PHYSICAL REVIEW D VOLUME 10, NUMBER 3 1 AUGUST 1974

Unitarity and crossing in Reggeon-particle amplitudes*

P. Hoyer†
State University of New York, Stony Brook, New York 11790

T. L. Trueman
Brookhaven National Laboratory, Upton, New York 11973
(Received 19 March 1974)

We discuss the single Regge limit of the process $a+b\to c+d+e$, in which the five-point amplitude is proportional to the Reggeon-particle amplitude $a+R\to c+d$. The partial-wave expansion in the cd system is given in the general case when all external particles have spin. Using this expansion we show that unitarity requires the phase of the Reggeon-particle amplitude to be the same as the phase of the amplitude for $c+d\to c+d$ in the elastic region. This implies relations between observable density matrix elements. We discuss the predictions for several physical reactions. We investigate the question of crossing for Reggeon-particle amplitudes by deriving the connection between the s-channel and t-channel helicity amplitudes. The helicity crossing matrix turns out to be the ordinary spin rotation matrix, the Reggeon being treated as a particle of continuous spin α_R.

I. INTRODUCTION

Consider the high-energy limit of the reaction $a+b\to c+d+e$ shown in Fig. 1. Just as in the reaction $a+b\to c+d$, it is expected that the contribution of a single Regge pole R to the amplitude factorizes into a part depending on the variables at the right vertex and a part depending on those at the left vertex. It is useful for many purposes to take the factor from the left vertex and to treat it by methods familiar from ordinary $2\to 2$ processes, that is, to consider it as a (Reggeon) $R+a\to c+d$ amplitude.

As is well known, the procedure for extracting this amplitude is a little more complicated than just suggested. Whereas, for the process $a+b\to c+d$, when $s_{ab}=(p_a+p_b)^2\to\infty$ the scattering amplitude goes as

$$(\cos\theta_t)^{\alpha(s_{a\bar{c}})}\beta_{ac}(s_{a\bar{c}})\beta_{bd}(s_{a\bar{c}}),$$

with $s_{a\bar{c}}=(p_a-p_{\bar{c}})^2$, for the process $a+b\to c+d+e$ the amplitude does *not* go as

$$(\cos\theta_t)^{\alpha(s_{b\bar{e}})}\beta_{be}(s_{b\bar{e}})f_{acd}(s_{cd},s_{b\bar{e}},s_{a\bar{c}}),$$

with $s_{cd}=(p_c+p_d)^2$, $s_{b\bar{e}}=(p_b-p_{\bar{e}})^2$. Rather, the amplitude retains in general a dependence on a fifth variable s_{de} or s_{ce}; alternatively, it depends on the Toller angle or a similar azimuthal variable (see, e.g., Ref. 1). Thus the amplitude in this limit continues to depend on a variable which connects the right and left vertices. Another way of seeing this is that in the $a+b\to c+d$ case the Reggeon is forced to have spin component $\lambda_a-\lambda_c$ along $\vec{p}_a-\vec{p}_{\bar{c}}$ and $\lambda_b-\lambda_d$ along $\vec{p}_{\bar{b}}-\vec{p}_d$ in the t-channel c.m. system. Thus the two sides are coupled just by a spin-rotation matrix

$$d^{\alpha}_{\lambda_a-\lambda_c,\lambda_b-\lambda_d}(\theta_t),$$

which factorizes in its index dependence as $\cos\theta_t\to\infty$. In the $a+b\to c+d+e$ case there is no definite spin along $\vec{p}_{\bar{a}}-(\vec{p}_c+\vec{p}_d)$ and so the helicity is summed on. It is this sum which brings in the extra dependence. Evidently, if a helicity projection of the cd subsystem were made this problem would not arise, and we should expect a simple factorizable behavior for each helicity state. All the complications thus lie in the helicity sum.

In the second section of this paper we do the helicity projection of the cd system. The Reggeon-particle amplitude can then be naturally defined through factorization for each helicity state of the Reggeon. Thus the Reggeon amplitude is simply related to the full five-point amplitude in the (physical) single Regge limit of Fig. 1.[2] Any properties we derive for the Reggeon amplitude (such as unitarity) therefore have direct consequences for the physical process $a+b\to c+d+e$.

In Sec. II we also give the partial-wave analysis in the cd system of the five-point amplitude, and derive the restrictions on the Reggeon-particle amplitude due to parity. This is done in the general case when the external particles $a,\ldots,e$ have arbitrary spin.

The constraints that arise from two-particle unitarity in the cd system are derived in Sec. III. These are particularly useful when c and d are spinless particles, or when c has spin $\frac{1}{2}$ and d is spinless. Then the phase of the Reggeon-particle amplitude $a+R\to c+d$ must be the same as that of the elastic $c+d\to c+d$ amplitude. Hence the phase is independent of the Reggeon helicity and mass (= momentum transfer), and is the same for all Reggeons R and particles a. Some evidence for

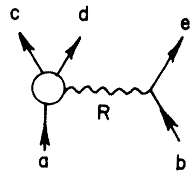

FIG. 1. The single-Regge limit of the reaction $a+b \to c+d+e$.

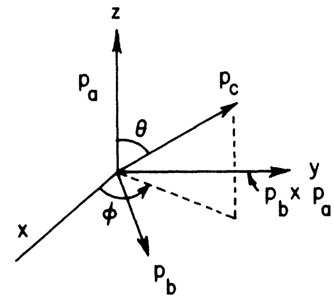

FIG. 2. The definition of the angles (θ, ϕ) in the Gottfried-Jackson (GJ) system.

this already exists from the reaction $\pi^- p \to \pi^- \pi^+ n$.[3,4] We also discuss the predictions for other observable reactions.

In Sec. IV we further consider the extent to which the Reggeon may be treated as a particle by investigating the behavior of the Reggeon helicity under crossing. The analysis of Sec. II was done in the Gottfried-Jackson frame, which corresponds to the s-channel c.m. system of the Reggeon-particle process. We now define the Reggeon helicity in the t-channel c.m. system, and derive the connection between the two representations. It is given by the ordinary spin-rotation matrices, the Reggeon being treated as a particle of spin α_R. Our conclusions are given in Sec. V.

II. KINEMATICS OF REGGEON-PARTICLE SCATTERING

The first objective of this section is to obtain a useful definition of a Reggeon-particle helicity amplitude, which we shall call $f_{\lambda_c \lambda_d, \lambda_a \lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}})$. As discussed in the Introduction, we shall take s_{ab} large for fixed s_{cd}, $s_{a\bar{c}}$, and $s_{b\bar{e}}$ in the amplitude for the process $a+b \to c+d+e$, project out a definite helicity λ_R for the exchanged Reggeon, and then use factorization to extract the amplitude $f_{\lambda_c \lambda_d, \lambda_a \lambda_R}$. The helicity projection is most easily done in the Gottfried-Jackson[5] (GJ) frame of cd (see Fig. 2). The polar angle θ is determined by s_{cd}, $s_{a\bar{c}}$, and $s_{b\bar{e}}$; the s_{de} dependence comes in only through the azimuthal angle ϕ. If we expand the full amplitude in a Fourier series of the form

$$\sum F_m e^{im\phi},$$

the coefficient F_m will not depend on s_{de}. Furthermore, the total angular momentum of the cd system along the z axis will be linearly related to m. The exact relation depends on the phase convention which we will spell out shortly.

In the Regge limit we may imagine the cd final state as resulting from a collision between particle a and the Reggeon R. Since $\vec{p}_a$ is along the z direction and $\vec{p}_c + \vec{p}_d = 0$, the Reggeon's momentum is in the negative z direction (see Fig. 3). Thus the angular momentum along the z direction is λ_a minus the Reggeon's helicity λ_R. Each amplitude F_m will therefore have a definite Reggeon helicity. Notice that the helicities $\lambda_a, \lambda_R, \lambda_c, \lambda_d$ are all defined in the center-of-mass system for the process $a+R \to c+d$. Hence they are the analogs of s-channel helicities for two-particle processes.

