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[25]We have used the fitted formula

$$E\frac{d^3\sigma}{d^3p} = AP_\perp{}^{-n} e^{-bP_\perp/\sqrt{s}}$$

with $A = (1.54 \pm 0.10)\times 10^{-26}$, $n = 8.24 \pm 0.05$, and $b = 26.10 \pm 0.5$ given in Ref. 11.

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Dispersion-relation calculation of the πN elastic amplitude A' for $0.20 \leqslant -t \leqslant 0.40$ (GeV/c)^2 and $0.87 \leqslant p \leqslant 20.0$ GeV/c

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A method previously presented in which dispersion relations were written for the logarithm of the scattering amplitude is extended to larger values of momentum transfer. The earlier technique was applied to πN elastic scattering but was restricted to $-t \lesssim 0.24$ (GeV/c)^2 because of the crossing of the t-dependent threshold energy and the nucleon pole. That limitation is here removed by writing the dispersion relations for the amplitude with its pole removed, and the thus extended method is again applied to πN scattering. The amplitudes $A'_\pm$ calculated at pion lab momenta of 2, 4, 8, 12, 15, 20, and 30 GeV/c for $-t = 0.20$, 0.25, 0.30, 0.35, 0.40 (GeV/c)^2 are presented.

I. INTRODUCTION

In a previous paper[1] (herein referred to as I) we applied the Hilbert transforms

$$\operatorname{Im} F(\nu, t) = -\frac{1}{\pi} \mathrm{P}\int \frac{\operatorname{Re} F(\nu', t)d\nu'}{\nu' - \nu} \tag{1}$$

to the logarithm of the pion-nucleon elastic scattering amplitude $A' = |A'|e^{i\phi}$,

$$F = \ln A' = \ln|A'| + i\phi, \tag{2}$$

to obtain a dispersion relation giving the phase of this amplitude in terms of its magnitude

$$\phi(\nu, t) = -\frac{1}{\pi} \mathrm{P} \int \frac{\ln|A'(\nu', t)| d\nu'}{\nu' - \nu} . \quad (3)$$

Here $\nu = (s-u)/M$ with s, u, and t the Mandelstam variables and M the nucleon mass. Two major problems arise in practical applications of this logarithmic dispersion relation (LDR). One, shared with conventional nonforward dispersion relations, is the presence of an unphysical region where no information about the amplitude's modulus can be obtained directly from experiment. The other, unique to the LDR, is that the zeros of the amplitude are singularities of its logarithm and therefore a knowledge of the location of these zeros is required for the evaluation of Eq. (3). The problem of the zeros will be discussed in Sec. III. Various methods for calculating nonforward scattering amplitudes including the LDR approach have recently been reviewed by Höhler and coworkers.[2]

In I the effect of the unphysical region on the integrals in (3) was minimized by arranging the calculation in such a way that its contribution was very small. This was accomplished by subtracting the dispersion relations at $\pm \nu_0$ near the upper end of the partial-wave region and inserting for F in Eq. (1) not $\ln A'$ but rather

$$F(\nu, t) = \frac{(\nu^2-\nu_1^2)^{1/2} \ln A'(\nu, t)}{(\nu-\nu_0)(\nu+\nu_0)} . \quad (4)$$

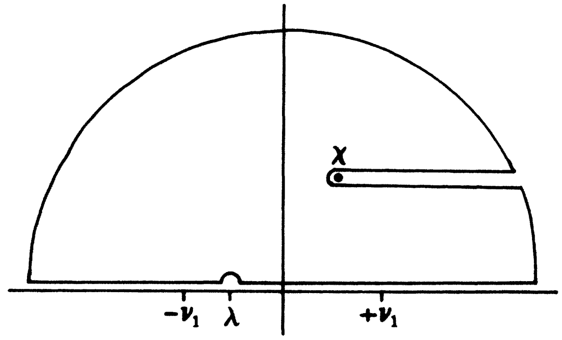

FIG. 1. Contour in the complex energy plane used in the derivation of Eqs. (6) and (9).

