ton, W. Fry, and N. Schgal, paper submitted to the XVI International Conference on High Energy Physics, Chicago-Batavia, Illinois, 1972. A more detailed description of the experiment is available by Douglas B. Clark, Ph.D. thesis, Univ. of Wisconsin, 1973 (unpublished); this paper contains a very thorough review of theoretical work on $K^+ \to \pi^+ e\bar{e}$. Among the papers cited, the following are of particular interest to us, since the approaches of these papers are complementary to ours: M. A. B. Bég, Phys. Rev. 132, 426 (1963); V. V. Geidt and I. B. Khriplovich, Yad. Fiz. 8, 960 (1968) [Sov. J. Nucl. Phys. 8, 558 (1969)]. The unitarity contribution has been calculated by G. Segrè and D. Wilkinson, Phys. Rev. D 8, 3056 (1973) and is negligible.

[23]S. Gjesdal, G. Presser, P. Steffen, J. Steinberger, F. Vannucci, H. Wahl, H. Filthuth, K. Kleinknecht, V. Lüth, and G. Zech, Phys. Lett. 44B, 217 (1973).

[24]G. D. Cable, R. H. Hildebrand, C. Y. Pang, and R. Stiening, Phys. Rev. D 8, 3807 (1973).

[25]B. W. Lee, J. R. Primack, and S. B. Treiman, Phys. Rev. D 7, 510 (1973); B. W. Lee and S. B. Treiman, Phys. Rev. D 7, 1211 (1973).

[26]K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D 6, 2923 (1972); J. R. Primack and H. Quinn, *ibid.* 6, 3171 (1972).

[27]I. Bars, M. B. Halpern, and M. Yoshimura, Phys. Rev. Lett. 29, 969 (1972); Phys. Rev. D 7, 1233 (1973).

[28]B. W. Lee, Phys. Rev. D 6, 1188 (1972); J. Prentki and B. Zumino, Nucl. Phys. B47, 99 (1972).

[29]B. W. Lee, Phys. Rev. D 9, 933 (1974).

[30]See for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Sec. 8.2.

PHYSICAL REVIEW D VOLUME 10, NUMBER 3 1 AUGUST 1974

$\eta'(958)$ branching ratio, linear matrix element, and dipion phase shift*

George R. Kalbfleisch
Brookhaven National Laboratory, Upton, New York 11973
(Received 28 February 1974)

The "world" data on $\eta'(958)$ in regard to the branching ratio $R = (\pi^+\pi^-\gamma/\pi^+\pi^-\eta_N)$, η_N denoting $\eta(549) \to$ all neutrals, and the dipion mass spectrum in the $\eta' \to \pi^+\pi^-\eta$ decay are reviewed. We find that $R = 0.97 \pm 0.08$ independent of energy, exclusive of the data of Aguilar-Benitez *et al.* Under the assumption that $J^P = 0^-$, the best value of the α parameter of the linear matrix element is $\alpha = -0.08 \pm 0.03$. Possible connections between the dipion mass spectrum and δ_{00}, the $I = J = 0$ phase shift, are discussed.

Many studies of the $\eta'(958)$ have been made. We comment here on the branching ratio $R = (\pi^+\pi^-\gamma/\pi^+\pi^-\eta_N)$, where η_N represents $\eta(549) \to$ all neutrals, and on the linear matrix element of the $\eta'(958)$. We base these comments on the data presented in Refs. 1–6; some additional data are not used.[7] We obtain the current "world" averages of R and of α, the parameter characterizing the linear matrix element. We also comment on possible connections with δ_{00}, the $I=J=0$ dipion phase shift. Additional comments on the $\eta'(958)$ are given in Ref. 8.

