Narrow resonances in statistical mechanics*

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It is shown that when the dynamics of a system is dominated by narrow-resonance formation it behaves like a noninteracting system with added species of free particles, corresponding to all the quantum numbers *and* the statistics of each of the resonances. This equivalence, widely assumed and used in practice, is established by explicitly constructing many-particle S matrices consistent with unitarity that correspond to purely resonance-dominated dynamics. These are then used in the S-matrix formulation of statistical mechanics to obtain the grand partition function, which is seen to reveal the above result. In the following paper, operational criteria are presented for determining when in general narrow resonances and loosely bound states may be treated as "elementary" particles.

I. INTRODUCTION

A popular notion, used in a variety of physical situations, asserts that narrow resonances may be treated to a good approximation on the same footing as stable elementary particles. We are quite accustomed to the idea of treating neutrons, pions, muons, etc., all of which are actually unstable, as if they were "elementary." It is assumed that in a collection of such objects, each species would obey its own separate statistics, and that one can ignore any constraints of symmetrization or antisymmetrization between wave functions of, say, a neutron and an electron. A possibly more debatable example of such use arises in certain calculations of the equation of state of neutron-star interiors. Here, not only the neutrons, the Λ, and the Σ, but often also the 3-3 resonance (Δ) is treated as a separate species. This assumption allows one to put, for instance, a Δ and a nucleon in the same momentum state, and leads to a great saving in kinetic energy at high baryon densities.

Yet, all these unstable particles are also expected to occur as resonances in the scattering of stable particles. This is certainly true of the hadronic resonance Δ, which occurs in the $N\pi$ channel, and presumably pions, muons, neutrons, etc. would also occur as resonances in the appropriate weak-interaction scattering experiments, if one could perform them. Thus, at least in part, the resonances may be composite objects. If so, one would expect some constraint between their wave functions, and those of the "constituent" stable particles.

The difficulty is that we do not know at present to what extent these resonances are really composite or elementary. It would therefore be nice to determine criteria under which narrow resonances may be considered approximately as elementary, even if in fact they were fully composite. Further these criteria, assuming one can find them, should be specified operationally, not involving ultimate knowledge of whether a resonance is truly composite, elementary, or both in any absolute sense. Finally, when a resonance may be treated as a separate elementary species, one should be able to justify the neglect of possible symmetrization constraints with its constituent stable particles.

We attempt, in this paper and the one following it, to throw some light on these issues, along with the corresponding problems for loosely bound composite systems.

We begin in Sec. II by considering a collection of two stable species, taking for generality one as a fermion and the other a boson. We assume that the two species interact only to the extent of forming a narrow resonance. This specification is made precise by giving a unitarily consistent S matrix for the scattering of m fermions with n bosons for all $\{n, m\}$. Then, using the S-matrix formulation of statistical mechanics,[1] we calculate the grand partition function Z for this interacting system at any temperature. The reason behind evaluating Z is that apart from giving all other thermodynamic functions, it also resolves questions of relative statistics. Since all many-particle S matrices are seen to exist even when the dynamics consists only of a narrow resonance, all virial coefficients also exist. However, in the limit of a narrow resonance, these virial coefficients can be summed. The resulting contribution to the partition function is shown to be exactly what a third ideal gas would give, if it had Fermi statistics and the mass and quantum numbers of the resonance. Note that for Boltzmann gases such a proof would be trivial, since only the second virial coefficient exists for a purely narrow-resonant scattering. This coefficient, given by the Beth-Uhlenbeck formula[2] in terms of the elastic phase shift, immediately reduces to an "ideal"-

Boltzmann-gas term corresponding to the resonance. However, the questions we posed earlier are meaningful only for (the more realistic) Bose and Fermi systems, in which case the calculation involves all virial coefficients and is more complicated.

In Sec. III we extend this result to cases where such narrow resonances "scatter" with other narrow resonances or stable particles. Of particular interest is the case where the scattering of narrow resonances with one another and with stable particles is again dominated by narrow-resonance formation. Such dynamics, dominated by tree graphs involving a hierarchy of narrow resonances, is used in both the dual models[3] and the thermodynamic theories of elementary particles.[4] We find that such dynamics can once again be replaced by a sum of ideal-gas terms, one for each resonance.

Reconciliation of these results with the need to symmetrize (or antisymmetrize) the constituents of the resonance with other stable particles involves the use of Levinson's theorem. The same theorem also throws light on the status of "loosely" bound states. By loosely bound states we mean states bound by potentials whose strength and inverse range are small compared to the temperature and the density. These and related topics form the content of the succeeding paper. It has been presented separately in order not to drown the physics there amongst the technical details of narrow-resonance algebra in this paper.

II. TWO SPECIES INTERACTING THROUGH A NARROW RESONANCE

We will calculate the exact grand partition function, from which other thermodynamic functions can be derived, for the following system:

There are two stable species of particles, say, one a boson and the other a fermion, each carrying a separately conserved quantum number. To facilitate discussion, let us call the boson a π^-, carrying charge, and the fermion as a neutron (n), carrying baryon number. Both particles are stable in the absence of weak interactions. These specifications are just for convenience. Our results are adaptable to cases of two fermions or two bosons, whether or not they carry separate quantum numbers.

Let us assume that there are no π^--π^- interactions and no n-n interactions. Further, let the π^--n interaction be such as to produce *only* a narrow S-wave resonance (called the N^*) in the elastic $\pi^- n$ scattering amplitude, both on and off the energy shell. That is, let [see Fig. 1(a)]

$$\langle p'k'|T(E)|p,k\rangle = \frac{(2\pi)^3 g^2 \delta^3(\vec{p}+\vec{k}-\vec{p}'-\vec{k}')}{E-[M^2+(\vec{p}+\vec{k})^2]^{1/2}+i\Gamma} \tag{2.1}$$

and the S matrix

$$\langle p'k'|S|pk\rangle = \langle p'k'|1-2\pi i\delta(e+\omega-e'-\omega')\,T(e+\omega)|pk\rangle\ , \tag{2.2}$$

where

$$M = \text{mass of the resonance } N^*,\qquad 2\Gamma = (2\pi)^4 g^2 \sum_{p\,k} \delta^3(\vec{p}+\vec{k})\,\delta(e+\omega-M)\ , \tag{2.3}$$

and we will always use the combinations $p_i = (\vec{p}_i, e_i)$ and $k_i = (\vec{k}_i, \omega_i)$ for the four-momenta of neutrons and pions, respectively.