We will use the Jacob and Wick phase convention.[6] Consider the cd state in the GJ frame. Then according to Jacob and Wick

$$|p_c\lambda_c, p_d\lambda_d\rangle \equiv |s_{cd}, \theta, \phi, \lambda_c, \lambda_d\rangle = R_z(\phi) e^{i(\lambda_c - \lambda_d)\phi} |s_{cd}, \theta, 0, \lambda_c, \lambda_d\rangle . \tag{2.1}$$

Hence

$$|s_{cd}, \theta; \lambda, \lambda_c, \lambda_d\rangle \equiv \frac{1}{2\pi} \int d\phi' e^{i[\lambda - (\lambda_c - \lambda_d)]\phi'} \times |s_{cd}, \theta, \phi', \lambda_c, \lambda_d\rangle \tag{2.2}$$

satisfies

$$R_z(\phi)|s_{cd}, \theta; \lambda, \lambda_c, \lambda_d\rangle = \frac{1}{2\pi} \int d\phi' e^{i[\lambda - (\lambda_c - \lambda_d)]\phi'} R_z(\phi' + \phi) |s_{cd}, \theta, 0, \lambda_c, \lambda_d\rangle e^{i[\lambda_c - \lambda_d]\phi'} = e^{-i\lambda\phi}|s_{cd}, \theta; \lambda, \lambda_c, \lambda_d\rangle ,$$

so $|s_{cd}, \theta, \lambda, \lambda_c, \lambda_d\rangle$ has angular momentum λ along the z axis and

$$\lambda_R = \lambda_a - \lambda . \tag{2.3}$$

Let us now consider the amplitude in the t-channel center-of-mass system appropriate to the Regge expansion for large s_{ab}; i.e., the $b\bar{e}$ center-of-mass system. One important feature of the GJ frame is that the Lorentz transformation to that t-channel c.m. system is along $\vec{p}_a$, and so

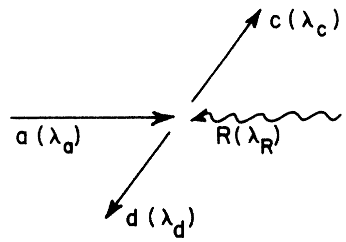

FIG. 3. The Reggeon-particle process $a + R \to c + d$, as seen in the GJ system.

the helicities λ_a and λ_R are the same in both frames[5] (see Fig. 4). Repeating the argument given in the Introduction for two-body processes, now applied to amplitudes of definite λ, we conclude that the amplitude is proportional to the spin-rotation matrix

$$d^{\alpha(s_{b\bar{e}})}_{\lambda_a-\lambda,\lambda_b-\lambda_e}(\theta_t)\,.$$

If we then resum the Fourier series in ϕ and take the asymptotic behavior of the d functions, we find the Regge-pole behavior to be

$$\begin{aligned}F^{(t)}_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s_{ab}, s_{cd}, s_{b\bar{e}}, s_{a\bar{c}}, s_{de}) &= \beta_{\lambda_e\lambda_b}(s_{b\bar{e}})\xi(s_{b\bar{e}})(-\cos\theta_t)^{\alpha(s_{b\bar{e}})} \\ &\times \sum_{\lambda_R=-\infty}^{\infty} e^{i[(\lambda_a-\lambda_R)-(\lambda_c-\lambda_d)]\phi} \\ &\times \frac{f_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}})}{[(\alpha+\lambda_R)!(\alpha-\lambda_R)!]^{1/2}}\,, \end{aligned} \quad (2.4)$$

where

$$\xi(s_{b\bar{e}}) = \frac{\tau + e^{-i\pi[\alpha(s_{b\bar{e}})-v]}}{\sin\pi[\alpha(s_{b\bar{e}})-v]} \quad (2.5)$$

and $v=0$ ($\frac{1}{2}$) for meson (fermion) trajectories of signature τ. Equation (2.4) serves as our definition

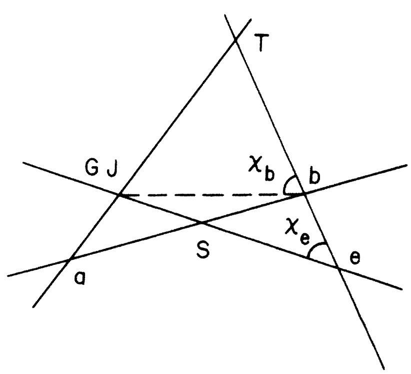

FIG. 4. A velocity diagram showing the relation between the ab c.m. system S, the $b\bar{e}$ c.m. system (Reggeon rest frame) T, and the Gottfried-Jackson system GJ. The crossing angles χ_b and χ_e refer to the Lorentz transformation from T to GJ.

of the Reggeon-particle amplitude for the process $a+R \to c+d$. Notice that we are using a mixed notation with helicities λ_c and λ_d referred to the GJ frame, λ_b and λ_e to the $b\bar{e}$ rest frame. (λ_a and λ_R are the same in both frames.) One could define all helicities in the same frame with the aid of the appropriate spin-rotation matrices. However, this is not necessary. See the detailed discussion at the end of this section.

The asymptotic relations needed to obtain (2.4) are, as $s_{ab} \to \infty$,

$$\cos\theta_t \sim \frac{2s_{ab}s_{b\bar{e}}}{[\lambda(s_{b\bar{e}}, m_b^2, m_e^2)\lambda(s_{b\bar{e}}, m_a^2, s_{cd})]^{1/2}} \to -\infty\,, \quad (2.6)$$

where

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

as usual, and

$$d^{\alpha}_{\mu'\mu}(\theta_t) = \frac{(2\alpha)!\,e^{i\pi(\mu-\mu')/2}(\cos\theta_t)^{\alpha}}{2^{\alpha}[(\alpha+\mu')!(\alpha-\mu')!(\alpha+\mu)!(\alpha-\mu)!]^{1/2}}\,. \quad (2.7)$$

Perhaps we should emphasize that our arguments leading to Eq. (2.4) are not intended to prove the Regge behavior of the $a+b \to c+d+e$ amplitude. To do that would require a discussion of the Froissart-Gribov continuation for the $2 \to 3$ amplitude and the multiparticle t-channel unitarity relations.[7,8] Rather, they are intended to show the relation between the Fourier coefficients in ϕ of a production amplitude with Regge behavior and the helicity of the Reggeon. From all we know about Regge poles, we believe the asymptotic behavior in s_{ab}, for fixed $s_{cd}, s_{b\bar{e}}, s_{a\bar{c}}$,

$$F \sim (\cos\theta_t)^{\alpha(s_{b\bar{e}})}\beta_{be}(s_{b\bar{e}})f_{acd}(s_{cd}, s_{b\bar{e}}, s_{a\bar{c}}, \phi)\,,$$

to be reasonable. If this asymptotic form is (piecewise) continuous in ϕ in the physical region $0 \le \phi < 2\pi$, the Fourier series converges and the infinite helicity sum introduces none of the problems which are sometimes found in other contexts.[8] Note that one does not need to assume the absence of singularities of F in $\cos\phi$; such singularities may well occur and at such points a *power* series in $\cos\phi$ would necessarily diverge while the *Fourier* series remains convergent. Note also that the singular function

$$\kappa^{\alpha} \propto [m_d^2 - s_{b\bar{e}} - s_{a\bar{c}} + 2(-s_{a\bar{c}})^{1/2}(-s_{b\bar{e}})^{1/2}\cos\phi]^{\alpha}\,,$$

which one frequently encounters in multi-Regge formulas, is only singular outside the physical region.

The factor $1/[(\alpha+\lambda_R)!(\alpha-\lambda_R)!]^{1/2}$ has been kept explicit in order that $f_{\lambda_c\lambda_d,\lambda_a\lambda_R}$ transform like an ordinary helicity amplitude (see Sec. IV). However, we have absorbed the phase factor $e^{i\pi\lambda_R/2}$ coming from d^α into the definition of f in order that it satisfy the normal reality (time reversal) conditions (see Sec. III). This factor is associated with the spacelike momentum of the Reggeon. It is a familiar feature that may be seen in the behavior of the polarization vector for a spin-1 particle when continued from timelike to spacelike momentum.[9] As a result of this choice the phase factors in parity relations are changed.

Parity conservation for the full amplitude requires that

$$F^{(t)}_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s_{ab}, s_{cd}, s_{b\bar{e}}, \theta, \phi) = (-1)^{\sum_i (s_i-\lambda_i)}\eta_{\bar{a}}\eta_c\eta_d\eta_b\eta_{\bar{e}} \times F^{(t)}_{-\lambda_c-\lambda_d-\lambda_e,-\lambda_a-\lambda_b}(s_{ab}, s_{cd}, s_{b\bar{e}}, \theta, -\phi),$$

where s_a, λ_a, η_a denote the spin, helicity, and intrinsic parity of particle a, and so on. (We omit all "particle 2" phase factors of Jacob and Wick.) This is seen most simply by using the reflection operator in the x-z plane. We have here made an obvious change of notation, expressing $s_{a\bar{c}}$ and s_{de} in terms of the angles θ, ϕ in the GJ frame. Then from (2.4)

$$\beta_{\lambda_e\lambda_b}(s_{b\bar{e}})f_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}}) = (-1)^{s_b+s_e-\lambda_b-\lambda_e}\beta_{-\lambda_e-\lambda_b}(s_{b\bar{e}})\eta_{\bar{e}}\eta_b(-1)^{s_a-\lambda_a+s_c-\lambda_c+s_d-\lambda_d}\eta_{\bar{a}}\eta_c\eta_d \times f_{-\lambda_c-\lambda_d,-\lambda_a-\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}}),$$

from which we infer that

$$f_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}}) = \sigma_R(-1)^{s_a-\lambda_a+s_c-\lambda_c+s_d-\lambda_d} \times \eta_{\bar{a}}\eta_c\eta_d f_{-\lambda_c-\lambda_d,-\lambda_a-\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}}). \quad (2.8)$$

Notice the absence of a factor $(-1)^{\lambda_R}$ for the Reggeon; this is the modification to which we referred. At the same time

$$\beta_{\lambda_e\lambda_b}(s_{b\bar{e}}) = \sigma_R^*(-1)^{s_b+s_e-\lambda_b-\lambda_e}\eta_{\bar{e}}\eta_b\beta_{-\lambda_e,-\lambda_b}(s_{b\bar{e}}). \quad (2.9)$$

Here σ_R is the analog of $(-1)^s\eta$ for the Reggeon.