BRANCHING RATIO

The data of Refs. 1–6 on the branching ratio $R = (\pi^+\pi^-\gamma/\pi^+\pi^-\eta_N)$ are given in Table I and are shown in Fig. 1 versus the momentum of the incident beam in the experiments. Included is the branching ratio of the $M(953)$ of Ref. 3. The values of the η' branching ratio R from Refs. 1, 2, and 4–6 are consistent; the weighted average is $\bar{R} = 0.97 \pm 0.08$ and is indicated also in Fig. 1. The value of R for the $\eta'(958)$ from Ref. 3 is in disagreement with this average value. Since there is no evidence that the $\pi^+\pi^-\gamma$ and $\pi^+\pi^-\eta$ states at 958 MeV are different,[5] we must attribute the disagreement between $\bar{R}$ and R (Ref. 3) to some unknown systematic effect in that experiment. The evidence that the $M(953)$ is not identical to $\eta'(958)$ rests in the $M(\pi^+\pi^-)$ distribution of the $\pi^+\pi^-\gamma$ decay mode. The data in the ρ^0 region of both the $\eta'(958)$ and $M(953)$ data samples have the characteristic $\sin^2\theta_{\pi+\gamma}$ distribution of $\eta'(958)$ events.[3] Therefore, the evidence for an $M(953)$ should be obtained from the $M(\pi^+\pi^-)$ mass distribution, Fig. 46(a) of Ref. 3, where events containing a $\rho^0(765)$ were excluded. The data yield 37 ± 10 events outside the ρ band, of which about 8–15 should be attributed to the tail of the ρ^0 in $\eta' \to \rho^0\gamma$ decay. Thus, the evidence for an $M(953)$ is about 2 to 3 standard deviations, attributing all the other "M" events to $\eta'(958)$; then the branching ratio R of the η' in the "M" sample is about 0.8 ± 0.2.

TABLE I. Summary of branching ratio $R = (\pi^+\pi^-\gamma/\pi^+\pi^-\eta_N)$, η_N denoting $\eta(549) \to$ all neutrals, from Refs. 1–6. The momentum of the incident beam of the experiment is denoted P_{lab}. The average value of R for data of Refs. 1, 2, 4–6 is $\bar{R} = 0.97 \pm 0.08$.

Data of Ref.	P_{lab} (GeV/c)	R
1	2.1 K^-	1.39 ± 0.27 [a]
	2.45–2.85 K^-	0.89 ± 0.14 [a]
2	1.8 and 1.95 K^-	0.53 ± 0.30 [b]
3	3.9 and 4.6 K^-	[$\eta'(958)$] 0.54 ± 0.10 [c]
		[$M(953)$] 1.2 ± 0.3
4	2.89 K^-	1.11 ± 0.18
5	2.18 K^-	0.92 ± 0.14
6 (Badier *et al.*)	3.0 K^-	1.3 ± 0.4
6 (Mott *et al.*)	5.5 K^-	0.7 ± 0.3

[a] 0.99 ± 0.19 (2.1 GeV/c) and 0.63 ± 0.10 (2.65 GeV/c) divided by 0.71, the branching ratio $\eta_N(549)/\eta(549)$.

[b] $R = (0.475)^{-1}R'$, where $R' = (\pi^+\pi^-\gamma/\pi\pi\eta) = 0.25 \pm 0.14$.

[c] A correction upwards of about 10% should be made to account for $\eta' \to \pi^0\pi^0\eta_c$.

LINEAR MATRIX ELEMENT

The spin-parity J^P of the η', 0^- or 2^-, is still an open question.[5,8,9] However, under the assumption that $J^P = 0^-$, the mass distribution $M(\pi^+\pi^-)$ of the decay mode $\eta' \to \pi^+\pi^-\eta$ is expected to be phenomenologically described by a "linear matrix element"; the event distribution should be

$$(|1+\alpha y|^2 + cx^2) \text{ times phase space}, \qquad (1)$$

where x, y are the Dalitz coordinates,[10] and where α, c are parameters. The "slope" α describes the deviation of the $M(\pi^+\pi^-)$ distribution from phase space; the c parameter describes any anisotropy of the $\cos\theta_{\pi^+\eta}$ distribution. For a discussion of