We will work in the narrow-resonance limit, i.e., $g^2 \to 0$, when the width Γ also goes to zero and the amplitude in Eq. (2.1) is unitary. Also, since the interaction in (2.1) is spin-independent, the "neutron" is effectively spinless in our problem.

Turning to the many-particle T matrix, we make a corresponding assumption consistent with Eq. (2.1) for the $\pi^- n$ amplitude. We assume that it consists of, and *only of,* diagrams where real (or virtual) pions repeatedly form narrow elastic N^* resonances with other real (or virtual) neutrons. Unitarity forbids us from using a two-particle amplitude of Eq. (2.1) without permitting

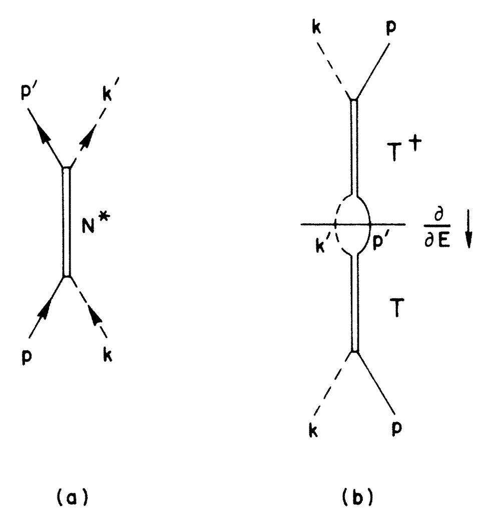

FIG. 1. (a) Narrow resonance N^* in the $\pi^- n$ channel and (b) its contribution to the lowest virial coefficient. (b) stands for a $T^\dagger(\partial/\partial E)T$ form as occurring in Eq. (2.9), where the horizontal line separates $T^\dagger$ from T and an operation $\partial/\partial E$ is to be applied on the piece T below the horizontal line. In this and subsequent diagrams, dashed lines, regular lines, and double lines stand for pions, neutrons, and the resonance N^*, respectively.

many-particle amplitudes as in Fig. 2(a). Thus the T matrix for the scattering of 2 pions and 3 nucleons would be the sum of all terms of the type shown in Fig. 2(a). [Note that several disconnected pieces like this are also permitted. However, a diagram such as Fig. 2(b) is *not* permitted since two successive resonances occur between the same $\pi^- n$ pair. Since the resonance in Eq. (2.1) is already unitary in two-particle space, Fig. 2(b) would lead to double counting.]

Diagrams such as Fig. 2(a) stand here for terms in a Lippmann-Schwinger expansion of the $T(E)$ matrix. To be precise, a diagram such as Fig. 3 stands for the expression

$$\text{Fig. } 3 = \frac{(2\pi)^3 g^2 \delta^3(\vec{p}_1 + \vec{k} - \vec{k}_5 - \vec{p}_3)}{E - e_2 - [M^2 + (\vec{p}_1 + \vec{k})^2]^{1/2} + i\Gamma} \, \frac{1}{E - \omega_5 - e_3 - e_2 + i\epsilon} \, \frac{(2\pi)^3 g^2 \delta^3(\vec{k}_5 + \vec{p}_2 - \vec{k}_4 - \vec{p}_2{}')}{E - e_3 - [M^2 + (\vec{k}_5 + \vec{p}_2)^2]^{1/2} + i\Gamma}$$
$$\times \frac{1}{E - \omega_4 - e_3 - e_2' + i\epsilon} \, \frac{(2\pi)^3 g^2 \delta^3(\vec{k}' + \vec{p}_1{}' - \vec{k}_4 - \vec{p}_3)}{E - e_2' - [M^2 + (\vec{p}_3 + \vec{k}_4)^2]^{1/2} + i\Gamma} \,. \tag{2.4}$$

From this illustration, the corresponding expression for any diagram for the many-particle $T(E)$ can be written down, and the assumed dynamics of one narrow-resonance system is fully specified.

With this information, the full grand partition function Z can in principle be evaluated using the S-matrix formulation of statistical mechanics.[1] For our system,

$$\frac{pV}{kT} = \ln Z = \ln Z_0 + \frac{1}{2\pi i} \sum_{n_1, n_2} e^{\beta(\mu_1 n_1 + \mu_2 n_2)} \int dE\, e^{-\beta E} \left[\mathrm{Tr}_{n_1, n_2} A S^{-1}(E) \frac{\partial}{\partial E} S(E) \right]_c \,, \tag{2.5}$$

where $S(E) = 1 - 2\pi i \delta(E - H_0)\, T(E)$, H_0 = free Hamiltonian, μ_1 = chemical potential for the π^-, corresponding to charge conservation, μ_2 = chemical potential for the neutron, for baryon conservation, A = boson symmetrization and fermion antisymmetrization operator, n_1 (n_2) is the number of pions (nucleons) in a state, and the subscript c stands for the connected part of the *combination* $\mathrm{Tr}[AS^{-1}(\partial/\partial E)S]$. Note that two apparently disconnected pieces can be "connected" through their final-state labels, thanks to the exchange operator A.[1] Z_0 in Eq. (2.5) is the noninteracting value of Z for the two species, viz.,

$$\ln Z_0 = V \int \frac{d^3p}{(2\pi)^3} \ln(1 + \exp\{-\beta[(\vec{p}^{\,2} + m_n{}^2)^{1/2} - \mu_2]\})$$
$$- V \int \frac{d^3k}{(2\pi)^3} \ln(1 - \exp\{-\beta[(\vec{k}^2 + m_\pi{}^2)^{1/2} - \mu_1]\}) \,. \tag{2.6}$$

Given an S matrix for n_1 pions and n_2 nucleons as specified for all n_1 and n_2, all virial coefficients in the double expansion in (2.5) exist. In the limit $g^2 \to 0$, we show that these can all be evaluated and summed.