$A'(\nu, t)$ at fixed t, regarded as a function of the complex energy ν, has cuts along the real axis from thresholds at $\pm \nu_1$ to $\pm\infty$, and the usual nucleon poles at $\lambda_\pm$ on the axis between thresholds.[3] These features as well as the contour used in deriving (3) are shown in Fig. 1. In terms of t and the masses of the pion and nucleon, μ and M respectively, the threshold and pole locations are

$$\nu_1 = \mu + \frac{t}{4M}, \quad \lambda_\pm = \mp \left(-\frac{\mu^2}{2M} + \frac{t}{4M} \right). \quad (5)$$

If the subscripts $\pm$ refer to $\pi^\pm p$ scattering, the complete LDR, written here for ϕ_- with a similar expression holding for ϕ_+, is

$$\begin{aligned}
\phi_-(\nu, t) = {} & \left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2} \left[\frac{(\nu+\nu_0)}{2\nu_0} \phi_-(\nu_0, t) - \frac{(\nu-\nu_0)}{2\nu_0} \phi_+(\nu_0, t) \right] \\
& - \frac{\nu^2-\nu_0^2}{\pi(\nu^2-\nu_1^2)^{1/2}} \left[\mathrm{P}\int_{\nu_1}^{\infty} \frac{(\nu'^2-\nu_1^2)^{1/2} \ln|A'_+(\nu', t)| d\nu'}{(\nu'+\nu)(\nu'^2-\nu_0^2)} + \mathrm{P}\int_{\nu_1}^{\infty} \frac{(\nu'^2-\nu_1^2)^{1/2} \ln|A'_-(\nu', t)| d\nu'}{(\nu'-\nu)(\nu'^2-\nu_0^2)} \right] \\
& + \frac{1}{2}\left[1 - \left(\frac{\nu}{\nu_0}\right)\left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right) \right] [\phi_-(\nu_1, t) - \phi_+(\nu_1, t)] - \sin^{-1}\left[\frac{\nu_1^2 - \nu\lambda_-}{\nu_1(\lambda_- - \nu)}\right] \\
& - \left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2} \left\{ \frac{(\nu_0-\nu)}{2\nu_0} \sin^{-1}\left[\frac{\nu_1^2+\nu_0\lambda_-}{\nu_1(\nu_0+\lambda_-)}\right] + \frac{(\nu_0+\nu)}{2\nu_0} \sin^{-1}\left[\frac{\nu_1^2-\nu_0\lambda_-}{\nu_1(\nu_0-\lambda_-)}\right] \right\} \\
& - \pi \frac{(\nu-\nu_0)}{\nu_0} \left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2} - 2\,\mathrm{Im}\ln\left[\frac{-\nu_1^2 + \nu\chi + (\nu^2-\nu_1^2)^{1/2}(\chi^2-\nu_1^2)^{1/2}}{(\nu^2-\nu_1^2)^{1/2}(\chi-\nu)}\right] \\
& + \left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2} \left\{ \frac{(\nu+\nu_0)}{\nu_0} \mathrm{Im}\ln\left[\frac{-\nu_1^2+\nu_0\chi+(\nu_0^2-\nu_1^2)^{1/2}(\chi^2-\nu_1^2)^{1/2}}{(\nu_0^2-\nu_1^2)^{1/2}(\chi-\nu_0)}\right] \right. \\
& \left. + \frac{(\nu-\nu_0)}{\nu_0} \mathrm{Im}\ln\left[\frac{-\nu_1^2-\nu_0\chi-(\nu_0^2-\nu_1^2)^{1/2}(\chi^2-\nu_1^2)^{1/2}}{(\nu_0^2-\nu_1^2)^{1/2}(\chi+\nu_0)}\right] \right\},
\end{aligned} \quad (6)$$

which is the equation used in the calculation reported in I.[4]

In deriving (6) we made the assumption, to be clarified in Sec. III, that A' has no real zeros and a single complex zero located at χ in the upper half plane. This expression was used in I to calculate the phases of A' for high-energy nonforward scattering, specifically, for the four-momentum transfer squared t and the pion lab momenta p in the ranges $-t \leq 0.20$ (GeV/c)^2 and $0.575 \leq p \leq 20.0$ GeV/c.

II. AN LDR FOR A' WITH THE POLE REMOVED

In the present investigation we have raised the lower limit to $p = 0.87$ GeV/c, as will be explained in Sec. III A ii, and have extended the calculation of the phase of A' to $0.20 \leqslant -t \leqslant 0.40$ (GeV/c)^2. However, this cannot be done with the LDR in the form (6). The difficulty that arises can be seen by referring to Eq. (5) and Fig. 1. The pole λ_- of A'_- is between the thresholds $\pm\nu_1$ for $|t|$ sufficiently small, and Eq. (6) is derived on this assumption. However, as $|t|$ increases the left-hand threshold moves to the right and the pole moves to the left, the two crossing at

$$t \approx -0.24 \ (\text{GeV}/c)^2 .$$

For $\lambda_- < -\nu_1$ the argument of the term

$$\sin^{-1}\left[\frac{\nu_1^2-\nu\lambda_-}{\nu_1(\lambda_- -\nu)}\right]$$

in (6) is < -1.