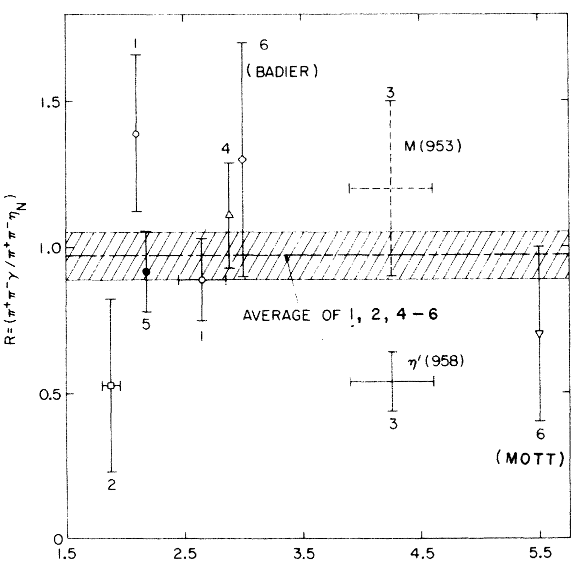

FIG. 1. Branching ratio $R = (\pi^+\pi^-\gamma/\pi^+\pi^-\eta_N)$, $\eta(549) \to$ all neutrals (N), vs laboratory momentum P_{lab} of incident particle producting $\eta'(958)$ [or $M(953)$]. The data are from Refs. 1–6 as indicated (see Table I for values of R). The average value of R, from the data of Refs. 1, 2, 4–6, $\bar{R} = 0.97 \pm 0.08$, is shown as the dash-dot line and the shaded area.

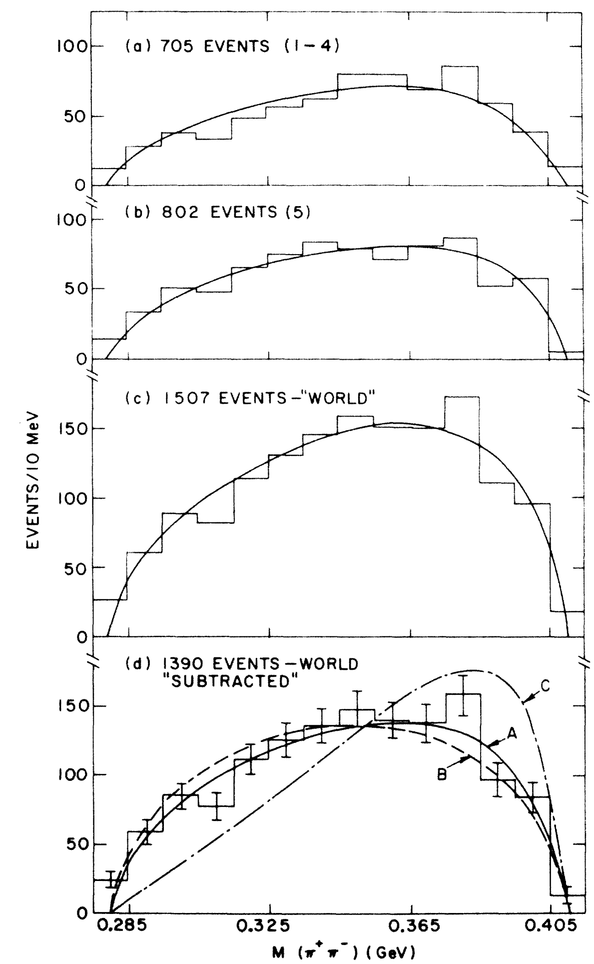

FIG. 2. The $M(\pi^+\pi^-)$ distributions of $\eta' \to \pi^+\pi^-\eta$ for (a) 705 events from other experiments (Refs. 1–4), (b) 802 events from Ref. 5, (c) the "world" data, the sum of (a) and (b), and (d) the "subtracted" world data, the data of (c) with the estimated contribution of 117 $\eta' \to \pi^0\pi^0\eta_c$ events subtracted [using data from Fig. 4(b) of Ref. 5]. The solid curves shown normalized to the data of (a), (b), and (c) correspond to the best fit [α(real) = −0.11 ± 0.03] to the data of (c). The solid curve A shown on the data of (d) corresponds to the best fit, α(real) = −0.08 ± 0.03; the dashed curve B corresponds to $\alpha = 0$, which is satisfactory; and the dash-dot curve C corresponds to the "predicted" value $\alpha = -0.43$, which does not fit the data at all.

