor behaves like $|\vec{q}|^{-5}$.

III. RELATIVISTIC MODELS

A. Two-body case

Next we consider the asymptotic behavior of the form factor of relativistic two- and three-body bound states. Again, we only consider s-wave bound states and always assume that the masses are equal, $m=1$, and only one particle is charged.

For the two-body bound state, the electromagnetic current in the ladder approximation is shown in Fig. 1(a) and it can be written

$$\langle \psi | J_\mu | \psi \rangle = 2i \int d^4p\, \varphi_Q(p) \frac{1}{(\frac{1}{2}Q+p)^2-1} (Q+2p+q)_\mu \times \frac{1}{(\frac{1}{2}Q-p)^2-1} \frac{1}{(\frac{1}{2}Q+p+q)^2-1} \times \varphi_{Q+q}(p+\tfrac{1}{2}q) , \tag{26}$$

where φ_Q is the vertex function satisfying the Bethe-Salpeter equation [cf. Fig. 1(b)]

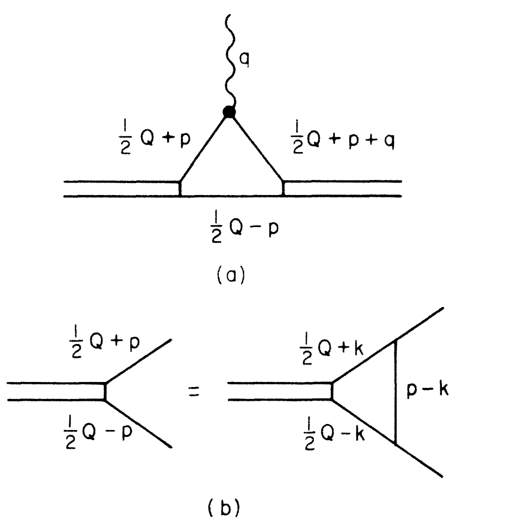

FIG. 1. (a) The electromagnetic form factor in the ladder approximation for a two-body bound state. (b) The Bethe-Salpeter equation in the ladder approximation for the wave (vertex) function of a two-body bound state. The elementary two-body interaction is defined in formula (29).

$$\varphi_Q(p) = (-i\lambda) \int d^4k\, V(p-k) G_0(k) \varphi_Q(k) \tag{27}$$

and

$$G_0(k) = [(\tfrac{1}{2}Q+k)^2-1]^{-1}[(\tfrac{1}{2}Q-k)^2-1]^{-1} . \tag{28}$$

We assume the interaction of the form

$$V(p) = \int_0^\infty d\mu^2 \frac{\sigma(\mu^2)}{(p^2-\mu^2)} , \qquad \sigma(\mu^2) \underset{\mu^2\to\infty}{\simeq} (\mu^2)^{-\theta} , \quad 0<\theta\le 1 \tag{29}$$

so that

$$V(p) \underset{p^2\to\infty}{\simeq} (p^2)^{-\theta} . \tag{30}$$

Here, the $\lambda\varphi^3$ and the $\lambda\varphi^4$ theories are described by $\theta=1$ and $\theta=0$, respectively. By means of a simple consistency argument it is straightforward to derive from Eq. (27) the following asymptotic behavior of the two-body wave function[1-4]:

$$\varphi_Q(p) \underset{p^2\to\infty}{\simeq} (p^2)^{-\theta} , \tag{31}$$

which, inserted in Eq. (26), gives for the form factor[1-4]

$$F_2(q^2) \underset{q^2\to\infty}{\simeq} (q^2)^{-1-\theta} . \tag{32}$$

From Eqs. (30) and (32) it follows that with a $\lambda\varphi^3$ theory we obtain $F_2(q^2) \simeq (q^2)^{-2}$, whereas we reach the $1/q^2$ behavior in the limiting case $\theta\to 0$, $\theta>0$. For $\theta=0$ the consistency argument does not apply any longer, and this reflects the well-known fact that in the $\lambda\varphi^4$ theory, which corresponds to the case $\theta=0$, the large-momentum-transfer behavior of the form factor depends on the coupling constant[2,4,19] [cf. the potential $(-\lambda/r^2)$ in Sec. II].

The use of the parameter θ in the definition of the potential is essentially the procedure applied in the analytic regularization[24]; on account of the possible nonanalytic dependence of the renormalizable theories on this parameter, however, we should not be surprised at this discontinuity.

B. Three-body case

For the three-body case we shall assume a pairwise interaction between the constituents and we shall consider the ladder graphs given in Fig. 2

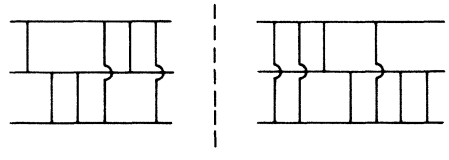

FIG. 2. The general graph in the ladder approximation for a system of three particles interacting with a two-body interaction.

only.