The problem caused by the crossing of the pole and threshold can be remedied by removing the pole from the amplitude. That is, in the derivation of the LDR we make the replacement

$$A'(\nu,t) \to (\nu-\lambda)A'(\nu,t) . \tag{7}$$

The subtraction procedure remains the same and instead of the F given by Eq. (4) we insert into Eq. (1)

$$F(\nu,t) = \frac{(\nu^2-\nu_1^2)^{1/2}\ln[(\nu-\lambda)A'(\nu,t)]}{(\nu-\nu_0)(\nu+\nu_0)} . \tag{8}$$

The new LDR analogous to (6) is

$$\begin{aligned}
\phi_-(\nu,t) = {} & -\arg(\nu-\lambda_-) + \left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2}\left[\frac{(\nu+\nu_0)}{2\nu_0}\,\phi_-(\nu_0,t) - \frac{(\nu-\nu_0)}{2\nu_0}\,\phi_+(\nu_0,t)\right] \\
& - \frac{\nu^2-\nu_0^2}{\pi(\nu^2-\nu_1^2)^{1/2}}\left[\mathrm{P}\int_{\nu_1}^{\infty}\frac{(\nu'^2-\nu_1^2)^{1/2}\ln|(\nu'-\lambda_+)A'_+(\nu',t)|\,d\nu'}{(\nu'+\nu)(\nu'^2-\nu_0^2)}\right. \\
& \left. \qquad + \mathrm{P}\int_{\nu_1}^{\infty}\frac{(\nu'^2-\nu_1^2)^{1/2}\ln|(\nu'-\lambda_-)A'_-(\nu',t)|\,d\nu'}{(\nu'-\nu)(\nu'^2-\nu_0^2)}\right] \\
& + \left[1-\frac{\nu}{\nu_0}\left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2}\right]\phi_-(\nu_1,t) - \pi\,\frac{(\nu-\nu_0)}{2\nu_0}\left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2} \\
& -2\,\mathrm{Im}\ln\left[\frac{-\nu_1^2+\nu\chi+(\nu^2-\nu_1^2)^{1/2}(\chi^2-\nu_1^2)^{1/2}}{(\nu^2-\nu_1^2)^{1/2}(\chi-\nu)}\right] \\
& + \left(\frac{\nu_0^2-\nu_1^2}{\nu^2-\nu_1^2}\right)^{1/2}\left\{\frac{(\nu+\nu_0)}{\nu_0}\,\mathrm{Im}\ln\left[\frac{-\nu_1^2+\nu_0\chi+(\nu_0^2-\nu_1^2)^{1/2}(\chi^2-\nu_1^2)^{1/2}}{(\nu_0^2-\nu_1^2)^{1/2}(\chi-\nu_0)}\right]\right. \\
& \left. \qquad + \left(\frac{\nu-\nu_0}{\nu_0}\right)\mathrm{Im}\ln\left[\frac{-\nu_1^2-\nu_0\chi-(\nu_0^2-\nu_1^2)^{1/2}(\chi^2-\nu_1^2)^{1/2}}{(\nu_0^2-\nu_1^2)^{1/2}(\chi+\nu_0)}\right]\right\}
\end{aligned} \tag{9}$$

The derivation follows that given elsewhere.[5]