Equation (2.9) defines the parity of the Reggeon by way of its contribution to two-particle processes. Note that $(-1)^s \equiv \tau$ for bosons and $(-1)^{s-1/2} \equiv \tau$ for fermions, so

$$\sigma_R = e^{i\pi v}\tau_R\eta_R, \quad (2.10)$$

where η_R is the Reggeon's parity and $v=0$ ($\frac{1}{2}$) for bosons (fermions).

Just as definite helicities are projected out by the azimuthal ϕ integration, the definite total angular momentum j of the cd system can be projected out by integration over the GJ polar angle θ. This forms the basis of a Jacob-Wick expansion for $f_{\lambda_c\lambda_d,\lambda_a\lambda_R}$:

$$\int d\Omega\, F^{(t)}_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s_{ab}, s_{cd}, s_{b\bar{e}}, \theta, \phi)\mathfrak{D}^j_{\lambda_a-\lambda_R,\lambda_c-\lambda_d}(\phi, \theta, -\phi)$$
$$\sim 2\pi\beta_{\lambda_e\lambda_b}(s_{b\bar{e}})\xi(s_{b\bar{e}})\frac{(-\cos\theta_t)^{\alpha(s_{b\bar{e}})}}{[(\alpha+\lambda_R)!(\alpha-\lambda_R)!]^{1/2}}\int_{-1}^{1} d(\cos\theta)d^j_{\lambda_a-\lambda_R,\lambda_c-\lambda_d}(\theta)f_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}})$$
$$\equiv \beta_{\lambda_e\lambda_b}(s_{b\bar{e}})\xi(s_{b\bar{e}})\frac{(-\cos\theta_t)^{\alpha(s_{b\bar{e}})}}{[(\alpha+\lambda_R)!(\alpha-\lambda_R)!]^{1/2}}f^j_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd}, s_{b\bar{e}}). \quad (2.11)$$

With this definition of f^j,

$$f_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}}) = \sum \frac{2j+1}{4\pi}d^j_{\lambda_a-\lambda_R,\lambda_c-\lambda_d}(\theta)f^j_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd}, s_{b\bar{e}}). \quad (2.12)$$

The amplitudes f^j are very useful for discussing low-energy properties in the s_{cd} channel, just as partial-wave amplitudes are useful in two-particle processes. In Sec. III we shall obtain unitarity relations for them. In the remainder of this section we shall discuss our use of a "mixed" helicity notation, where some helicities (λ_c, λ_d) are defined in the GJ system and others (λ_b, λ_e) in the Reggeon rest frame. This is a natural way of describing the kinematic situation we are studying. It is a straightforward generalization of the helicity notation for two-body processes.

In Eq. (2.11) it can be seen that λ_c and λ_d, as well as s_{cd}, play the role of internal variables. This results from our mixed notation. Corresponding to (2.12), we have in the GJ frame

$$|\vec{p}_c\lambda_c, \vec{p}_d\lambda_d\rangle = \sum_{j,\lambda}\left(\frac{2j+1}{4\pi}\right)^{1/2}|s_{cd}, j; \lambda; \lambda_c\lambda_d\rangle \times\mathfrak{D}^j_{\lambda,\lambda_c-\lambda_d}(\phi, \theta, -\phi). \quad (2.13)$$

The most important characteristic of $|s_{cd}, j; \lambda; \lambda_c\lambda_d\rangle$

is its transformation properties under rotations:

$$R\,|s_{cd}, j; \lambda; \lambda_c, \lambda_d\rangle = \sum_{\lambda'} \mathfrak{D}^j_{\lambda'\lambda}(R)\,|s_{cd}, j; \lambda'; \lambda_c\lambda_d\rangle, \tag{2.14}$$

as may be easily verified from the definition (2.13). Notice that λ_c and λ_d are unchanged by R. Now treat $|s_{cd}, j; \lambda; \lambda_c, \lambda_d\rangle$ as a single-particle state in order to define helicity states for c and d in a frame in which $\vec{p}_{cd} = \vec{p}_c + \vec{p}_d \neq 0$. Just as Jacob and Wick do for a single-particle state, define

$$|\vec{p}_{cd}, s_{cd}, j; \lambda; \lambda_c, \lambda_d\rangle \equiv H(\vec{p}_{cd})\,|s_{cd}, j; \lambda; \lambda_c, \lambda_d\rangle. \tag{2.15}$$

The helicity transformation $H(\vec{p}_{cd})$ is given by

$$H(\vec{p}_{cd}) = R_z(\gamma)R_y(\beta)R_z(-\gamma)B_z(|\vec{p}_{cd}|), \tag{2.16}$$

i.e., a boost along the z axis of magnitude required to give the cd system a momentum of magnitude $|\vec{p}_{cd}|$ followed by a rotation through an angle β about an axis in the x-y plane making an angle γ with the y axis. This brings the momentum of cd to

$$\vec{p}_{cd} = (|\vec{p}_{cd}|\sin\beta\cos\gamma,\ |\vec{p}_{cd}|\sin\beta\sin\gamma,\ |\vec{p}_{cd}|\cos\beta)$$

by a well-defined sequence of transformations. With this definition and Eq. (2.14), the states $|\vec{p}_{cd}, s_{cd}, j; \lambda; \lambda_c, \lambda_d\rangle$ transform just like a single-particle state of mass $(s_{cd})^{1/2}$, spin j, and helicity λ, *independent of* λ_c *and* λ_d. Thus the amplitudes

$$F_{\lambda\lambda_e,\lambda_a\lambda_b}(j\lambda_c\lambda_d) \equiv \langle \vec{p}_{cd}, s_{cd}\, j, \lambda, \lambda_c\lambda_d; \vec{p}_e, \lambda_e|\,T\,|\vec{p}_a\lambda_a, \vec{p}_b\lambda_b\rangle$$

are kinematically just like $2\to 2$ amplitudes and can be transformed in the same way between the s-channel c.m. system and the GJ system or crossed to the t-channel c.m. system. Correspondingly, in any frame the full amplitude is given by

$$F_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b} = \sum_{j,\lambda}\left(\frac{2j+1}{4\pi}\right)^{1/2} F_{\lambda\lambda_e,\lambda_a\lambda_b}(j\lambda_c\lambda_d) \times \mathfrak{D}^{j*}_{\lambda,\lambda_c-\lambda_d}(\phi, \theta, -\phi),$$

where, we emphasize, λ_c, λ_d, θ, and ϕ are always defined in the GJ frame. Equation (2.11) is just a special case of this expansion, in the $b\bar{e}$ center-of-mass frame.

Because each j transforms differently under Lorentz transformation, the full amplitude will generally have complicated transformation properties. However, in the important case of going from the $b\bar{e}$ c.m. frame to the GJ frame, λ and λ_a are not changed and

$$F^{\mathrm{GJ}}_{\lambda\lambda_e,\lambda_a\lambda_b}(j\lambda_c\lambda_d) = \sum_{\lambda_e',\lambda_b'} F^{(t)}_{\lambda\lambda_e',\lambda_a\lambda_b'}(j\lambda_c\lambda_d) \times d^{s_e}_{\lambda_e'\lambda_e}(\chi_e)\, d^{s_b}_{\lambda_b'\lambda_b}(\chi_b).$$

χ_e and χ_b are shown geometrically in Fig. 4. χ_e is the usual s-t crossing angle for e, and χ_b becomes the s-t crossing angle for b in the high-s limit[10]:

$$\cos\chi_e = -\frac{(s_{b\bar{e}} + m_e^2 - m_b^2)}{[\lambda(m_b^2, m_e^2, s_{b\bar{e}})]^{1/2}},$$

$$\cos\chi_b \approx \frac{m_b^2 + s_{b\bar{e}} - m_e^2}{[\lambda(m_b^2, m_e^2, s_{b\bar{e}})]^{1/2}}.$$

III. UNITARITY

A. Derivation

Consider the unitarity relation for the five-point amplitude[11] shown in Fig. 5:

$$\mathrm{Disc}_{cd} F_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s_{ab}, s_{cd}, s_{b\bar{e}}, \theta, \phi) = \sum_n \langle s_{cd}, \theta, \phi, \lambda_c\lambda_d|\,T^\dagger\,|n\rangle\langle n; \vec{p}_e, \lambda_e|\,T\,|\vec{p}_a\lambda_a, \vec{p}_b\lambda_b\rangle. \tag{3.1}$$

If we let $s_{ab}\to\infty$, keeping s_{cd}, $s_{b\bar{e}}$, θ, and ϕ fixed, all the processes $a+b\to n+e$ will Reggeize. Using factorization, Eq. (3.1) then becomes a general unitarity relation for the Reggeon amplitude $a+R\to c+d$, expressing its discontinuity in terms of the processes $a+R\to n$ and $n\to c+d$.