We make the Faddeev decomposition of the bound-state vertex function, i.e., $\varphi=\varphi^1+\varphi^2+\varphi^3$, where φ^1 is related to all the graphs in which the interaction between particles 2 and 3 comes first. Graphically, the once-iterated relativistic Faddeev equations are shown in Fig. 3, where the zig-zag lines stand for a two-body t matrix with a three-body propagator (cf. Fig. 4). It is easy to see that the iteration of Fig. 3 reproduces all the (uncrossed) ladder graphs. The integral equation for the vertex function can be written in a symbolic form similar to the nonrelativistic equation [Eq. (10)]:

$$
\begin{aligned}
\varphi^1 &= T_1G_0T_2G_0\varphi^1 + T_1G_0T_3G_0\varphi^1 \\
&\quad + T_1G_0T_2G_0\varphi^3 + T_1G_0T_3G_0\varphi^2 , \\
\varphi^2 &= T_2G_0T_1G_0\varphi^2 + T_2G_0T_3G_0\varphi^2 \\
&\quad + T_2G_0T_1G_0\varphi^3 + T_2G_0T_3G_0\varphi^1 , \\
\varphi^3 &= T_3G_0T_1G_0\varphi^3 + T_3G_0T_2G_0\varphi^3 \\
&\quad + T_3G_0T_1G_0\varphi^2 + T_3G_0T_2G_0\varphi^1 ,
\end{aligned}
\tag{33}
$$

where now, symbolically,

$$
\begin{aligned}
\langle p_1p_2p_3 \,|\, T_1 \,|\, p_1'p_2'p_3'\rangle
&= (p_1{}^2-1)\delta^4(p_1-p_1')\delta^4(p_2+p_3-p_2'-p_3') \\
&\quad \times t_1(p_2,p_3;p_2',p_3') .
\end{aligned}
\tag{34}
$$

Here t_1 is the usual two-body Bethe-Salpeter scattering matrix between particles 2 and 3 in the ladder approximation, and

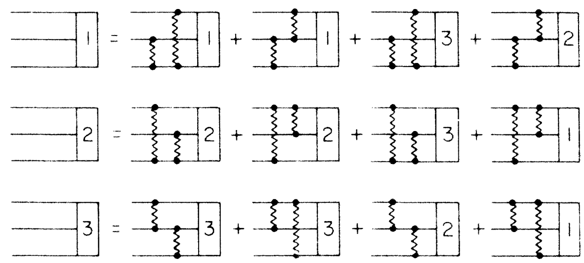

FIG. 3. The once-iterated relativistic Faddeev equations for the wave (vertex) function of a three-body bound state. The symbol $\boxed{i}$ means the Faddeev components ψ^i (φ^i). The ziz-zag lines represent the two-body Bethe-Salpeter T matrix in the ladder approximation.

$$
\begin{aligned}
\langle p_1p_2p_3 \,|\, G_0 \,|\, p_1'p_2'p_3'\rangle
&= \frac{1}{p_1{}^2-1}\,\frac{1}{p_2{}^2-1}\,\frac{1}{p_3{}^2-1} \\
&\quad \times \delta^4(p_1-p_1')\delta^4(p_2-p_2')\delta^4(p_3-p_3') .
\end{aligned}
\tag{35}
$$

[Relativistic Faddeev equations have been written down by many authors in different approximations. See for example Refs. 12–15 and 21.]

As we did in the nonrelativistic case, we introduce the four-momenta

$$
\begin{aligned}
Q &= p_1+p_2+p_3 , \\
k_1 &= \tfrac{1}{2}(p_2-p_3) , \\
q_1 &= \frac{2p_1-(p_2+p_3)}{3}
\end{aligned}
\tag{36}
$$

and their cyclic permutations k_2,q_2 and k_3,q_3 [the relation among them is the same as given in Eq. (16)]. Equation (33) now reads explicitly