While Eq. (9) has been worked out to treat the situation where the pole crosses the threshold, it is also valid for the pole in the gap, $-\nu_1 < \lambda < \nu_1$, and when this is the case Eqs. (6) and (9) must agree. They can be shown to be the same by writing the integrals which occur in (9) as

$$\begin{aligned}
\int \frac{(\nu'^2-\nu_1^2)^{1/2}\ln|(\nu'-\lambda)A'(\nu',t)|\,d\nu'}{(\nu'-\nu)(\nu'^2-\nu_0^2)}
&= \int \frac{(\nu'^2-\nu_1^2)^{1/2}\ln|A'(\nu',t)|\,d\nu'}{(\nu'-\nu)(\nu'^2-\nu_0^2)} \\
&\quad + \int \frac{(\nu'^2-\nu_1^2)^{1/2}\ln|\nu'-\lambda|\,d\nu'}{(\nu'-\nu)(\nu'^2-\nu_0^2)} ,
\end{aligned} \tag{10}$$

and noting that the second integral on the right is the one which will arise if we write a subtracted LDR for $(\nu-\lambda)$ by itself. If this is done, more precisely if we set A' equal to unity in (8) and put the resulting F into (1) the equation obtained when combined with (10) shows that (6) and (9) are the same for $-\nu_1 < \lambda < \nu_1$. Requiring (9) to reproduce results already found in I provided a necessary check on the present calculation. Actually the results obtained with (9) for $-t \leqslant 0.20$ (GeV/c)^2 are somewhat more reliable than those presented in I due to improvements in the computational procedure.

The presence of the factor $(\nu-\lambda)$ in the logarithms in the integrands of Eq. (9) causes no convergence difficulties because[1,3]

$$\nu^{-1}|A'(\nu,t)| < K \quad \text{as } \nu\to\infty,\ K \text{ constant}$$

although it weights the integrals slightly more toward high energies. Because (9) is quite complicated, a breakdown of the contributions of the various types of terms at several energies is given in Table I. By using this table together with the analogous one (Table II) in I, a comparison between (6) and (9) can be made. In doing this it

TABLE I. Contributions to the total phase $\phi_\pm$ of the different types of terms appearing in Eq. (9). The terms are grouped as follows: Those depending on the (A) threshold phase $\phi_\pm(\nu_1, t)$, (B) location of the zero $\chi(t)$, (C) subtraction-point phase $\phi_\pm(\nu_0, t)$, (D) direct integrals over $\ln|(\nu-\lambda_\pm)A'_\pm(\nu,t)|$, and (E) crossed integrals over $\ln|(\nu-\lambda_\mp)A'_\mp(\nu,t)|$. All entries are in degrees.

p (GeV/c)	$-t$ [(GeV/c)^2]	A	B	C	D	E	ϕ_-	ϕ_+
0.87	0.0	0.0	−85.0	105.6	−7.3	63.0	76	
		−1.2	−81.7	94.4	47.7	58.6		118
	0.2	0.0	−86.3	127.1	20.9	56.2	118	
		−0.5	−83.4	123.7	35.4	57.5		133
	0.4	0.0	−70.2	140.6	49.2	45.0	165	
		−0.1	−79.5	167.4	−6.2	57.7		139
1.88	0.0	0.0	58.3	99.6	−4.2	−55.3	98	
		0.6	56.8	88.6	18.3	−52.4		112
	0.2	0.0	59.2	107.5	−2.2	−48.6	116	
		0.2	57.6	104.0	0.7	−49.2		113
	0.4	0.0	51.4	104.6	3.4	−38.9	121	
		0.0	59.6	131.3	−38.9	−46.6		106
8.00	0.0	0.0	152.6	96.0	95.7	−246.7	98	
		1.0	149.9	85.0	106.5	−239.7		103
	0.2	0.0	153.4	95.2	77.9	−217.1	109	
		0.4	150.4	91.7	77.9	−217.8		103
	0.4	0.0	141.6	82.9	59.0	−180.2	103	
		0.1	160.2	109.6	37.5	−197.6		110

should be remembered that the $\sin^{-1}$ terms of (6) are incorporated in the integrals and threshold terms of (9).

III. INPUT AND RESULT

A. Input

To evaluate (9) numerically we must use the subtraction-point phases $\phi_\pm(\nu_0, t)$, the location of the zero or zeros of the amplitude, and the numerical values of $|A'_\pm(\nu, t)|$ for all energies from threshold to infinity. As in I the subtraction point was taken at $p = 1.278$ GeV/c and the phases there were obtained by reconstruction from the partial-wave results.[6] The other pieces of input require somewhat more discussion.

i. Location of the zero. It is known that $A'(\nu, t)$ has only one zero for $t=0$,[7] and it is reasonable to assume that this situation persists for those small values of t for which the LDR is to be used. This single zero will in general depend on t, however, and in I its location in the complex plane was determined by the requirement that the LDR correctly reproduce the phases of A' in the resonance region, $\nu \leq 2.0$ GeV. The phase in this region was reconstructed from existing partial-wave analyses.[6]