TABLE II. Matrix element parameters obtained for $\eta'(958)$ under the assumption that $J^P=0^-$. The matrix element squared is $(|1+\alpha y|^2+cx^2)$, where x,y are the usual Dalitz variables (see text). All fits to the data had acceptable confidence levels.

Data of ref.	No. of events	α (real) ($c\equiv 0$)	α (complex) Reα	|Imα|	c
1	278	-0.11 ± 0.05	-0.11 ± 0.05	0 ± 0.3	...
2	57	...			
3 $\eta'(958)$	109	$-0.34^{+0.17}_{-0.15}$			
$M(953)$	60	$-0.46^{+0.22}_{-0.19}$			
4	201	-0.19 ± 0.07			
5 [a] "all"	802	-0.05 ± 0.03			
"subtracted"	534	0 ± 0.10			
5 [b]	802	-0.039 ± 0.033	-0.06 ± 0.04	$0.4^{+0.15}_{-0.21}$	...
			-0.06 ± 0.04	$0.38^{+0.16}_{-0.25}$	-0.06 ± 0.16
			-0.04 ± 0.03	...	-0.11 ± 0.14
1–4	705	$-0.16\ \pm 0.04$	-0.16 ± 0.04	0 ± 0.8	...
World (1–5)	1507	$-0.11\ \pm 0.03$	-0.11 ± 0.03	0 ± 1.0	...
"Subtracted" world (1–5) [c]	1390	$-0.08\ \pm 0.03$	-0.08 ± 0.03	0 ± 0.3	...
7 [d]	392	$-0.28\ \pm 0.06$	$(1+2\alpha y+\beta y^2)$ $\alpha=-0.25\pm 0.08$ $\beta=-0.03\pm 0.04$		

[a] As reported in Ref. 5.

[b] Newly reported here; values from a maximum-likelihood analysis of the "normalized" Dalitz plot coordinates (see Ref. 5).

[c] The estimated contribution of 117 $\eta'\to\pi^0\pi^0\eta_c$ events has been subtracted [using data of Fig. 4(b) of Ref. 5].

[d] Counter experiment data including about 25% background; the y distribution has been "corrected" for instrumental biases.

the event distribution (1) see Ref. 11. The α parameters for $\eta(549)$ and K^0 decay $\alpha\simeq -0.5$ are nonzero and in reasonable agreement with theoretical predictions (see, e.g., Refs. 8 and 11). For $\eta'\to\pi^+\pi^-\eta$ decay, Schwinger[12] proposed that $\alpha=-0.41$ and Dolgov, Vainshtein, and Zakharov[13] give a value $\alpha=-0.45$. Scadron[14] has given a phenomenological discussion of chiral symmetry breaking including $\eta'\to\pi^+\pi^-\eta$ decay; others have considered $\eta'\to\pi\pi\eta$ from the standpoint of finite dispersion relations.[15]

The $M(\pi^+\pi^-)$ mass distributions from the data of Refs. 1–5 are shown in Fig. 2. The data, where necessary, were apportioned to the mass bins shown assuming uniformity in the bins of the original data; this is statistically satisfactory since the mass distributions are smoothly varying and nearly like phase space. Figure 2(a) shows the data from Refs. 1–4 and Fig. 2(b) shows those from Ref. 5. The combined "world" data, exclusive of those in Ref. 7, are shown in Fig. 2(c). The solid curves shown in Figs. 2(a), 2(b), and 2(c) represent the best fit to the data of Fig. 2(c) normalized to each data set respectively. Since background from $\eta'\to\pi^0\pi^0\eta_c$, $\eta_c\to\pi^+\pi^-\pi^0$ is included in these data, a "subtraction" due to an estimated 117 $\pi^0\pi^0\eta_c$ events has been made, as in Ref. 5; the "subtracted" world data are shown in Fig. 2(d). The three curves shown are (a) the best fit, α(real) $=-0.08\pm 0.03$, (b) a fit with $\alpha\equiv 0$, which is satisfactory, and (c) a fit to $\alpha=-0.43$ (average of the predictions of Refs. 12 and 13), which is unsatisfactory. The results of various fits with α(real), α(complex), and (α and c) are given in Table II in addition to the values deduced in the original references. We find that the parameter c is not needed to describe the data (i.e., $c=0$) and that the best value of α(real) $=-0.08\pm 0.03$. Also, $\alpha\equiv 0$ gives an acceptable fit. Thus the decay mode $\eta'\to\pi^+\pi^-\eta$ does not require a large value for the slope parameter α as do $\eta(549)$ and $K(498)$.