[Incidentally, we have added an Appendix at the end of the following paper, clarifying what the $\partial/\partial E$ means, given a general S-matrix element as a function of several energy variables. In this paper we have postulated S-matrix elements with

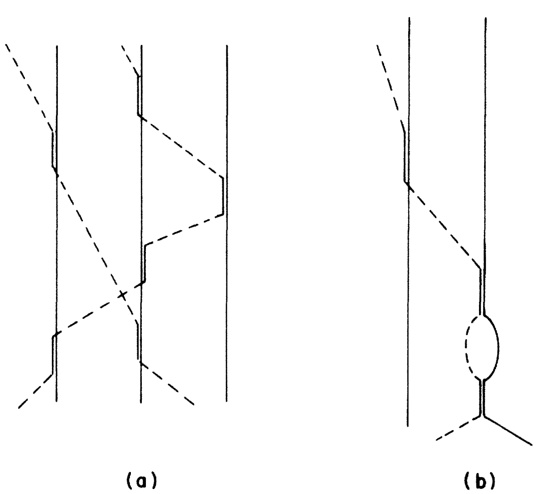

FIG. 2. (a) A typical many-particle S-matrix element in resonance-dominated dynamics. This is a term in a two-pion–three-neutron scattering. (b) A typical T-matrix diagram not permitted since two successive resonances occur between the same $\pi^- n$ pair. A unitary resonance as in (2.1) already includes such iterations.

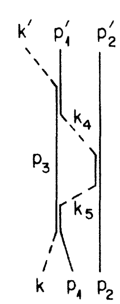

FIG. 3. A contribution to the one-pion–two-neutron T matrix, which stands for the expression in Eq. (2.4).

explicit off-energy shell (E) dependence, so that $\partial/\partial E$ is unambiguous.]

To begin, note that every contribution to any many-particle T matrix is, apart from momentum-conserving δ functions, a product of free Green's functions $1/(E-H_0+i\epsilon)$, and "resonance propagators" of the form $g^2/(x_i+i\Gamma)$ where x_i are some energy variables made up of E, the resonance mass M, and the energy-momenta of other particles in the diagram. [See, for example, Eq. (2.4).] Remember that $\Gamma \propto g^2$ from (2.3), and that

$$\lim_{\epsilon\to 0}\left(\frac{\epsilon}{x+i\epsilon}\right)^n = 0 = \frac{1}{x+i\epsilon}\left(\frac{\epsilon}{x+i\epsilon}\right)^n , \tag{2.7a}$$

$$\lim_{\epsilon\to 0}\left(\frac{\epsilon}{x-i\epsilon}\right)^m\left(\frac{\epsilon}{x+i\epsilon}\right)^n = 0 , \tag{2.7b}$$

while

$$\lim_{\epsilon\to 0}\left(\frac{2\epsilon}{x-i\epsilon}\right)^n \frac{\partial}{\partial x}\left(\frac{2\epsilon}{x+i\epsilon}\right)^m = \frac{2\pi i}{(i)^{m+n}}(-1)^n\frac{(n+m-1)!}{(n-1)!\,(m-1)!}\,\delta(x) \tag{2.8}$$

for all integral $n, m>0$. From this one can see that all the T matrices vanish as $g^2\to 0$, since they all have g^2 dependence of the form

$$\prod_i \frac{g^2}{x_i+i\Gamma} ;$$

so does $(\partial/\partial E)\,T(E)$ and, for that matter, $T^\dagger T$. However, the combination $T^\dagger(\partial/\partial E)\,T$ from Eq. (2.5) can survive since poles from $x=i\epsilon$ and $x=-i\epsilon$ can pinch the real axis for some energy variable x, giving a $\delta(x)$ as per (2.8). Note while using (2.5) that in the combination

$$T^\dagger(E)\frac{\partial}{\partial E}\left[\delta(E-H_0)\,T(E)\right] ,$$

when $\partial/\partial E$ acts on the $\delta(E-H_0)$, that is to say on n-particle phase space $\rho_n(E)$, giving

$$T^\dagger(E)\,T(E)\frac{\partial}{\partial E}\,\rho_n(E) ,$$

the result is again zero as $g^2\to 0$, because of (2.7b). Thus, Eq. (2.5) reduces to

$$\begin{aligned}\ln Z &= \ln Z_0 + \frac{(2\pi)^2}{2\pi i}\sum_{n_1,n_2} e^{\beta(\mu_1 n_1+\mu_2 n_2)}\int dE\, e^{-\beta E}\left[\mathrm{Tr}_{n_1,n_2} A\,\delta(E-H_0)\,T^\dagger(E)\,\delta(E-H_0)\frac{\partial}{\partial E}\,T(E)\right]_c \\ &\equiv \ln Z_0 + \sum_{n_1,n_2} e^{\beta(\mu_1 n_1+\mu_2 n_2)}\,b(n_1,n_2) .\end{aligned} \tag{2.9}$$

The physics of the singular nature of such a narrow resonance can be understood better by looking at the lowest virial terms in the expansion (2.9), corresponding to $n_1=n_2=1$. It involves the elastic $\pi^- n$ amplitude given in (2.1), which as $g^2\to 0$ vanishes everywhere in E, except at the resonance energy, where it is *finite*. The same is true of the cross section related to $|T(E)|^2$, which also has a spike of finite height and decreasing width as $g^2\to 0$. Neither the cross section nor the amplitude, if integrated over the energy, will contribute anything. However, the partition function is an integral over $S^{-1}(\partial/\partial E)S$, which may be interpreted as the cross section multiplied by the "time delay" due to interaction. For a narrow resonance, this time delay clearly becomes infinite. As a result, $S^{-1}(\partial/\partial E)S$ develop enough strength at the resonance point to behave like a δ function in energy. This is seen when $b(1,1)$ is obtained by substituting (2.1) into (2.9). We get

$$b(1,1) = \frac{\delta^3(0)}{2\pi i}\int d^3P\int dE\, e^{-\beta E}[2\Gamma(E,P)]^2\,\frac{1}{E-(M^2+\vec{P}^2)^{1/2}-i\Gamma}\,\frac{\partial}{\partial E}\,\frac{1}{E-(M^2+\vec{P}^2)^{1/2}+i\Gamma} ,$$

where, analogous to the definition of the width Γ in Eq. (2.3), we define

$$2\Gamma(E,P) = g^2(2\pi)^4\sum_{k,p}\delta^3(\vec{k}+\vec{p}-\vec{P})\,\delta(e+\omega-E) .$$