In the present calculation we have again assumed a single t-dependent zero $\chi_1 + i\chi_2$, and the values (χ_1, χ_2) which were found to give best agreement with the phases all across the partial-wave region were (0.05, 0.45), (0.05, 0.50), (0.04, 0.50), (−0.15, 0.90), and (−0.20, 1.0) for $-t = 0.20$, 0.25, 0.30, 0.35, and 0.40 (GeV/c)^2, respectively. As in I the agreement obtained was better for ϕ_+ where the difference between the LDR and partial-wave phases was less than 5% at all energies. For ϕ_- the difference reached 9% for $-t = 0.30$, 0.40 (GeV/c)^2 and $p = 0.874$, 1.053 GeV/c, respectively. This may in part be due to uncertainty in the $\pi^- p$ data. We will return to the zeros of the amplitude in Sec. IV.

ii. Modulus of $A'(\nu, t)$. In the main, the procedures followed here were as before and I should be consulted for details. For resonance energies, $\nu \leq 2$ GeV, the moduli were reconstructed from partial-wave results.[6] Near the forward direction in the diffraction region, $2 < \nu \leq 24$ GeV, the differential cross section involves only A' and is observed to fall off exponentially with t:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{t=0} e^{at+bt^2}$$

$$\sim \left(\frac{M}{4\pi W}\right)^2 \left(1 - \frac{t}{4M^2}\right) |A'|^2,$$

where $W = s^{1/2}$. Therefore,

$$|A'(\nu,t)| = \frac{|A'(\nu,0)|\, e^{(at+bt^2)/2}}{(1-t/4M^2)^{1/2}}\,, \tag{11}$$

where a and b are weakly energy-dependent. As in I we fitted the $\pi^- p$ (2–24 GeV) and the $\pi^+ p$ (2–16 GeV) cross-section data[8] to the form (11), and values of $A'(\nu, 0)$, $a(\nu)$, and $b(\nu)$ were obtained.

At very high energies, $\nu > 24$ GeV, few angular data are available and some assumptions must be made about the amplitude in this range. We have supposed Eq. (11) to be valid at these high energies. For $p > 8$ GeV/c a and b are observed to be nearly constant, and their average values are used. Above 25 GeV the contribution of the real part of $A'(\nu, 0)$ is less than 2%, and we have therefore approximated A' by its imaginary part.[9] This was obtained through the optical theorem from Carter's fit to the total cross section.[10]

The unphysical region, $\nu_1 < \nu < \mu$, is an interval of energy just above threshold for which $\cos\theta < -1$ when $t \neq 0$. It increases in size as $-t$ becomes larger. In this region the scattering is not a physically realizable process and the input for $|A'|$ cannot be obtained from experiment. Compared with the $\simeq 20$ GeV/c range over which $|A'|$ is known the unphysical interval is relatively small, starting at zero and running up to $p = 0.293$, 0.450 GeV/c for $-t = 0.2$, 0.4 (GeV/c)^2, respectively. However, it is quite troublesome because there is no reliable way to determine the amplitude there. The approach developed in I and followed again here is to subtract the LDR in such a way that the contribution of the unphysical region is minimized. The actual input $|A'|$ is then obtained by the Lehmann procedure[11] for the analytic continuation of the partial-wave expansion although the values of ν and t are outside the range for which this method has been proven to converge.

Integrals of the type that enter Eqs. (6) and (9) are strongly influenced by that portion of the range of integration lying closest to the point at which the principal value is taken. Therefore, the influence of the unphysical region tends to be greatest at lowest energies. Because of the uncertainty in $|A'|$ for unphysical energies, we have adopted the practice, both in this calculation and in I, of regarding the results of our methods as unreliable when the unphysical region contributes more than 5% to the total phase. In I, where the largest momentum transfer is $-t = 0.20$ (GeV/c)^2, the unphysical region contributed 5% (π^+) at $p = 0.575$ GeV/c and decreased steadily thereafter. The lower limit of validity in the present calculation has been raised to $p = 0.87$ GeV/c, corresponding to an increase in $-t$ to 0.40 (GeV/c)^2, and with this new lower limit the contribution of the unphysical region to the total phase is always less than 4%. Again the largest contribution is to ϕ_+.