DIPION PHASE SHIFT δ_{00}

If the spin-parity of the $\eta'(958)$ is indeed 0^-, the $(\pi^+\pi^-)$ pair in the $\eta' \to \pi^+\pi^-\eta$ decay is predominantly in the $I=J=0$ state. The deviation of $M(\pi^+\pi^-)$ from phase space might then be assumed to be proportional to $\sin^2\delta_{00}$ for the $\pi\pi$ system, if the $(\pi^\pm\eta)$ system does not show any structure. The $(\pi^\pm\eta)$ system is not known to exhibit any structure below about 950 MeV.[16] The $M(\pi^\pm\eta)$ distributions from Refs. 1 and 5 are given in Fig. 3; the data are well represented by the solid phase-space curves drawn over the data.

Parameterizing δ_{00} by the usual s-wave scattering length a, effective range r approximation,[17]

$$k\cot\delta = a^{-1} + \tfrac{1}{2}k^2 r\,, \qquad (2)$$

where k is the pion momentum in the dipion rest frame, we can fit the data of Fig. 2(d) as phase space times $\sin^2\delta$. Part of the two-standard-deviation region of δ_{00} vs $M(\pi^+\pi^-)$ about the best solution is shown shaded in Fig. 4(a). The lower dashed envelope corresponds to $(a, r) = (2.5, 3.0)$ (in fermis). The δ_{00} phase shifts from K_{l4} decay of Zylbersztejn *et al.*[18] and Beier *et al.*[19] are shown as the open and solid circles, respectively. The prediction of the Weinberg[20] scattering length $(a, r) = (0.2\,m_\pi^{-1}, 0)$ is also shown. The (extrapolated) δ_{00} phase-shift solutions of Protopopescu *et al.*[21] (not shown) lie somewhat higher than the Weinberg curve. We see that "δ_{00}" from the η' does not agree with these other values. Figure 4(b) shows the data of Fig. 2(d), without error bars, with some additional curves. Curve D corresponds to the lower envelope labeled D in Fig. 4(a); curve E is that corresponding to the Weinberg scattering length,[20] which is essentially the "linear matrix element" curve of Fig. 2(d). Curve F is a fit for $J^P = 2^-$ with no phase-shift effects, as usually done. We can improve the 2^- fit by including $\sin^2\delta$ using the scattering length of Weinberg. Curve G shows such a fit, which is quite satisfactory; somewhat better fits could be obtained using the extrapolated Protopopescu