Then as g^2 [and hence Γ and $\Gamma(E,P)$] go to zero, using (2.8),

$$\begin{aligned}b(1,1) &= \delta^3(0)\int d^3P\int dE\, e^{-\beta E}\left(\frac{\Gamma(E,P)}{\Gamma}\right)^2\delta(E-(M^2+\vec{P}^2)^{1/2}) \\ &= V\int\frac{d^3P}{(2\pi)^3}\exp[\beta(M^2+\vec{P}^2)^{1/2}] .\end{aligned} \tag{2.10}$$

The result (2.10), where the resonance behaves just like another free particle of mass M, is very familiar. It is derived in Ref. 2 and is also easily available from the old Beth-Uhlenbeck formula.[2] For a Boltzmann gas, this is the only contribution in the narrow-resonance limit to the whole virial series, and is the basis for the notion that narrow resonances behave like stable particles. For a realistic Bose or Fermi gas, all the higher virial coefficients in the series (2.9) also exist and must be calculated and summed. This is really the only new element in our work in this section. As hoped for, these higher virial contributions merely provide the "N^* species" with the appropriate (in this example, Fermi-Dirac) statistics. But this simple result evolves only after some interesting cancellations between classes of diagrams, which amount to a self-consistent change in the resonance width in the presence of the medium.

To evaluate the higher virial coefficients, we again rely on the identities (2.7) and (2.8). Every n-particle T-matrix diagram, when put on the energy shell [as required by the $\delta(E-H_0)$ in 2.9], involves a product of resonance propagators $\prod_i g^2/(x_i+i\Gamma)$. Each x_i is just $E_i-E_i^r$, where E_i is the energy of the $\pi^- n$ pair forming that resonance and E^r is the resonance energy for that particular total momentum. The combination $T^\dagger(\partial/\partial E)T$ will then be of the form

$$\left[\prod_i\left(\frac{g^2}{x_i-i\Gamma}\right)\right]\left[\prod_j\left(\frac{g^2}{x_j+i\Gamma}\right)\right]\left[\sum_j\left(\frac{-1}{x_j+i\Gamma}\right)\right]. \tag{2.11}$$

Note that the sum over state labels (phase-space integrals) can be written as integrals over independent pair energies x_i and any remaining variables, with a suitable Jacobian. Apart from the resonance propagators, $T^\dagger(\partial/\partial E)T$ involves momentum-conserving δ functions, energy-conserving δ functions, and Green's functions $1/(E-H_0-i\epsilon)$ whose imaginary parts again conserve appropriate energies.

As a result of these δ functions, two things may happen to any given $T^\dagger(\partial/\partial E)T$ contribution:

(a) *All* the energy variables x_i and x_j in (2.11) get constrained to be the same. In that case (2.11) has the form

$$\left(\frac{g^2}{x_i-i\Gamma}\right)^m\frac{\partial}{\partial x_i}\left(\frac{g^2}{x_i+i\Gamma}\right)^n,$$

which by (2.8) survives on integration over x_i. Such a diagram will thus survive.

(b) At least two independent x's remain in (2.11). In that case, in the expansion in (2.11) at least one of them (x_1) will involve

$$\left(\frac{g^2}{x_1-i\Gamma}\right)^m$$

or

$$\left(\frac{g^2}{x_1+i\Gamma}\right)^n$$

or

$$\left(\frac{g^2}{x_1-i\Gamma}\right)^m\left(\frac{g^2}{x_1+i\Gamma}\right)^n,$$

any of which make the integral over x_1 and hence the contribution of the entire diagram vanish.

The interested reader can verify this argument by considering some examples. A diagram such as Fig. 3, when inserted either for $T^\dagger$ or for T in $T^\dagger(\partial/\partial E)T$ will destroy that contribution. On the other hand, the lowest virial coefficient $b(1,1)$ survived since essentially only one energy resonated [Fig. 1(b)] for a given total momentum in both $T^\dagger(E)$ and $T(E)$.

The criterion then is that only those contributions to the n-particle $[\mathrm{Tr}\,A\,T^\dagger(\partial/\partial E)T]_c$ combination survive in which *all* the resonance propagators are constrained by energy momentum conservation to be the *same*. *This important technical criterion, along with identities (2.7) and (2.8), will be used repeatedly.*

Before we collect all the terms that satisfy the criterion and survive, let us start with a simple example that does [Fig. 4(a)]. Its contribution to $\ln Z$ in Eq. (2.9) is

$$C\equiv-\frac{e^{+\beta(\mu_1+2\mu_2)}}{2\pi i}\sum_{\substack{k,p_1,p_2\\k'',p',k'}}e^{-\beta(\omega+e_1+e_2)}\left(\frac{(2\pi)^4g^2\delta^3(\vec{k}+\vec{p}_1-\vec{k}''-\vec{p}_2)}{e_2+\omega''-[M^2+(\vec{k}''+\vec{p}_2)^2]^{1/2}-i\Gamma}\right)$$

$$\times\frac{1}{e_1+\omega+e_2-(e_2+e_2-\omega'')-i\epsilon}\left(\frac{(2\pi)^3g^2\delta^3(\vec{p}_2+\vec{k}''-\vec{p}'-\vec{k}')}{e'+\omega'-[M^2+(\vec{p}'+\vec{k}')^2]^{1/2}-i\Gamma}\right)$$

$$\times 2\pi\delta(e'+\omega'+e_2-e_1-e_2-\omega)\left(\frac{(2\pi)^3g^2\delta^3(\vec{p}_1+\vec{k}-\vec{p}'-\vec{k}')(-1)}{\{e_1+\omega-[M^2+(\vec{p}_1+\vec{k})^2]^{1/2}+i\Gamma\}^2}\right). \tag{2.12}$$

The minus sign arises because the p_1, p_2 lines have been exchanged. Retain only the $+i\pi\delta(e_1+\omega-e_2-\omega'')$ in the Green's function in (2.12). We will come to the principal part in a moment. Note that because of momentum conservation, all resonance propagators are the same. Let $\vec{k}+\vec{p}_1=\vec{P}$ and $\omega+e_1=E$. Then the expression in (2.12) is merely

$$C \equiv \frac{\delta^3(0)}{2\pi i}\, e^{+\beta(\mu_1+\mu_2)} \sum_{k,p_1} e^{-\beta(\omega+e_1)} \frac{(2\pi)^4 g^2}{[E-(M^2+\vec{P}^2)^{1/2}-i\Gamma]^2}\, \frac{-i\tilde{\gamma}_N(E,P)\, 2\Gamma(E,P)}{[E-(M^2+\vec{P}^2)^{1/2}+i\Gamma]^2}\,, \tag{2.13}$$

where $2\Gamma(E,P)$ as before equals

$$\sum_{k',p'} \delta^3(\vec{k}'+\vec{p}'-\vec{P})\, \delta(e'+\omega'-E)\,(2\pi)^4 g^2$$

and

$$2\tilde{\gamma}_N(E,\vec{P},\mu_2) = \sum_{k'',p_2} e^{-\beta(e_2-\mu_2)}\, \delta^3(\vec{k}''+\vec{p}_2-\vec{P}) \times \delta(\omega''+e_2-E)\,(2\pi)^4 g^2\,. \tag{2.14}$$

It is clear from the identity (2.8) that as $g^2 \to 0$, (2.13) will give a term proportional to $\delta(E-(\vec{P}^2+M^2)^{1/2})$, which will survive upon integration over $E=\omega+e$.