B. Results

With the procedures and input just discussed, the LDR (9) for $\phi_\pm$ can now be evaluated. As in I, we are interested in finding these phases at energies above the partial-wave region and have carried out the calculation for $0 \leq -t \leq 0.4$ (GeV/c)^2 at intervals of 0.05 (GeV/c)^2 for each of the momenta $p = 2.0$, 4.0, 8.0, 12.0, 15.0, 20.0, and 30.0 GeV/c. The main results are for the momentum range $2.0 \leq p \leq 12.0$ GeV/c; above this the phases are much more dependent on the assumptions made about the asymptotic region. All results, including some for the partial-wave region, are given in Table II. They vary slowly with energy, and interpolations within the momentum intervals can be made with reasonable confidence. Because of an improvement in computational procedure, we have recalculated some of the phases for $0 \leq -t \leq 0.2$ (GeV/c)^2 which were originally presented in I. They differ only slightly from the earlier results. Finally, Table I lists the contributions to the total phase $\phi_\pm$ of the various types of terms entering Eq. (9).

IV. CONCLUSIONS

This is the second in a projected sequence of three papers in which we attempt to make practical application of the logarithmic dispersion relation (3). The LDR is intrinsically attractive because it requires as input amplitude moduli and these are often more available at high energies than other forms of the amplitude. With these moduli we aim to calculate the phases at nonforward angles for energies lying above the partial-wave region. Because input, moduli, and subtraction-point phases change as experiments are refined and new ones performed, we do not wish to overly stress the particular numerical values we have found, but rather to emphasize the LDR approach generally. The calculation can be redone as better input becomes available.

Several difficulties arise in attempts to utilize the LDR. Two of these are major and were dealt with in I. They are the presence of a t-dependent unphysical region and the necessity of knowing the location of the zeros of the amplitude. A less serious difficulty, treated in the present investigation, arises because the locations of the nucleon pole and the threshold are t-dependent and move toward one another with increasing $|t|$, crossing at $-t \approx 0.24$ (GeV/c)^2 for πN scattering. This problem has been remedied by using an LDR writ-

TABLE II. The phases $\phi_\pm(\nu,t)$ of the amplitudes $A'_\pm(\nu,t)$ as given by Eq. (9) using the zeros $\chi(t)$. At each p and t the upper number is ϕ_+ and the lower is ϕ_-, both in degrees.

p (GeV/c) \ $-t$ [(GeV/c)^2]	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
0.87	118	123	127	130	133	135	134	138	139
	76	84	94	105	118	129	139	160	165
1.05	102	107	112	117	121	125	128	131	133
	103	105	107	109	112	116	121	134	147
1.88	112	113	113	113	113	112	111	109	106
	98	103	107	112	116	120	124	120	121
2.0	109	110	111	111	111	111	110	108	105
	100	104	109	114	118	123	127	124	125
4.0	108	109	109	110	110	110	110	114	113
	100	104	108	111	113	116	117	109	107
8.0	103	103	103	103	103	102	102	109	110
	98	101	105	107	109	111	113	105	103
12.0	102	102	102	102	101	101	101	107	108
	97	100	103	105	107	109	111	103	101
15.0	105	104	103	102	102	101	100	107	108
	103	104	106	108	108	110	111	104	102
20.0	104	104	103	102	100	99	98	106	108
	100	101	102	104	104	105	106	99	97
30.0	95	94	93	92	91	89	89	97	98
	100	101	102	104	105	106	107	100	98

ten for the amplitude multiplied by a factor which removes the pole. With this new LDR the calculation has been extended to $-t=0.4$ (GeV/c)^2. The subtraction procedure introduced in I continues to be effective in limiting the contribution of the unphysical region to the total phase. For example, the contribution of the unphysical region was less than 0.5% for $p>4.0$ GeV/c and the $0\leq -t\leq 0.20$ (GeV/c)^2 range considered in I and does not exceed 2.5% for $2<p<20$ GeV/c in the present calculation although the unphysical region is now much larger. Although the pole-removal procedure tends to shift the weighting of the calculation somewhat toward the high-energy side, this effect is not large. With the original LDR Eq. (6) the contribution of the asymptotic region was 15% of the total phase at $p=2.07$ GeV/c, whereas here it is 19% at the roughly comparable momentum 1.88 GeV/c.