We shall only discuss the constraints of two-particle unitarity here. Thus, we assume that s_{cd} is small enough so that only the state cd itself contributes to the sum over n in Eq. (3.1). We have then

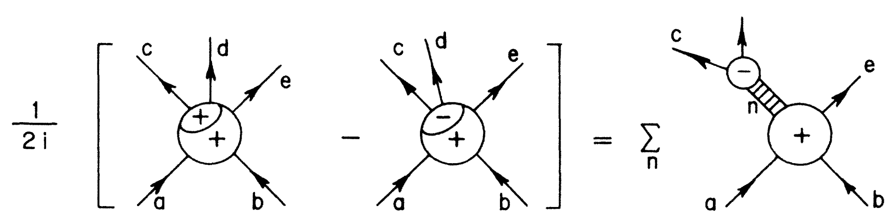

FIG. 5. The discontinuity relation corresponding to Eq. (3.1) in the text.

$$\mathrm{Disc}_{cd}F_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s_{ab},s_{cd},s_{b\bar{e}},\theta,\phi)$$
$$=\sum_{\lambda_c'\lambda_d'}\int d\Omega'\langle s_{cd},\theta\phi,\lambda_c\lambda_d|T^\dagger|s_{cd},\theta'\phi',\lambda_c'\lambda_d'\rangle$$
$$\times F_{\lambda_c'\lambda_d'\lambda_e,\lambda_a\lambda_b}(s_{ab},s_{cd},s_{b\bar{e}},\theta',\phi')\,. \tag{3.2}$$

When $s_{ab}\to\infty$, we can use Eq. (2.4) to express the five-point amplitude F in terms of the Reggeon amplitude f. We diagonalize the relation (3.2) by expanding f into partial waves according to (2.12). Note that for each Reggeon helicity λ_R, the partial-wave expansion of $f_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd},s_{b\bar{e}},\theta)$ is the same as for an ordinary helicity amplitude. Denoting the partial-wave amplitudes of the elastic process $c+d\to c'+d'$ by $T^j_{\lambda_c'\lambda_d',\lambda_c\lambda_d}$, we find

$$\mathrm{Disc}_{cd}f^j_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd},s_{b\bar{e}})$$
$$=\sum_{\lambda_c'\lambda_d'}f^j_{\lambda_c'\lambda_d',\lambda_a\lambda_R}(s_{cd},s_{b\bar{e}})T^{j*}_{\lambda_c'\lambda_d',\lambda_c\lambda_d}(s_{cd})\,. \tag{3.3}$$

As in two-body collisions, the left-hand side of Eq. (3.3) is real, because $f^j_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd},s_{b\bar{e}})$ is a real analytic function of s_{cd}. This follows from two properties which we assume the full five-point amplitude to possess:

(i) Real analyticity. Arbitrary phases can be chosen so that

$$F_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s_{ab},s_{cd},s_{b\bar{e}},\theta,\phi)$$
$$=F^*_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s^*_{ab},s^*_{cd},s_{b\bar{e}},\theta,-\phi) \tag{3.4}$$

for $s_{b\bar{e}}$, θ, ϕ real. [Note that $s_{de}(s_{ab},s_{cd},s_{b\bar{e}},\theta,\phi)=s^*_{de}(s^*_{ab},s^*_{cd},s_{b\bar{e}},\theta,\phi)$.] The change in sign of ϕ results from the behavior of helicity states under time reversal $\mathcal{T}$.[6] For a single-particle helicity state of momentum p in the θ, ϕ direction

$$\mathcal{T}|p,\theta,\phi,\lambda\rangle=\mathcal{T}e^{-iJ_z\phi}e^{-iJ_y\theta}e^{iJ_z\phi}|p,0,0,\lambda\rangle$$
$$=e^{-iJ_z\phi}e^{-iJ_z\theta}e^{iJ_z\phi}\mathcal{T}|p,0,0,\lambda\rangle$$
$$=e^{-iJ_z\phi}e^{-iJ_y\theta}e^{iJ_z\phi}e^{-iJ_y\pi}|p,0,0,\lambda\rangle$$
$$=e^{-iJ_y\pi}|p,\theta,-\phi,\lambda\rangle\,.$$

Under time reversal the scattering operator $T\to\mathcal{T}T\mathcal{T}^{-1}=T^\dagger$. Below thresholds in all channels, in the Euclidean region, the matrix elements of $T^\dagger$ equal the matrix elements of T, and so $F(\phi)=F^*(-\phi)$ there. Then Eq. (3.4) follows from the Schwarz reflection principle.

(ii) Signature symmetry. For large s_{ab}, fixed s_{cd}, θ, ϕ,

$$F_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(s_{ab},s_{cd},s_{b\bar{e}},\theta,\phi)$$
$$\sim e^{i\pi v}\tau F_{\lambda_c\lambda_d\lambda_e,\lambda_a\lambda_b}(-s_{ab},s_{cd},s_{b\bar{e}},\theta,\phi) \tag{3.5}$$

if the amplitude is dominated by a Regge pole of signature τ. (Note that $s_{de}\to -s_{de}$ when $s_{ab}\to -s_{ab}$, with the other variables fixed.)

Using these two assumptions with Eq. (2.4) we obtain

$$f_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s_{cd},s_{a\bar{c}},s_{b\bar{e}})=f^*_{\lambda_c\lambda_d,\lambda_a\lambda_R}(s^*_{cd},s_{a\bar{c}},s_{b\bar{e}})\,, \tag{3.6}$$

and so Disc$f$ = Imf and the left-hand side of Eq. (3.3) is real.

The unitarity relation (3.3) is particularly interesting in the case when there is only one term on the right-hand side. This is so when particles c and d are spinless, or when one is spinless and the other has spin $\frac{1}{2}$ (provided we choose states of definite parity to describe the cd system). From the reality of the left-hand side of (3.3) it then follows that

$$\arg f^{j\eta}_{\lambda_a\lambda_R}=\arg T^{j\eta}\,, \tag{3.7}$$

where we have labeled the state cd by its angular momentum j and parity η.

B. Discussion

According to (3.7) the phase of the Reggeon-particle amplitude for $a+R\to c+d$ is equal to the phase of the $cd\to cd$ elastic amplitude. It is not surprising that there is a connection between the two, in particular when the cd system is resonating.[12] However, Eq. (3.7) shows that the phases must be exactly equal, also for nonresonant cd states, whenever a pure Regge pole is exchanged (and s_{cd} is below the first inelastic threshold). It is also remarkable that the phase of $f^{j\eta}_{\lambda_a\lambda_R}(s_{cd},s_{b\bar{e}})$ consequently does not depend on the nature of particle a or Reggeon R, nor on their helicities λ_a, λ_R. The phase is furthermore independent of the Reggeon "mass," the momentum transfer $s_{b\bar{e}}$.

It is possible in many cases to measure experimentally the relative phases of Reggeon amplitudes. By testing the validity of Eq. (3.7) one can then find out whether any corrections to the Regge-pole exchange are needed. The situation here is quite analogous to that in two-body scattering, where the polarization has to vanish when a single Regge pole is exchanged. Note, however, that the relation (3.7) determines the relative phase also between different partial waves j, and that it must hold for all values of s_{cd} (in the elastic region). Furthermore, it is interesting that Eq. (3.7) can be tested experimentally in some cases without using polarized targets or beams (see below).

On the other hand, if one assumes Regge-pole exchange, the constraint (3.7) reduces significantly the number of unknowns to be determined from the data. This may make a Reggeon-particle amplitude analysis possible. We shall next discuss briefly the processes for which such an analysis seems most attractive.