Next, we add to Fig. 4(a) a sequence of terms [such as in Fig. 4(b)] where an arbitrary number of noninteracting neutron lines are "exchanged"[1] with the nucleon p_2. This will only replace in (2.13) the factor $2\tilde{\gamma}_N$ by a factor

$$2\gamma_N(E,\vec{P},\mu_2) = \sum_{k'',p_2} \delta^3(\vec{k}''+\vec{p}_2-\vec{P})\, \delta(\omega''+e_2-E) \times \frac{e^{-\beta(e_2-\mu_2)}\,(2\pi)^4 g^2}{1+e^{-\beta(e_2-\mu_2)}}\,. \tag{2.15}$$

$2\gamma_N(E,\vec{P},\mu_2)$ is clearly just the "reduced" phase space for two particles of energy-momentum $(E,\vec{P})$ weighted by the neutron occupation probability.

Similarly, the diagrams depicted in Fig. 5 will give a term similar to (2.13) except that $-i\tilde{\gamma}_N$ will now be replaced by $+i\gamma_\pi(E,\vec{P},\mu_1)$ given by

$$2\gamma_\pi(E,\vec{P},\mu_1) = \sum_{k'',p_2} \delta^3(\vec{k}''+\vec{p}_2-\vec{P})\, \delta(\omega''+e_2-E) \times \frac{e^{-\beta(\omega''-\mu_1)}}{1-e^{-\beta(\omega''-\mu_1)}}(2\pi)^4 g^2\,. \tag{2.16}$$

Rather than evaluate these contributions separately, we now identify a whole sequence of such diagrams, which effectively involve only one resonance denominator. These diagrams are generated as follows. Consider Fig. 6, in which the shaded blob on the right-hand side is iterated to give a sequence of T-matrix diagrams. As illustrated in Eq. (2.4) for the case of Fig. 3, the precise algebraic expression represented by any T-matrix diagram is defined. Now insert the shaded blob in Fig. 6 into all the $T^\dagger(\partial/\partial E)T$ combinations depicted in Fig. 7. In both Figs. 6 and 7, the noninteracting pion (or neutron) lines are meant to be arbitrary in number, beginning with zero, and are exchanged with the k_1 (or p_1) line. Their effect is merely to introduce the Bose (or Fermi) occupation probability in the phase-space integral of the k_1 (or p_1) line. See, for example, Eqs. (2.15) or (2.16) and Ref. 1.

Clearly, Fig. 7 represents a particular sequence of terms in the series (2.9). While the notation of Figs. 6 and 7 is rather compact, the reader should

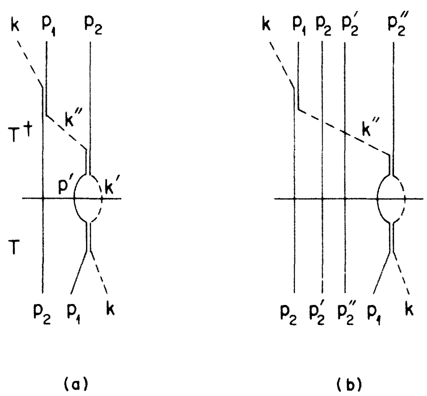

FIG. 4. Examples of diagrams which have only one independent resonance energy, consequently surviving in the virial series. Such diagrams will end up reducing the N^* width in the presence of other neutrons in the medium.

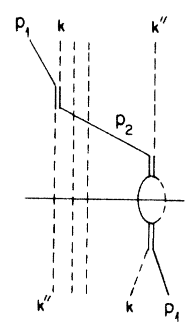

FIG. 5. Corresponding examples to Fig. 4, but where the pions are "exchanged." These will help increase the N^* width due to the presence of other pions in the medium.

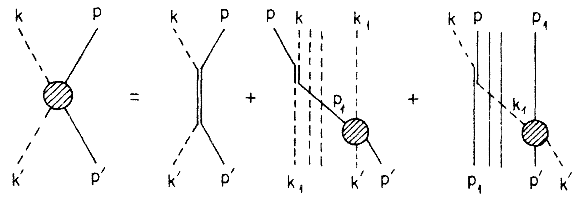

FIG. 6. An iterative sequence which generates T-matrix elements with only one independent resonance energy. This exhausts all linked T matrices having this property. We use the phrase "linked" to include trivially disconnected pieces consisting of noninteracting lines which are exchanged.

have no trouble generating this sequence. A typical term is shown in Fig. 8. Its contribution to Eq. (2.9) has the form $T_2^\dagger(E)\,\delta(E-H_0)\,(\partial/\partial E)\,T_1(E)$, where T_2 (and T_1) are unambiguously defined by the parts above (and below) the horizontal line in Fig. 8. The simple examples we did earlier in Figs. 4 and 5 also belong to the series in Fig. 7.

It may be checked that every term contained in Fig. 7 involves resonance propagators in only one independent energy variable, once full energy-momentum conservation is imposed, just as in the examples (Figs. 4 and 5) we explicitly worked out. [We continue to retain only the imaginary part $-i\pi\delta(E-H_0)$ in every Green's function $1/(E-H_0+i\epsilon)$ in the T matrices. It will be seen that these alter the width Γ. The real part of $1/(E-H_0+i\epsilon)$ will alter the resonance mass M by an amount which vanishes as $g^2\to 0$.] Thus, every term in the expansion of Fig. 7 will survive as $g^2\to 0$.