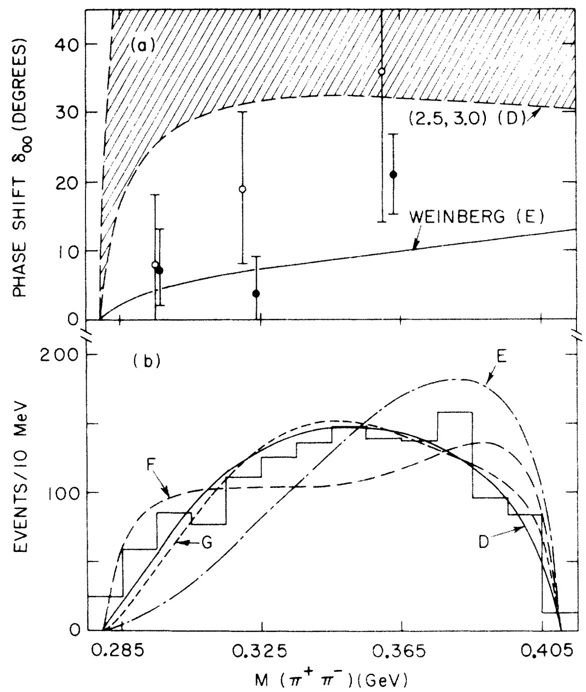

FIG. 4. (a) The "phase shift" δ_{00} (degrees) vs dipion mass $M(\pi^+\pi^-)$. The shaded region corresponds to the assumption that $M(\pi^+\pi^-)$ in $\eta' \to \pi^+\pi^-\eta$ is proportional to $\sin^2\delta_{00}$ (see text). The shaded region is the 95% confidence level region allowed by the data of Fig. 2(d). The δ_{00} of Zylbersztejn *et al.*, open circles (Ref. 18), and of Beier *et al.*, solid points (Ref. 19), and the phase shift corresponding to the scattering length of Weinberg (Ref. 20), $a_{00} = 0.2m_\pi^{-1}$, are also shown. (b) The $M(\pi^+\pi^-)$ distribution of Fig. 2(d) with four additional fits. Curves D and E for $J^P = 0^-$ are phase space times $\sin^2\delta$, with δ corresponding to the curves D and E in part (a), respectively. Curves F and G are for $J^P = 2^-$; curve F corresponds to the best fit to the usual mixture of matrix elements: curve G corresponds to modifying the $(l_{\pi\pi}, l_\eta) = (0, 2)$ part of the mixture using Weinberg's scattering length.

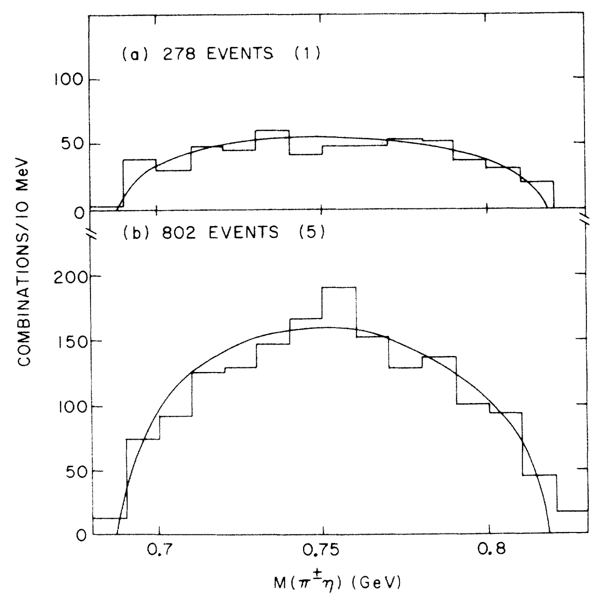

FIG. 3. The $M(\pi^\pm\eta)$ distributions of $\eta' \to \pi^+\pi^-\eta$ for (a) the 278 events of Ref. 1 and (b) the 802 events of Ref. 5. There are two $\pi\eta$ combinations per event. The solid phase-space curves shown are a good representation of the data.

phase-shift solutions. Also, for $J^P = 0^-$, some double d-wave amplitude ($l_{\pi\pi} = l_\eta = 2$) can be added to the usual double s-wave amplitude ($l_{\pi\pi} = l_\eta = 0$) which predominates; very little double d-wave amplitude is expected due to the small Q value of the $\eta' \to \pi^+\pi^-\eta$ decay. The ($l_{\pi\pi} = l_\eta = 0$) part of the matrix element can then be modified by the factor $e^{i\delta}\sin\delta$ using the Weinberg phase shift. Such a fit was made (not shown) to the data of Fig. 25 of Ref. 5; it does not describe the data. The (essentially) isotropic $\cos\theta_{\pi^+\eta}$ distribution constrains the interference term to cancel the angular correlations of the ($l_{\pi\pi} = l_\eta = 2$) amplitude. The resulting $M(\pi^+\pi^-)$ distribution deviates almost as strongly from phase space as does curve E in Fig. 4(b).