(a) $\pi + R \to \pi\pi$ *(or* $K + R \to K\pi$*).* These processes can be conveniently studied using the abundant data on $\pi N \to \pi\pi N$ and $\pi N \to \pi\pi\Delta$. In the elastic region ($m_{\pi\pi} \lesssim 1$ GeV) S and P waves dominate. For a given S-wave isospin ($I = 0$ or 2) there are 3 amplitudes $f^j_{\lambda_R}(s_{cd}, s_{b\bar{e}})$ (j, $|\lambda_R| = 0, 1$) describing unnatural parity (π) exchange and one natural parity amplitude $f^1_1 = +f^1_{-1}$ (ω-A_2 exchange). Assuming their phases to be given by the elastic $\pi\pi$ amplitude, we are left with four unknown magnitudes. Experimentally, six quantities can be measured from the $\pi\pi$ angular distribution. The phase relation (3.7) therefore allows us to determine the magnitudes of all amplitudes, and in addition imposes two constraints on the density matrix. In the GJ (or helicity) frame the constraints are, in the standard notation,

$$\frac{\rho_{00} - (\rho_{11} - \rho_{1\,-1})}{\mathrm{Re}\rho_{10}} = \frac{\mathrm{Re}\rho_{0s}}{\mathrm{Re}\rho_{1s}} - 2\frac{\mathrm{Re}\rho_{1s}}{\mathrm{Re}\rho_{0s}}, \tag{3.8a}$$

$$\cos^{-2}\Delta = \frac{\mathrm{Re}\rho_{10}}{\mathrm{Re}\rho_{1s}}\left[\frac{1 + 2(\rho_{00} - \rho_{11})}{\mathrm{Re}\rho_{0s}} - 3\frac{\mathrm{Re}\rho_{10}}{\mathrm{Re}\rho_{1s}}\right], \tag{3.8b}$$

where Δ is the relative angle between the S- and P-wave amplitudes in $\pi\pi \to \pi\pi$.

For the reaction $\pi^- p \to \pi^-\pi^+ n$ an amplitude analysis can, in fact, be done[4] even without assuming the phase relation (3.7). This is because there is no interference between π and A_2 exchange (when the nucleons are unpolarized), so that only two relative angles (Δ and $\phi \equiv \arg f^1_0 - \arg f^1_1$) matter. Hence the number of unknowns equals the number of observables (assuming the absence of A_1 exchange). The amplitude analysis has been done in the ρ region of $\pi\pi$ mass at several incident momenta.[4] The resulting angles ϕ, Δ are in agreement with the prediction (3.7) ($\phi = 0°$ or $180°$, Δ small[13]) for all momentum transfers covered ($|s_{b\bar{e}}| \lesssim 0.5$). It has also been shown[3] that the positivity conditions on the density matrix require the amplitudes to be nearly phase-coherent.

(b) $N + R \to \pi\pi$. For a given (baryon) exchange R there are four independent amplitudes $f^j_{\lambda_a\lambda_R}$ (j, $|\lambda_R| = 0, 1$). Assuming Eq. (3.7), their magnitudes can be calculated and two additional constraints obtained on the six observables describing the $\pi\pi$ angular distribution. If we express the moments of the $\pi\pi$ angular distribution in terms of density matrix elements exactly as for the reaction $\pi N \to \pi\pi N$, the constraints assume the form (3.8). An example of a physical reaction for which this kind of analysis could be done is $\pi^- p \to p + (\pi^0\pi^-)$, which should be dominated by Δ exchange.[14]

(c) $N + R \to N\pi$. The number of amplitudes with $j \leq \frac{3}{2}$ is 12. In an experiment with unpolarized nucleons 10 quantities can be determined from the angular distribution. Thus, even assuming the phases to be known, the polarization of one of the nucleons must be measured to get an overconstrained situation. However, it is likely that many of the 12 amplitudes are small. An investigation of this reaction would be interesting because the elastic phases are accurately known from $\pi N \to \pi N$. Also, the isolation of a given Regge trajectory [e.g., ρ in $\pi^+ p \to \pi^0 + (\pi^+ p)$] is straightforward in many physical reactions.

IV. CROSSING FOR REGGEON HELICITY AMPLITUDES

When s_{cd} becomes large for fixed $s_{a\bar{c}}$, the process $a + R \to c + d$ is probably more simply discussed in terms of Regge exchange in the crossed ($\bar{d} + R \to c + \bar{a}$) channel (see Fig. 6). In this situation it is simpler to work directly with the analog of t-channel amplitudes for Reggeon-particle scattering rather than the s-channel amplitudes we have used so far. These will have a simpler t-channel partial-wave expansion, the analog of Eq. (2.12), which can be Reggeized via a Sommerfeld-Watson transformation. These amplitudes correspond to those defined by Bali, Chew, and Pignotti.[1]

To obtain the t-channel amplitudes one defines the helicity of the Reggeon in terms of an expansion in the Toller angle ω (Refs. 15, 1) instead of the azimuthal angle ϕ used earlier. This is well known, but we would like to give a geometrical interpretation of this fact since it will be useful in understanding the relation between the two kinds of amplitudes. We will ignore the spins of the external particles in this section for the sake of simplicity; it is straightforward to include them.

The problem here is almost entirely one of kinematics. Mainly, we wish to express the energy variable s_{de} in terms of the angle ϕ or ω and the other invariants s_{ab}, s_{cd}, $s_{b\bar{e}}$, and $s_{a\bar{c}}$ (or θ in the GJ frame). These relations can all be worked out algebraically, but a very pretty picture of the relations can be obtained geometrically if we work in the Reggeon's rest frame (the $b\bar{e}$ center-of-mass system). In order for the angles involved to be real angles, it is necessary

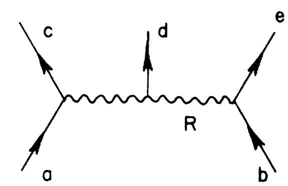

FIG. 6. The double-Regge limit of the reaction $a + b \to c + d + e$.

to work in a region of the invariants where the process $b+\bar{e}\to\bar{a}+c+d$ is physical. This involves an analytic continuation in the four-momenta $p_a\to -p_{\bar{a}}$, $p_b\to p_b$, $p_c\to p_c$, $p_d\to p_d$, $p_e\to -p_{\bar{e}}$. There are many paths in the invariants along which the continuation can be made, but since we are deriving purely kinematical relations it cannot matter which path is chosen.

It is simplest to do the continuation in a reference frame whose definition is unaffected by the continuation; for example, the cd center-of-mass frame or the d rest frame rather than the ab center-of-mass frame. Then no complex Lorentz transformation is needed in going to the $b\bar{e}$ c.m. frame. Correspondingly, the kinematic structure of the momenta in terms of the invariants is very simple. For example, in the cd c.m. frame

$$p_a^0=\frac{s_{cd}+m_a^{\,2}-s_{b\bar{e}}}{2(s_{cd})^{1/2}},$$

$$(\vec{p}_a^{\,2})^{1/2}=+\frac{[\lambda(m_a^{\,2},s_{cd},s_{b\bar{e}})]^{1/2}}{2(s_{cd})^{1/2}},$$

so $p_a^0<-m_a$ when $s_{b\bar{e}}>[(s_{cd})^{1/2}+m_a]^2$. If the continuation is done so that $(\lambda)^{1/2}\to -|(\lambda)^{1/2}|$ and the angles are all kept fixed, then we have simply $p_a\to -p_{\bar{a}}$, with the usual physical expression for $p_{\bar{a}}$:

$$p_{\bar{a}}^0=\frac{s_{b\bar{e}}-s_{cd}-m_a^{\,2}}{2(s_{cd})^{1/2}},$$

$$(\vec{p}_{\bar{a}}^{\,2})^{1/2}=+\frac{[\lambda(m_a^{\,2},s_{cd},s_{b\bar{e}})]^{1/2}}{2(s_{cd})^{1/2}}.$$

We will continue all momenta in this way so that $(\vec{p}_e^{\,2})^{1/2}\to -(\vec{p}_{\bar{e}}^{\,2})^{1/2}$, $(\vec{p}_b^{\,2})^{1/2}\to +(\vec{p}_b^{\,2})^{1/2}$ and all orientations remain fixed.