So far Fig. 7 has not included "exchanges" of spectator neutrons (or pions) with the initial p (or k) line. Let us now add, for every term in Fig. 7, also terms where *either* the neutron line p *or* the pion line k is exchanged with arbitrary numbers of spectators of the respective species. Thus, in the place of Fig. 7(a), we use the whole sequence in Fig. 9. Similarly, Figs. 7(b) and 7(c) are expanded to include a trivially larger set when either p or k is exchanged an arbitrary number of times. Let us call the sum of all contributions in Fig. 7 *along with such exchanges* as S. S does not

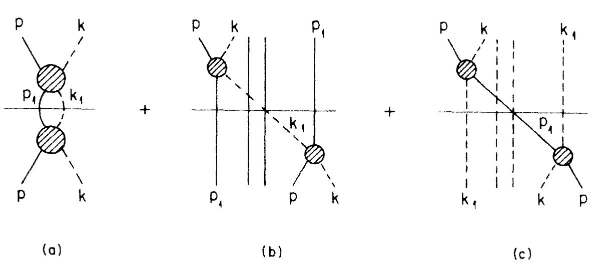

FIG. 7. An iterative sequence for $T^\dagger(\partial/\partial E)T$ obtained by substituting the sequence in Fig. 6 for the shaded blob.

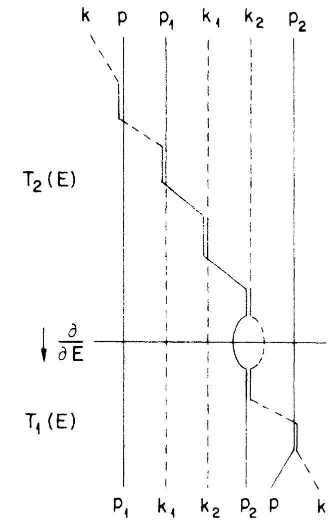

FIG. 8. An example of a typical term contained in Fig. 7. This particular example is contained in Fig. 7(a).

include terms, where *both* p and k are exchanged with noninteracting lines. We will sum those separately and they will be seen to provide the "statistics" for the N^* species.

The set S has a simple sum, which has to be obtained by examining the structure of a general term such as Fig. 8, and evaluating it as we did the simple case of Fig. 4. It can be seen that the sum of *all* terms in S involving n resonances above the horizontal line and m resonances below the line ($n=4$, $m=2$ in Fig. 8) is

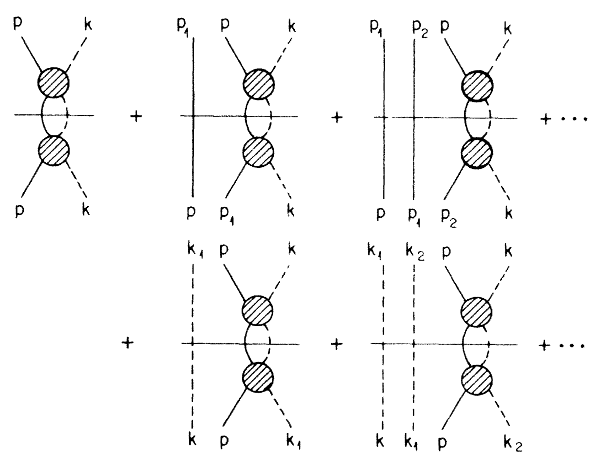

FIG. 9. This is essentially the sequence in Fig. 7(a), where, in addition, either the initial neutron or pion (but not both) is exchanged with an arbitrary number of noninteracting spectators. A similar enlargement on Fig. 7(b) and Fig. 7(c) is also to be made. The net sum is called S.

$$S_{nm}=\frac{\delta^3(0)}{2\pi i}\sum_{p,k} e^{-\beta(E-\mu_1-\mu_2)}\left[1-\frac{e^{-\beta(e-\mu_2)}}{1+e^{-\beta(e-\mu_2)}}+\frac{e^{-\beta(\omega-\mu_1)}}{1-e^{-\beta(\omega-\mu_1)}}\right](2\pi)^4 g^2\frac{(-i\gamma_N+i\gamma_\pi)^{n-1}}{[E-(M^2+\vec{P}^2)^{1/2}-i\Gamma]^n}$$
$$\times\sum_{p_1,k_1}\left[1-\frac{e^{-\beta(e_1-\mu_2)}}{1+e^{-\beta(e_1-\mu_2)}}+\frac{e^{-\beta(\omega_1-\mu_1)}}{1-e^{-\beta(\omega_1-\mu_1)}}\right]\delta(E_1-E)\,\delta^3(\vec{P}_1-\vec{P})$$
$$\times\frac{\partial}{\partial E}(2\pi)^4 g^2\frac{(+i\gamma_N-i\gamma_\pi)^{m-1}}{[E-(\vec{P}^2+M^2)^{1/2}+i\Gamma]^m}\ , \tag{2.17}$$

where (p,k) and (p_1,k_1) are just the labels in Fig. 7,

$E=e+\omega,\quad \vec{P}=\vec{p}+\vec{k},\quad E_1=e_1+\omega_1,\quad \vec{P}_1=\vec{p}_1+\vec{k}_1$,

$\gamma_N=\gamma_N(E,\vec{P},\mu_2)$ as in (2.15) ,

$\gamma_\pi=\gamma_\pi(E,\vec{P},\mu_1)$ as in (2.16) .

Let us define a reduced width

$$\sigma(E,P)\equiv\tfrac{1}{2}\sum_{\vec{k},\vec{p}}\left[1-\frac{e^{-\beta(e-\mu_2)}}{1+e^{-\beta(e-\mu_2)}}+\frac{e^{-\beta(\omega-\mu_1)}}{1-e^{-\beta(\omega-\mu_1)}}\right]\delta(\omega+e-E)\,\delta^3(\vec{p}+\vec{k}-\vec{P})\,(2\pi)^4 g^2$$
$$=[\Gamma(E,P)-\gamma_N(E,P)+\gamma_\pi(E,P)]\,. \tag{2.18}$$

Thus,

$$S_{nm}=\frac{\delta^3(0)}{2\pi i}\int d^3P\int dE\, e^{-\beta(E-\mu_1-\mu_2)}[2\sigma(E,P)]^2\frac{(-i\gamma_N+i\gamma_\pi)^{n-1}}{[E-(M^2+\vec{P}^2)^{1/2}-i\Gamma]^n}\frac{\partial}{\partial E}\frac{(i\gamma_N-i\gamma_\pi)^{m-1}}{[E-(M^2+\vec{P}^2)^{1/2}+i\Gamma]^m}\ .$$