Weinberg[20] has noted that the meaning of any procedure such as we used above is ambiguous in decay processes such as K, η, and η' decay. However, the "linear matrix element" expectation and the Weinberg scattering length appear to be equivalent (curves C [Fig. 2(d)] and E (Fig. 4(b)]) and appear to explain $\eta(549) \to \pi^+\pi^-\pi^0$ decay.[11] The same expectation is not realized for $\eta' \to \pi^+\pi^-\eta$ if $J^P = 0^-$. Whether this is due to the largely singlet nature of the η', to the possibility that $J^P = 2^-$, or to some unknown $\pi^\pm\eta$ interaction, or is an accident in the case of $\eta(549)$, is not known.

ACKNOWLEDGMENTS

We thank Professor M. Scadron and Dr. S. D. Protopopescu for helpful discussions and Dr. V. VanderBurg for help with some of the computations.

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

[1]A. Rittenberg, LRL Report No. UCRL-18863, 1969 (unpublished).

[2]P. M. Dauber *et al.*, Phys. Rev. Lett. 13, 449 (1964).

[3]M. Aguilar-Benitez *et al.*, Phys. Rev. D 6, 29 (1972).

[4]S. M. Jacobs *et al.*, Phys. Rev. D 8, 18 (1973).

[5]J. S. Danburg *et al.*, Phys. Rev. D 8, 3744 (1973).

[6]J. Badier *et al.*, Phys. Lett. 17, 337 (1965); J. Mott *et al.*, Phys. Rev. 177, 1966 (1969).

[7]G. W. London *et al.*, Phys. Rev. 143, 1034 (1966); the mass distribution of their 49 events is not given separately from the combined BNL-Syracuse plus LRL sample of 102 events. J. P. Dufey *et al.*, Phys. Lett. 29B, 605 (1969); their 392 events, including about 25% background, have not been used; they do not present either the raw or corrected mass distributions. See also footnote 16 of Ref. 8.

[8]G. R. Kalbfleisch *et al.*, Phys. Rev. Lett. 31, 333 (1973).

[9]D. Cohen *et al.*, Bull. Am. Phys. Soc. 19, 80 (1974).

[10]R. H. Dalitz, Philos. Mag. 44, 1068 (1953).

[11]H. R. Hicks and P. Winternitz, Phys. Rev. D 4, 2339 (1971); 5, 2877 (1972); see, e.g., D. W. Carpenter *et al.*, *ibid.* 1, 1303 (1970) and references cited therein.

[12]J. Schwinger, Phys. Rev. 167, 1432 (1968).

[13]A. D. Dolgov, A. I. Vainshtein, and V. I. Zakharov, Phys. Lett. 24B, 425 (1967); or A. I. Vainshtein and V. I. Zakharov, Usp. Fiz. Nauk. 100, 225 (1970) [Sov. Phys.–Usp. 13, 73 (1970)].

[14]M. D. Scadron and H. R. Jones, Phys. Rev. D 10, 967 (1974).

[15]See, e.g., R. H. Graham and T. Ng, Phys. Rev. D 8, 2957 (1973). They expect a value of $\alpha \simeq -0.2$ (-0.28 to -0.10).

[16]Structure in $\pi^\pm\eta$ would be due to $\pi_N^\pm(980)$ and $A_2^\pm(1310)$; see, e.g., Particle Data Group, Rev. Mod. Phys. 45, S1 (1973).

[17]We use the Chew-Low form rather than the Serber modified form. See, e.g., J. D. Jackson, *The Physics of Elementary Particles* (Princeton Univ. Press, Princeton, N. J., 1958), p. 23.

[18]A. Zylbersztejn *et al.*, Phys. Lett. 38B, 457 (1972).

[19]E. W. Beier *et al.*, Phys. Rev. Lett. 30, 399 (1973).

[20]S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

[21]S. D. Protopopescu *et al.*, Phys. Rev. D 7, 1279 (1973).