The cd c.m. frame is appropriate for the ϕ expansion, the d rest frame for the ω expansion. Let us imagine that the continuation in one or the other of these frames has been done and the resultant amplitude for $b+\bar{e}\to\bar{a}+c+d$ Lorentz-transformed to the $b\bar{e}$ c.m. frame. The angles between the various vectors in this frame are defined in Fig. 7 ($\theta_{\bar{a}b}=\theta_t$ of Sec. II). As discussed in the Introduction, the Reggeon's spin in the direction $\vec{p}_b$ is $\lambda_b-\lambda_e$, here taken to be zero. Its helicity λ_R in the $a+R$ c.m. frame is equal to its spin along $\vec{p}_{\bar{a}}$ in Fig. 7: On continuing back to the $a+b\to c+d+e$ physical region $\vec{p}_{\bar{a}}\to -\vec{p}_a$ and the Lorentz transformation to the $a+R$ c.m. frame is along $\vec{p}_a$. Thus the overlap between these two spin projections is $d_{0\lambda_R}^{\alpha(s_{b\bar{e}})}(\theta_{\bar{a}b})$ as given in Sec. II. By

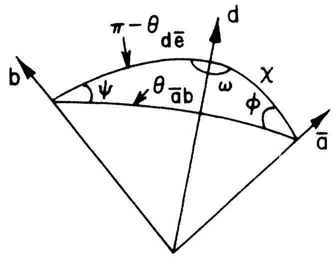

FIG. 7. The definition, in the $b\bar{e}$ c.m. system (Reggeon rest frame), of the angles used in the text.

the same argument, its helicity μ_R in the $R+\bar{d}$ c.m. frame will be equal to the spin projection in the $\vec{p}_d$ direction in Fig. 7 and hence the overlap function is $d_{0\mu_R}^{\alpha(s_{b\bar{e}})}(\pi-\theta_{d\bar{e}})$. The remaining angle χ in Fig. 7 is defined so that it corresponds to the definition of the crossing angle for particles when $\alpha(s_{b\bar{e}})$ goes through a physical integer or half-integer: It is the angle between $\vec{p}_{\bar{a}}=-\vec{p}_a$ and $\vec{p}_d$ in the rest frame of particle R. (Note that R is an uncrossed particle in going from $a+R\to c+d$ to $\bar{d}+R\to c+\bar{a}$.)

The angles ϕ and ω are also shown in Fig. 7. These can be seen to correspond to the definition of ϕ as shown in Fig. 2 and to the usual definition of ω which is shown in the d rest frame in Fig. 8: (i) ϕ is the angle from $\vec{p}_{\bar{a}}\times\vec{p}_b$ to $\vec{p}_{\bar{a}}\times\vec{p}_d$ about $-\vec{p}_{\bar{a}}$, in the right-handed sense. The cd rest system is reached by a Lorentz transformation along $-\vec{p}_{\bar{a}}=\vec{p}_c+\vec{p}_d$; since the planes intersect along that line, ϕ is unchanged by the transformation, remaining the angle between $\vec{p}_{\bar{a}}\times\vec{p}_b$ and $\vec{p}_{\bar{a}}\times\vec{p}_d$ with the momenta evaluated in that frame. Under the continuation back to $a+b\to c+d+e$, $\vec{p}_{\bar{a}}\to -\vec{p}_a$, and so ϕ is the angle from $\vec{p}_b\times\vec{p}_a$ to $\vec{p}_d\times\vec{p}_a=\vec{p}_a\times\vec{p}_c$. (ii) ω is the angle from $\vec{p}_d\times\vec{p}_b=\vec{p}_b\times(\vec{p}_c+\vec{p}_{\bar{a}})$ to $\vec{p}_d\times\vec{p}_{\bar{a}}$

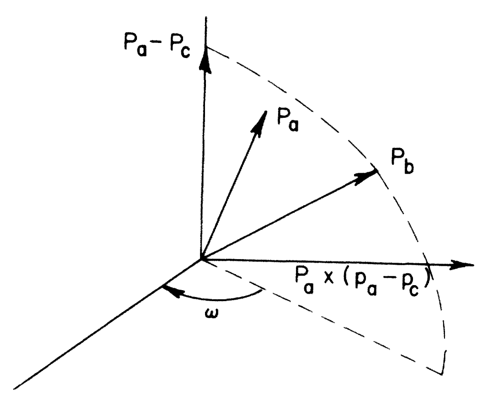

FIG. 8. The definition of the Toller angle ω in the rest frame of particle d.

$=\vec{p}_{\bar{a}}\times(\vec{p}_c+\vec{p}_{\bar{a}})$ about $\vec{p}_d=-(\vec{p}_c+\vec{p}_{\bar{a}})$. The Lorentz transformation to d at rest is along the intersection of the planes and leaves the definition of ω unchanged. Under the continuation back to $a+b\to c+d+e$, ω becomes the angle from $(\vec{p}_a-\vec{p}_c)\times\vec{p}_b$ to $\vec{p}_a\times(\vec{p}_a-\vec{p}_c)$ about $\vec{p}_a-\vec{p}_c$. This checks with Fig. 8; i.e., ω is the angle from $\vec{p}_b\times\vec{p}_e$ to $\vec{p}_a\times\vec{p}_c$ about $\vec{p}_a-\vec{p}_c$.

A definite spin λ_R of the Reggeon along $\vec{p}_{\bar{a}}$ is obtained by integrating over the azimuthal angle ϕ weighted by $e^{i\lambda_R\phi}$. (It is not $e^{-i\lambda_R\phi}$ because ϕ is measured in a right-handed sense about $-\vec{p}_{\bar{a}}$.) A definite spin μ_R along $\vec{p}_d$ is obtained by integrating over ω weighted by $e^{-i\mu_R\omega}$. We are thus led to the two alternative expansions:

$$F=\beta\xi\sum_{\lambda_R}(-\cos\theta_{\bar{a}b})^{\alpha(s_{b\bar{e}})}\times\frac{e^{-i\lambda_R\phi}}{[(\alpha+\lambda_R)!(\alpha-\lambda_R)!]^{1/2}}f_{\lambda_R},\qquad(4.1\text{a})$$

which is just (2.4) again, and

$$F=\beta\xi\sum_{\mu_R}(-\cos\theta_{d\bar{e}})^{\alpha(s_{b\bar{e}})}\times\frac{e^{i\mu_R\omega}}{[(\alpha+\mu_R)!(\alpha-\mu_R)!]^{1/2}}g_{\mu_R}.\qquad(4.1\text{b})$$

The relation between f_{λ_R} and g_{μ_R} is clearly that of a crossing relation. We will now see that it has the same form as crossing for particles with spin, with the complication that the crossing matrix is infinite-dimensional.

Application of the laws of spherical trigonometry to Fig. 7 gives the relations

$$\cos\theta_{d\bar{e}}=-[\cos\chi\cos\theta_{\bar{a}b}+\sin\chi\sin\theta_{a\bar{b}}\cos\phi],\qquad(4.2\text{a})$$

$$\cos\theta_{\bar{a}b}=[-\cos\theta_{d\bar{e}}\cos\chi+\sin\theta_{d\bar{e}}\sin\chi\cos\omega],\qquad(4.2\text{b})$$

$$\frac{\sin\theta_{\bar{a}b}}{\sin\omega}=\frac{\sin\theta_{d\bar{e}}}{\sin\phi}=\frac{\sin\chi}{\sin\psi}.\qquad(4.2\text{c})$$

As s_{ab} and s_{de} become large

$$\cos\theta_{\bar{a}b}\sim\frac{2s_{ab}s_{b\bar{e}}}{[\lambda(s_{b\bar{e}},m_e^2,m_b^2)\lambda(s_{b\bar{e}},m_a^2,s_{cd})]^{1/2}},$$

$$\cos\theta_{d\bar{e}}\sim\frac{2s_{de}s_{b\bar{e}}}{[\lambda(s_{b\bar{e}},m_e^2,m_b^2)\lambda(s_{b\bar{e}},s_{a\bar{c}},m_d^2)]^{1/2}}.\qquad(4.3)$$

A choice of the branch of $\sin\theta_{\bar{a}b}$ must be made; we take $\sin\theta_{\bar{a}b}\sim i\cos\theta_{\bar{a}b}$. Equation (4.2c) then constrains the phase of $\sin\theta_{d\bar{e}}$ and $\sin\chi$ to be $-\frac{1}{2}\pi$. (All of the angles ω, ϕ, ψ are less than π or greater than π together.) Then

$$\frac{\cos\theta_{d\bar{e}}}{\cos\theta_{\bar{a}b}}=\frac{s_{de}}{s_{ab}}\frac{[\lambda(s_{b\bar{e}},m_a^2,s_{cd})]^{1/2}}{[\lambda(s_{b\bar{e}},s_{a\bar{c}},m_d^2)]^{1/2}}=-(\cos\chi+i\sin\chi\cos\phi)=-1/(\cos\chi-i\sin\chi\cos\omega).\qquad(4.4)$$

This equation provides the desired relation between s_{de}, ϕ, and ω.