Therefore

$$S=\sum_{m,n=1}^{\infty}S_{mn}=\frac{\delta^3(0)}{2\pi i}\int d^3P\int dE\, e^{-\beta(E-\mu_1-\mu_2)}\frac{2\sigma(E,P)}{E-(M^2+\vec{P}^2)^{1/2}-i(\Gamma-\gamma_N+\gamma_\pi)}$$
$$\times\frac{-2\sigma(E,P)}{[E-(M^2+\vec{P}^2)^{1/2}+i(\Gamma-\gamma_N+\gamma_\pi)]^2}\ .$$

Remembering that the only contribution, as $g^2\to 0$ comes at $E=(M^2+\vec{P}^2)^{1/2}$, we see that Γ in the denominators, which equals $\Gamma(M)=\Gamma((M^2+\vec{P}^2)^{1/2},P)$ can be replaced by $\Gamma(E,P)$. Then

$$S=\frac{\delta^3(0)}{2\pi i}\int d^3P\int dE\, e^{-\beta(E-\mu_1-\mu_2)}\frac{2\sigma(E,P)}{[E-(M^2+\vec{P}^2)^{1/2}-i\sigma(E,P)]}\frac{-2\sigma(E,P)}{[E-(M^2+\vec{P}^2)^{1/2}+i\sigma(E,P)]^2}\ . \tag{2.19}$$

Note that (2.19) has the same simple form as the lowest virial coefficient evaluated in (2.10) corresponding to Fig. 1(b), except that the full width $\Gamma(E,P)$ has been replaced by a reduced width $\sigma(E,P)$. From its definition in (2.18), we see that $\sigma(E,P)$ has a reduced-phase-space integral, where the occupation probability of the neutron has been subtracted, and that of the pion added. Such a reduction in width due to occupied fermions in the domain of decay and an enhancement due to occupied bosons would be expected on physical grounds, when an N^* decays in a medium. Our choice of diagrams contained in S conspire to merely alter the width in a self-consistent way. This is intuitively clear when a typical term in S, such as Fig. 8 is redrawn as in Fig. 10. Figure 10 shows that all we are doing in the sequence S is to expand the unitary series of the resonance in powers of the width Γ, and antisymmetrize the neutron and symmetrize the pion in the N^* wave function.

Finally, in the narrow ($g^2\to 0$) resonance limit, where all the widths Γ, γ_N, γ_π, and σ go to zero anyway, this self-consistent change in width should make no difference. This is clear from the fact that as g^2 [and hence $\sigma(E,P)]\to 0$, using (2.8)

$$S=\delta^3(0)\int d^3P\int dE\, e^{-\beta(E-\mu_1-\mu_2)}\,\delta(E-(\vec{P}^2+M^2)^{1/2})$$
$$=e^{+\beta(\mu_1+\mu_2)}\,V\int\frac{d^3P}{(2\pi)^3}\,e^{-\beta(M^2+\vec{P}^2)^{1/2}}\ . \tag{2.20}$$

Thus, S merely reduces to the lowest virial term $e^{-\beta(\mu_1+\mu_2)}b(1,1)$. In the narrow-resonance limit, all the other terms in S, which are individually nonzero and belong to higher virial coefficients, simply cancel, leaving only the lowest contribution in (2.10) due to the two-particle $\pi^- n$ scattering [Fig. 1(b)].

Let us now see what other contributions survive as $g^2\to 0$, other than S. It can be checked that

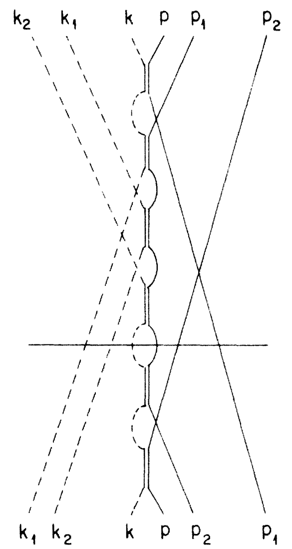

FIG. 10. This is Fig. 8 redrawn to bring out the reason why such diagrams merely alter the width.

only two types of diagrams remain:

(1) Diagrams where *both* the p and k line in Fig. 6 are exchanged with spectator particles. Remember that this was *not* included in S. Figure 11 gives the sequence of such diagrams. Note that diagrams such as in Fig. 12 will also individually exist, but they add up to zero exactly for the same reason that higher virial terms in S canceled. Thus the two examples in Fig. 12, apart from the extra exchanged pair which merely add a factor $e^{-\beta(e_1+\omega_1-\mu_1-\mu_2)}$ to both terms, are just two of the higher virial terms in S, all of which cancel to every order in μ_1 and μ_2. This leaves only terms of the type in Fig. 11. The reader can convince himself that diagrams in Fig. 11, along with the sequence S [of which only Fig. 1(b) remains] exhaust all "linked" diagrams which survive as $g^2 \to 0$. By linked we mean that all resonances in the diagram are manifestly connected by internal lines.

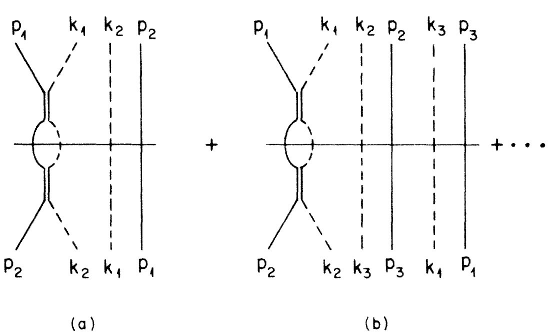

FIG. 11. The sequence where both the initial pion and neutron are exchanged with spectators, starting from the primitive diagram in Fig. 1(b).