$$
\begin{aligned}
\varphi_Q^1(k_1,q_1) &= \int\!\!\int dk'dq'\, t_1(\tfrac{1}{3}Q-\tfrac{1}{2}q_1+k_1,\, \tfrac{1}{3}Q-\tfrac{1}{2}q_1-k_1;\, \tfrac{1}{3}Q-\tfrac{1}{2}q'+k',\, \tfrac{1}{3}Q+\tfrac{1}{2}q'-k'-q_1) \\
&\qquad \times [(\tfrac{1}{3}Q-\tfrac{1}{2}q'+k')^2-1]^{-1}[(\tfrac{1}{3}Q+\tfrac{1}{2}q'-k'-q_1)^2-1]^{-1} \\
&\qquad \times t_2(\tfrac{1}{3}Q+q_1,\, \tfrac{1}{3}Q+\tfrac{1}{2}q'-k'-q_1;\, \tfrac{1}{3}Q+q',\, \tfrac{1}{3}Q-\tfrac{1}{2}q'-k') \\
&\qquad \times [(\tfrac{1}{3}Q+q')^2-1]^{-1}[(\tfrac{1}{3}Q-\tfrac{1}{2}q'-k')^2-1]^{-1}\varphi_Q^1(k',q') \\
&\quad + \int\!\!\int \cdots \varphi^1 + \int\!\!\int \cdots \varphi^2 + \int\!\!\int \cdots \varphi^3 , \\
\varphi_Q^2(k_2,q_2) &= \cdots , \\
\varphi_Q^3(k_3,q_3) &= \cdots .
\end{aligned}
\tag{37}
$$

In terms of the vertex function $\varphi=\varphi^1+\varphi^2+\varphi^3$, the electromagnetic current for the three-body bound state now reads (cf. Fig. 5)

$$
\begin{aligned}
\langle\psi \,|\, J_\mu \,|\, \psi\rangle &= \int\!\!\int dq_1'dk_1'\, \varphi_Q(k_1',q_1')\, [(\tfrac{1}{3}Q+q_1')^2-1]^{-1}(\tfrac{2}{3}Q+2q_1'+q)_\mu \\
&\qquad \times [(\tfrac{1}{3}Q-\tfrac{1}{2}q_1'+k_1')^2-1]^{-1}[(\tfrac{1}{3}Q-\tfrac{1}{2}q_1'-k_1')^2-1]^{-1}[(\tfrac{1}{3}Q+q_1'+q)^2-1]^{-1}\varphi_{Q+q}(k_1',q_1'+\tfrac{2}{3}q) .
\end{aligned}
\tag{38}
$$

In order to evaluate the asymptotic behavior of the form factor, we need the behavior of the vertex function for large momenta. In this limit, the t matrix reduces to the potential up to logarithms and the asymptotic integral equation reads

$$
\begin{aligned}
\varphi_Q^1(k_1,q_1) \simeq \int\int dq'dk' & [(\tfrac{1}{2}q_1 - k_1 - \tfrac{1}{2}q' + k')^2]^{-\theta}[(\tfrac{1}{3}Q - \tfrac{1}{2}q' + k')^2]^{-1} \\
& \times [(\tfrac{1}{3}Q + \tfrac{1}{2}q' - k' - q_1)^2]^{-1}[(q_1 - q')^2]^{-\theta}[(\tfrac{1}{3}Q + q')^2]^{-1}[(\tfrac{1}{3}Q - \tfrac{1}{2}q' - k')^2]^{-1}\varphi_Q^1(k',q') \\
& + \int\int \cdots \varphi^1 + \int\int \cdots \varphi^2 + \int\int \cdots \varphi^3 , \\
\varphi_Q^2(k_2,q_2) \simeq \cdots , & \\
\varphi_Q^3(k_3,q_3) \simeq \cdots &
\end{aligned}
\tag{39}
$$

If we first consider the integration over a finite volume, we immediately obtain the following behavior:

$$
\begin{aligned}
\varphi_Q^1(q_1,k_1) &\simeq [(\tfrac{1}{2}q_1 - k_1)^2]^{-\theta}(q_1{}^2)^{-1}(q_1{}^2)^{-\theta} + \cdots \\
&= 2[(\tfrac{1}{2}q_1 - k_1)^2]^{\theta}(q_1{}^2)^{-1}(q_1{}^2)^{-\theta} + 2[(\tfrac{1}{2}q_1 + k_1)^2]^{-\theta}(q_1{}^2)^{-1}(q_1{}^2)^{-\theta} .
\end{aligned}
\tag{40}
$$