Equation (4.4) is rather complicated when expressed in terms of the invariants but is quite simple when the angle χ is introduced. $\cos\chi$ can be calculated from its definition in Fig. 7:

$$\cos\chi=\frac{(s_{b\bar{e}}-m_a^2+s_{cd})(s_{b\bar{e}}-m_d^2+s_{a\bar{c}})+2s_{b\bar{e}}[m_d^2-s_{b\bar{e}}+m_a^2-m_c^2]}{[\lambda(s_{b\bar{e}},m_a^2,s_{cd})\lambda(s_{b\bar{e}},m_d^2,s_{ac})]^{1/2}}.\qquad(4.5)$$

Comparison of this with the standard crossing-angle formulas[10] will verify that it is the s-t crossing angle for a particle of mass $s_{b\bar{e}}$. The corresponding formula for $\sin\chi$ is

$$\sin\chi=\frac{(s_{b\bar{e}})^{1/2}}{(s_{cd})^{1/2}}\frac{[\lambda(s_{cd},m_d^2,m_c^2)]^{1/2}}{[\lambda(s_{b\bar{e}},s_{a\bar{c}},m_d^2)]^{1/2}}\sin\theta,\qquad(4.6)$$

with the phase determined to be $-\frac{1}{2}\pi$ from Eq. (4.2c). Notice that as $s_{cd}\to\infty$, too,

$$\kappa\equiv\frac{s_{cd}s_{de}}{s_{ab}}\sim-\frac{[\lambda(s_{b\bar{e}},s_{a\bar{c}},m_d^2)]^{1/2}}{\cos\chi-i\sin\chi\cos\omega}\sim\frac{\lambda(s_{b\bar{e}},s_{a\bar{c}},m_d^2)}{m_d^2-s_{b\bar{e}}-s_{a\bar{c}}+2(-s_{a\bar{c}})^{1/2}(-s_{b\bar{e}})^{1/2}\cos\omega}.\qquad(4.7)$$

This is the usual relation between κ and $\cos\omega$ in the double-Regge limit.

One further relation which can easily be worked out from Eq. (4.2) and (4.4) is

$$e^{i\omega}=\frac{e^{-i\phi/2}\cos(\frac{1}{2}\chi)+ie^{i\phi/2}\sin(\frac{1}{2}\chi)}{e^{i\phi/2}\cos(\frac{1}{2}\chi)+ie^{-i\phi/2}\sin(\frac{1}{2}\chi)}\qquad(4.8\text{a})$$

or

$$e^{i\phi}=\frac{e^{-i\omega/2}\cos(\frac{1}{2}\chi)-ie^{i\omega/2}\sin(\frac{1}{2}\chi)}{e^{i\omega/2}\cos(\frac{1}{2}\chi)-ie^{-i\omega/2}\sin(\frac{1}{2}\chi)}.\qquad(4.8\text{b})$$

Then from Eqs. (4.1), (4.4), and (4.8) we have

$$f_{\lambda_R}=\sum_{\mu_R}\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{i\lambda_R\phi}[-\cos\chi-i\sin\chi\cos\phi]^\alpha\left(\frac{e^{-i\phi/2}\cos(\frac{1}{2}\chi)+ie^{i\phi/2}\sin(\frac{1}{2}\chi)}{e^{i\phi/2}\cos(\frac{1}{2}\chi)+ie^{-i\phi/2}\sin(\frac{1}{2}\chi)}\right)^{\mu_R}\left(\frac{(\alpha+\lambda_R)!(\alpha-\lambda_R)!}{(\alpha+\mu_R)!(\alpha-\mu_R)!}\right)^{1/2}g_{\mu_R} \tag{4.9a}$$

or

$$g_{\mu_R}=\sum_{\lambda_R}\frac{1}{2\pi}\int_0^{2\pi}d\omega\, e^{-i\mu_R\omega}[-\cos\chi+i\sin\chi\cos\omega]^\alpha\left(\frac{e^{-i\omega/2}\cos(\frac{1}{2}\chi)-ie^{i\omega/2}\sin(\frac{1}{2}\chi)}{e^{i\omega/2}\cos(\frac{1}{2}\chi)-ie^{-i\omega/2}\sin(\frac{1}{2}\chi)}\right)^{-\lambda_R}\left(\frac{(\alpha+\mu_R)!(\alpha-\mu_R)!}{(\alpha+\lambda_R)!(\alpha-\lambda_R)!}\right)^{1/2}f_{\lambda_R}. \tag{4.9b}$$

For integral j the rotation matrices have the integral representation[16,17]

$$d^j_{mm'}(\chi)=\left(\frac{(j+m')!(j-m')!}{(j-m)!(j+m)!}\right)^{1/2}e^{i\pi(m-m')/2}\int_0^{2\pi}\frac{d\phi}{2\pi}e^{im'\phi}(\cos\chi+i\cos\phi\sin\chi)^j\left(\frac{e^{-i\phi/2}\cos(\frac{1}{2}\chi)+ie^{i\phi/2}\sin(\frac{1}{2}\chi)}{e^{i\phi/2}\cos(\frac{1}{2}\chi)+ie^{-i\phi/2}\sin(\frac{1}{2}\chi)}\right)^m. \tag{4.10}$$

Recall that $\cos\chi+i\cos\phi\sin\chi<0$, so we can make the simple continuation satisfying the Carlson conditions

$$e^{i\pi j}d^j\to\tau d^\alpha.$$

Then Eqs. (4.9) are formally equivalent to the ordinary crossing relations

$$f_\lambda=\tau\sum_\mu d^\alpha_{\mu\lambda}(\chi)g_\mu e^{-i\pi(\mu-\lambda)/2}, \tag{4.11a}$$

$$g_\mu=\tau\sum_\lambda d^\alpha_{\lambda\mu}(\chi)f_\lambda e^{-i\pi(\mu-\lambda)/2}. \tag{4.11b}$$

The extra helicity-dependent phase factors in Eq. (4.11) are related to the phase convention chosen in Sec. II because the Reggeon's momentum is spacelike. Formal relations similar to this have been written down by White for his definition of Reggeon amplitudes.[7]

It should be apparent from the preceding discussion that one could define Reggeon-particle amplitudes in arbitrary reference frames and that their Lorentz-transformation properties would be formally the same as those for ordinary amplitudes modulo factors like those in Eq. (4.11). In certain simple models, such as resonance dominance in s_{cd} or one-pion exchange in $s_{a\bar{c}}$ —the Deck effect—the sums over either λ or μ are finite, but in general the sums run over all integers $-\infty<\lambda,\ \mu<\infty$. The crossing relations will converge under very weak assumptions on the behavior of F. If F is a continuous, periodic function of ϕ on $0\le\phi\le 2\pi$, then[18]

$$|f_\lambda|\le\frac{c}{\lambda}[(\alpha+\lambda)!(\alpha-\lambda)!]^{1/2}.$$

By the same theorem on Fourier coefficients

$$|d^\alpha_{\lambda\mu}(\chi)|\le\frac{c}{\lambda}[(\alpha+\lambda)!(\alpha-\lambda)!]^{-1/2}.$$

Hence the crossing relations are absolutely convergent.

V. CONCLUSIONS

In this paper we have studied several aspects of four-point Reggeon-particle amplitudes. Insofar as Regge exchange is an important feature of high-energy scattering, such Reggeon amplitudes are likely to be useful for our understanding of the scattering mechanism. This is particularly true now, as high-statistics data are becoming available on few-body reactions. As we have seen, it is possible to derive many properties of the Reggeon amplitudes using only the general features of Regge exchange. Some of the results derived here are no doubt already known; they have been included for completeness of the presentation.

In our treatment we tried to emphasize the extent to which the Reggeon amplitudes resemble ordinary scattering amplitudes. Because the Reggeon carries a continuous spin α_R, it, in general, can have any helicity λ_R, $-\infty<\lambda_R<\infty$. In the Reggeon rest system its spin component in the direction of the external particles must, however, be equal to the difference of the helicities of those particles. The Reggeon helicity λ_R can thus assume only integer (or half-integer) values. This suggests that the Reggeon helicity amplitude, defined by projecting out a definite helicity of the Reggeon, may have many properties in common with ordinary helicity amplitudes.

The partial-wave expansion and parity relations of the Reggeon helicity amplitude are straightforward to derive and analogous to the usual ones. An interesting question is the behavior of the Reggeon helicity under crossing from the s- to the t-channel c.m. system. This is important because many features, such as resonances and unitarity, are best dealt with in the s-channel frame, whereas the high-energy Regge behavior is commonly studied in the t-channel frame. We found that the helicity crossing matrix of the Reggeon is the same as for an ordinary particle, with the particle spin equal to α_R.

In Sec. III we showed that the Reggeon can be treated as a particle also in deriving unitarity relations. White[7] previously found that additional terms contributed to his Reggeon-particle unitarity relation. We do not have any such extra terms, presumably because our Reggeon helicity amplitude is defined differently from his, which is obtained in a helicity-pole limit. Our basic partial-wave unitarity relations are the same as his for the five-point function.

Below the first inelastic threshold in $cd \to cd$ the Reggeon amplitude $a+R\to c+d$ is required by unitarity to have the same phase as the cd elastic amplitude (when the spins $s_c, s_d = 0, 0$ or $0, \frac{1}{2}$). This relation is particularly interesting because the Reggeon can carry many different quantum numbers. Also, owing to the unusual kinematics (the Reggeon is spacelike), one can study experimentally processes like $N+\overline{N}\to \pi+\pi$ below the 4π threshold. There are thus several reactions for which the phase prediction of the Regge-pole exchange has direct observable effects.