Despite the generally favorable outcome of the present calculation, we feel that at $-t=0.40$ (GeV/c)^2 we are approaching the limits of practical usefulness of the LDR under the assumptions we have used. There are several reasons for this conclusion.

i. Erratic zero behavior. The zeros of scattering amplitudes are t-dependent and are in general unknown. However, for $t=0$ the πN amplitude has only one zero and its location in the complex energy plane has been determined.[7] In I it was argued that it was reasonable to assume that for t small there was still only a single zero, although it might move in the complex energy plane as t varied. Our operating procedure both here and in I was to determine the location of this single zero by the requirement that the LDR reproduce the known partial-wave phases. For $-t\leq 0.20$ (GeV/c)^2 a smooth locus for the zero was thereby determined; this is shown in Fig. 5 of I. However, in the present calculation an inspection of the zero locations given in Sec. III A *i* shows that while this smooth behavior continues to $-t=0.30$ (GeV/c)^2, thereafter the zero "jumps" crossing the imaginary axis into the left upper quarter plane. We do not know what causes this; perhaps the amplitude has developed another zero. In any case it strongly suggests that the assumption of a single zero should not be used at even larger values of $-t$ without further study.

ii. The magnitude of B. It is clear from the linear factor of t entering the coefficient of $|B|$ in the unpolarized πN differential cross section[12]

$$\frac{d\sigma}{d\Omega}=\left(\frac{M}{4\pi W}\right)^2\left\{\left(1-\frac{t}{4M^2}\right)|A'|^2+\frac{t}{4M^2}\left[s-\frac{(\nu+M-t/4M)^2}{(1-t/4M^2)}\right]|B|^2\right\} \quad (12)$$

that this term is negligible provided t is small. Here, as in I, we assumed that t was small enough so that $|A'|$ was given by Eq. (11). To test this assumption we reconstructed $|B|$ from existing partial-wave analyses at $p=2.07$ GeV/c and compared the first and second terms of Eq. (12). The results, given in Table III, show that while the contribution to the cross section of its second term does not exceed 16% for the t interval of our two calculations, its significant increase between $-t=0.20$ and 0.40 (GeV/c)^2 suggests that this term cannot be safely neglected for larger values of $-t$.

iii. Growth of the unphysical region. While our subtraction procedure is still effective in damping out the contribution of the unphysical region, it continues to grow relentlessly as $-t$ increases. Although it seems to us less pressing than the previous two restrictions, the unphysical region will ultimately impose a limitation on the LDR as it will on any nonforward dispersion relation approach.

An interesting possibility for reducing the dependence of the phases on the unknown asymptotic region was noticed after this calculation was completed. For the range of integration in which $\nu'\gg\nu$, the magnitudes of the two integrals in Eq. (6) are nearly the same, the reason being that $|A'_-|\sim|A'_+|$ at high energies. Now the relative sign of these two terms is even or odd according as the number of subtractions is even or odd. In the present calculation there are two subtractions, $\nu=\pm\nu_0$, and the integrals add. For an odd number the integrals would tend to cancel and the influence of the asymptotic region therefore be decreased. For various reasons three is probably the least odd number of subtractions that could be used. The resulting calculation would be somewhat more complex than the present one.

TABLE III. Contributions to the unpolarized differential cross section at $p=2.07$ GeV/c of the two terms appearing in Eq. (12), $d\sigma/d\Omega=\alpha+\beta$. α and β refer to the terms involving $|A'|$ and $|B|$, respectively.

Interaction	$-t$ [(GeV/c)^2]	α [(F)^2]	β [(F)^2]	β/α %	$d\sigma/d\Omega$ [(F)^2]
π^+p	0.2	0.367	0.023	6.3	0.390
π^+p	0.4	0.096	0.015	15.5	0.111
π^-p	0.2	0.320	0.014	4.4	0.334
π^-p	0.4	0.043	0.005	11.6	0.048

Our next application of the LDR will be to πN polarization,[13]

$$P(\nu,t)=-\frac{\sin\theta}{16\pi s^{1/2}}\,\frac{|A'||B|\sin(\phi-\psi)}{d\sigma/dt}, \quad (13)$$

in which ψ is the phase of the amplitude B and θ is the scattering angle. Aside from the total cross section, the polarization and the unpolarized differential cross section (12) are the most important experimentally observed quantities in πN scattering. It is clear that the cross section involving amplitude moduli and the polarization involving amplitude phases are intimately connected through analyticity. The LDR provides the connection between phase and modulus and is a powerful tool in exploring the relationship between these two observables.

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