When the integration variables are big, we substitute this ansatz in Eq. (39) and evaluate the contribution to the asymptotic behavior coming from the other regions of integration[25]: k'_μ small and $q'_\nu = O(q_1)$ or $q'_\mu = O(\frac{1}{2}q_1 - k_1)$ and vice versa; $q'_\mu = O(q_1)$ and $(\frac{1}{2}q' - k')_\nu = O(\frac{1}{2}q_1 - k_1)$, etc. The behavior (40) turns out to be dominant. Collecting from Eq. (39) the missing terms, we obtain for the vertex function [cf. (13)]

$$
\begin{aligned}
\varphi_Q(q_1,k_1) \simeq (q_1{}^2)^{-1-\theta}\{[(\tfrac{1}{2}q_1 - k_1)^2]^{-\theta} + [(\tfrac{1}{2}q_1 + k_1)^2]^{-\theta}\} &+ [(\tfrac{1}{2}q_1 - k_1)^2]^{-1-\theta}\{(q_1{}^2)^{-\theta} + [(\tfrac{1}{2}q_1 + k_1)^2]^{-\theta}\} \\
&+ [(\tfrac{1}{2}q_1 + k_1)^2]^{-1-\theta}\{(q_1{}^2)^{-\theta} + [(\tfrac{1}{2}q_1 - k_1)^2]^{-\theta}\} .
\end{aligned}
\tag{41}
$$

From Eq. (41) it follows that the wave function

$$
\psi = G_0\varphi \to \psi_Q(q_1,k_1) = [(\tfrac{1}{3}Q + q_1)^2 - 1]^{-1}[(\tfrac{1}{3}Q - \tfrac{1}{2}q_1 + k_1)^2 - 1]^{-1}[(\tfrac{1}{3}Q - \tfrac{1}{2}q_1 - k_1)^2 - 1]^{-1}\varphi_Q(q_1,k_1)
\tag{42}
$$

is integrable. Furthermore,

$$
\varphi_{Q+q}(k_1, q_1 + \tfrac{2}{3}q) \underset{\substack{q^2\to\infty \\ k_1, q_1 \text{ fixed}}}{\simeq} (q^2)^{-1-2\theta} .
\tag{43}
$$

In order to evaluate the asymptotic behavior of the form factor (38), we first consider a finite region of integration, and from Eq. (43) we obtain

$$
F(q^2) \underset{q^2\to\infty}{\simeq} (q^2)^{-2-2\theta} .
\tag{44}
$$

The other regions of integration confirm the behavior (44) as the dominant one. The nucleon form factor would correspond to the limiting case $\theta \to 0$ as it was necessary to consider in the two-body case in order to recover the correct pion form factor. So far, we have a consistent (although spinless) model for the pion and the nucleon form factors.

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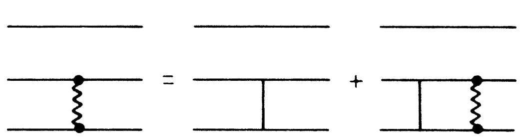

FIG. 4. The two-body Bethe-Salpeter T matrix with a three-particle propagator.

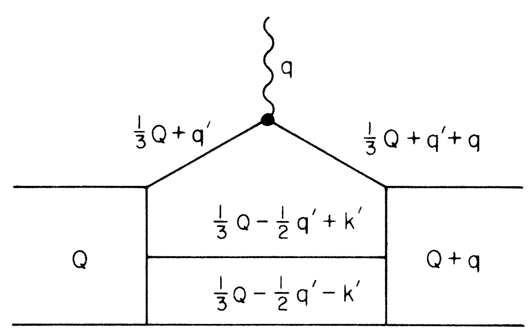

FIG. 5. The electromagnetic form factor in the ladder approximation for a three-body bound state.

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Phenomenological approach to chiral-symmetry breaking*

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It is shown that all meson-baryon and meson-meson σ terms which can be extracted from experiment can be reconciled with the $(3,\bar{3})$ chiral-symmetry-breaking scheme, $H' = u_0 + cu_8$, provided that c assumes the value $c \sim -1.0$.

I. INTRODUCTION

In a world in which many of the symmetries of nature are approximate, it is important to know how such symmetries are broken. For the case of the algebra of currents, a framework of $SU_3 \times SU_3$ breaking has been given by Gell-Mann[1] and elaborated upon by Gell-Mann, Oakes, and Renner[2] (GMOR) and by Glashow and Weinberg.[3] In the GMOR scheme, the symmetry-breaking part of the Hamiltonian density H' takes the form $u_0 + cu_8$, where u_0 and u_8 transform according to the $(3,\bar{3}) + (\bar{3},3)$ representation of $SU_3 \times SU_3$. The parameter c can be determined from the "σ terms" of meson-baryon and meson-meson scattering or from the pseudoscalar mass formula. In the latter case, GMOR assume that all the pseudoscalar mesons are Goldstone bosons which obey a quadratic mass formula. They conclude that $c \approx -1.25$, quite near the $SU_2 \times SU_2$ limit $c = -\sqrt{2}$, which in turn implies that the various pion σ terms should be small. However, $\sigma(\pi N)$ has been estimated[4] to be large, and a recent study of low-energy πN and KN scattering[5,6] has unified previously con-