We have seen that Reggeon amplitudes have many properties in common with ordinary scattering amplitudes. One should, however, remember that other features, such as the analytic structure, may be more complicated. An example of this is provided by the signature properties of Reggeon amplitudes. Consider the definition of the Reggeon amplitude f_{λ_R} (we take all external particles to be scalars)

$$\beta(s_{b\bar{e}})\frac{\tau+e^{-i\pi\alpha}}{\sin\pi\alpha}(-\cos\theta_t)^{\alpha}\frac{f_{\lambda_R}(s_{cd}, s_{a\bar{c}}, s_{b\bar{e}})}{[(\alpha+\lambda_R)!]^{1/2}[(\alpha-\lambda_R)!]^{1/2}} = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{i\lambda_R\phi}F(s_{ab}, s_{cd}, s_{de}, s_{a\bar{c}}, s_{b\bar{e}}) . \tag{5.1}$$

Let us take $|s_{cd}|\to\infty$ and continue s_{cd}, s_{ab} along a semicircle:

$$s_{cd}\to s_{cd}e^{i\pi}, \quad s_{ab}\to s_{ab}e^{i\pi}, \quad s_{cd}s_{de}/s_{ab} \text{ fixed} .$$

The complex conjugate of the continued amplitude can be related to the original amplitude in (5.1) assuming signature τ' in the $a\bar{c}$ channel:

$$F(-s_{ab}, -s_{cd}, s_{de}, s_{a\bar{c}}, s_{b\bar{e}}) = \tau' F(s_{ab}, s_{cd}, s_{de}, s_{a\bar{c}}, s_{b\bar{e}}) .$$

The result is

$$\tau\tau' e^{i\pi\alpha} f^*_{\lambda_R}(-s_{cd}+i\epsilon, s_{a\bar{c}}, s_{b\bar{e}}) = f_{\lambda_R}(s_{cd}+i\epsilon, s_{a\bar{c}}, s_{b\bar{e}}) - \left[\frac{2\pi\beta\xi}{[(\alpha+\lambda_R)!(\alpha-\lambda_R)!]^{1/2}}(-\cos\theta_t)^{\alpha}\right]^{-1}\int_0^{2\pi} d\phi\, e^{i\lambda_R\phi}2i\,\mathrm{Disc}_{de}F(s_{ab}, s_{cd}, s_{de}, s_{a\bar{c}}, s_{b\bar{e}}) . \tag{5.2}$$

If the Reggeon were an ordinary particle, the second term on the right-hand side of Eq. (5.2) would be absent. The phase of f_{λ_R} for $|s_{cd}|\to\infty$ would then be given by a signature factor. The presence of the second term in (5.2) reflects the complicated analytic structure of f_{λ_R} in s_{cd}, and precludes any general determination of its phase.

For the purposes of the present paper the precise analytic structure of f_{λ_R} was not essential. However, the singularity structure has interesting consequences when one wants, e.g., to derive finite-energy sum rules for Reggeon-particle amplitudes.[19]

ACKNOWLEDGMENT

This work was begun at the Department of Theoretical Physics, Oxford. One of us (T. L. T.) was a Fellow of the John Simon Guggenheim Foundation.

*Work performed under the auspices of U. S. Atomic Energy Commission.

†Research supported in part by the NSF under Grant No. GP-32998X, and by a grant from the University of Helsinki.

[1]N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967).

[2]Note that our Reggeon-particle amplitude is different from the one previously discussed by White (Ref. 7). His amplitude is defined by taking an (unphysical) helicity-pole limit, and so is not simply related to the full five-point amplitude in the physical region.

[3]G. Grayer *et al.*, Nucl. Phys. B50, 29 (1972).

[4]P. Estabrooks and A. D. Martin, in *π-π Scattering—1973*, proceedings of the international conference on π-π scattering and associated topics, Tallahassee, 1973, edited by P. K. Williams and V. Hagopian (A.I.P., New York, 1973), p. 357; D. S. Ayres *et al.*, *ibid.*, p. 284; J. A. Charlesworth *et al.*, *ibid.*, p. 317.

[5]K. Gottfried and J. D. Jackson, Nuovo Cimento 33, 309 (1964).

[6]M. Jacob and G. C. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

[7]A. R. White, Nucl. Phys. B67, 189 (1973).

[8]J. W. Dash, Phys. Rev. D 3, 1012 (1971).

[9]A. H. Mueller and T. L. Trueman, Phys. Rev. D 2, 953

(1970).
[10]T. L. Trueman and G. C. Wick, Ann. Phys. (N.Y.) 26, 322 (1964).
[11]R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966), p. 232; K. E. Cahill and H. P. Stapp, Phys. Rev. D 6, 1007 (1972).
[12]G. Fox, in *Experimental Meson Spectroscopy—1972*, proceedings of the third international conference on experimental meson spectroscopy, Philadelphia, 1972, edited by Kwan-Wu Lai and Arthur H. Rosenfeld (A.I.P., New York, 1972), p. 271.
[13]We neglect the small $I=2$ S wave.
[14]The ($I=2$) S wave is probably small in this reaction. If it is neglected, only one constraint on the nonzero density matrix elements is obtained.
[15]M. Toller, Nuovo Cimento 37, 631 (1965).
[16]This formula can be derived from Wigner's Eq. (15.19) (Ref. 17) by putting $\epsilon = e^{i\psi/2}$, $\zeta = e^{-i\psi/2}$, and integrating over ψ with $e^{-im'\psi}$ and substituting $\psi = \frac{1}{2}\pi - \phi$.
[17]E. Wigner, *Group Theory* (Academic, New York, 1959), p. 163.
[18]See, e.g., Courant and Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1953), Vol. 1, p. 74.
[19]P. Hoyer and J. Kwieciński, Nucl. Phys. B60, 26 (1973).

PHYSICAL REVIEW D VOLUME 10, NUMBER 3 1 AUGUST 1974

Dispersion calculation of the transition form factor $F_{\pi\omega\gamma}(t)$ with cut contributions*

G. Köpp
III. Physikalisches Institut, Technische Hochschule Aachen, Aachen, West Germany
(Received 8 April 1974)

In this paper we study the t dependence of the $\gamma\pi\omega$ vertex functions on the basis of partial-wave dispersion relations and unitarity. The right-hand cut is approximated by the 2π contribution and the left-hand cut by the nearest s- and u-channel poles. The electromagnetic $\pi\omega$ transition form factor is calculated as a function of t in the timelike and spacelike region and is compared with predictions of the ρ-dominance model and with recent experimental data for $e^+e^- \to \omega\pi^0 \to \pi^+\pi^-\pi^0\pi^0$ near threshold. The influence of the left-hand cut and the finite width of the ρ resonance is explicitly shown.

I. INTRODUCTION

The construction of electron-positron storage rings has opened a new field in high-energy physics. Interest is centered on $e^\pm$-$e^\pm$ interactions, both in annihilation and in scattering processes. In the high-energy region these reactions are accompanied by hadron production.

Experimental results from storage rings are already available, especially from the Orsay, Novosibirsk, and Frascati rings.[1-5] These give an idea what pion and kaon form factors look like in the timelike region and also yield annihilation cross sections for multimesonic final states (where the latter can partly be understood as quasi-two-body states).

The description of e^+e^- annihilation into hadronic two-body or quasi-two-body states leads in the one-photon approximation—which will be one of our basic assumptions—to the definition of form factors and transition form factors in timelike region. The kinematic structure of such reactions has been given in detail by Kramer and Walsh.[6]

The best known example, from the theoretical as well as the experimental point of view, is that of the two-pion final state which reduces to the description of the pion form factor. This problem has been studied many times, and we refer, for instance, to the calculations of Frazer and Fulco,[7] Gounaris and Sakurai, Schwarz, Aubrecht, Renard, and Bonneau and Martin.[8] $K\bar{K}$ production has been analyzed, for instance, by Renard.[8,9]

The next step in e^+e^- annihilation is the production of three-pion final states. The case of the $\pi^+\pi^-\pi^0$ channel has been analyzed by KW in a model based on $\rho^0\pi^0$ production. They also discuss 4π production by resonance formation using a vector-dominance model as has been done by other authors.[10]

We shall concentrate on the production channel $\pi^+\pi^-\pi^0\pi^0$. In contrast with KW and others, we study the influence of the left-hand cut and subsequent finite-width corrections on the resulting transition form factors.

Finite-width corrections in connection with analyticity have been the subject of many discussions in the past especially in studying pion and kaon form factors.[8,9] There exists also a dynamical model for the reaction $e^+e^- \to \rho^0 \to \pi^0 + \omega$ ($\pi^+\pi^-\pi^0$) due to Renard.[11] He assumes Breit-Wigner shapes