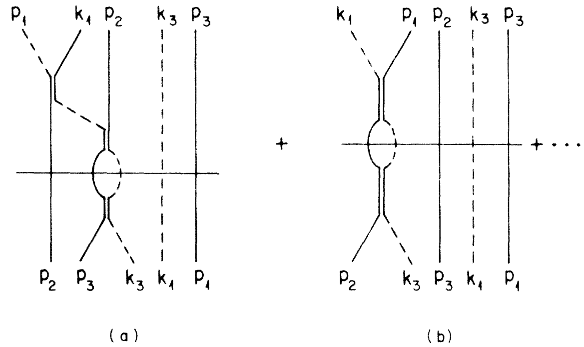

FIG. 12. The corresponding sequence for higher virial terms of S. These, however, cancel for the same reason that the higher virial terms in S canceled. See the text.

(2) In addition, one can also have unlinked terms which are connected because of the permutation of final-state labels, and consequently are permitted in the partition function expansion. For appropriate permutation of labels, these will also have only one independent energy which resonates. See the examples in Fig. 13. Note that the on-energy-shell S matrix of two unconnected pieces is just a product of the individual S matrices, each separately conserving energy momentum. It can be seen that, thanks to energy-momentum conservation, all the resonances in Fig. 13 occur in the same variable

$$\{e_1 + \omega_1 - [M^2 + (\vec{p}_1 + \vec{k}_1)^2]^{1/2} + i\Gamma\},$$

so that such terms will survive upon integration.

We assert that diagrams exemplified in Fig. 13 and Fig. 11, along with the sequence S, exhaust all possible linked and unlinked terms that effectively have only one independent resonant energy and therefore survive. To evaluate contributions in Fig. 11 and Fig. 13, let us consider for example

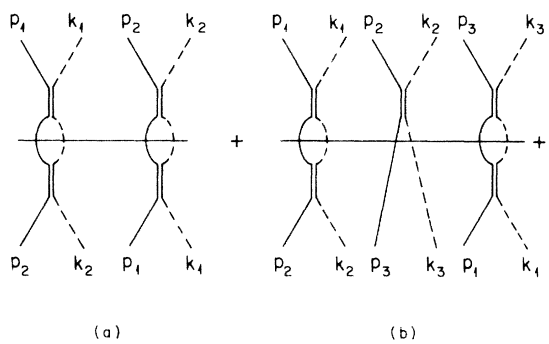

FIG. 13. Examples of unlinked contributions, which are, however, connected by the exchange of labels. Note that for the permutation of labels that connects such diagrams, all resonances again occur in the same energy variable. Thus every such contribution survives.

$n_1 = n_2 = 2$, when all terms of this type give

$$\frac{1}{2!}\,\frac{-1}{2\pi i}\sum_{\substack{AB\\CD}} e^{-\beta(e_A+e_C)+\beta(2\mu_1+2\mu_2)} \times \langle A|S_1^{-1}|B\rangle\langle C|S_2^{-1}|D\rangle \frac{\partial}{\partial(e_A+e_C)} \times(\langle B|S_1|C\rangle\langle D|S_2|A\rangle)\,, \qquad (2.21)$$

where labels A, B, C, D stand for pairs of $\pi^- n$ as in Fig. 14. We have used the fact that the on-energy-shell S matrices factorize for disconnected pieces—in this case into a product S_1S_2. We are using in (2.21) on-energy-shell S matrices as well as a derivative with respect to the total initial energy. Reference 1 shows that this is equivalent to the off-energy expression in (2.5) for unitary S matrices. Since S_1 and S_2 each conserve energy, it is clear for the particular set of exchanged labels in Fig. 14 that $e_A = e_B = e_D = e_C$. Further, since each S matrix is a sum of two terms,

$$\langle i|S|j\rangle = \delta_{ij} - 2\pi i\delta(e_i - e_j)\,T_{ij}\ ,$$

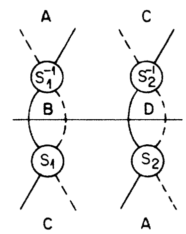

FIG. 14. The total contribution from two unlinked pieces.

Fig. 14 is a sum of 16 terms, of which Figs. 13(a) and 11(a) are examples. But the expression (2.21) equals

$$\frac{1}{2!}\,\frac{-1}{2\pi i}\sum_{AC} e^{-\beta(e_A+e_C-2\mu_1-2\mu_2)}\Bigg[\Big(\sum_B\langle A|S_1^{-1}|B\rangle\langle B|S_1|C\rangle\Big)\Big(\sum_D\langle C|S_2^{-1}|D\rangle\frac{\partial}{\partial(e_A+e_C)}\langle D|S_2|A\rangle\Big)$$
$$+\Big(\sum_B\langle A|S_1^{-1}|B\rangle\frac{\partial}{\partial(e_A+e_C)}\langle B|S_1|C\rangle\Big)\Big(\sum_D\langle C|S_2^{-1}|D\rangle\langle D|S_2|A\rangle\Big)\Bigg]$$
$$=\frac{1}{2!}\,\frac{-1}{2\pi i}\sum_{AC} e^{-\beta(2e_A-2\mu_1-2\mu_2)}\left(\left\langle C\left|S_2^{-1}\frac{\partial}{2\partial e_A}S_2\right|A\right\rangle\delta_{AC}+\delta_{AC}\left\langle A\left|S_1^{-1}\frac{\partial}{2\partial e_C}S_1\right|C\right\rangle\right),$$

where we have used

$$\frac{\partial}{\partial(e_A+e_C)} = \frac{\partial}{2\partial e_A} = \frac{\partial}{2\partial e_C}\ .$$

Thus, Fig. 14 merely gives

$$\frac{1}{2!}\,\frac{-1}{2\pi i}\sum_A e^{-2\beta(e_A-\mu_1-\mu_2)}\left\langle A\left|S^{-1}\frac{\partial}{\partial E}S\right|A\right\rangle$$
$$=\frac{-1}{2!}\,V\int\frac{d^3P}{(2\pi)^3}\exp\{-2\beta[(\vec{P}^2+M^2)^{1/2}-\mu_1-\mu_2]\}\ , \qquad (2.22)$$

following earlier algebra.

Incidentally, the $1/2!$ in (2.21) is the usual Boltzmann factor since there are two equivalent pairs in Fig. 14, while the minus sign in front is because a pair of fermions and a pair of bosons have been exchanged in the final state.

Following the same method, it can be seen that unlinked terms with $n_1 = m$ pions and $n_2 = m$ neutrons will yield

$$\frac{(-1)^{m-1}}{m}\,V\int\frac{d^3P}{(2\pi)^3}\exp\{-\beta m[(\vec{P}^2+M^2)^{1/2}-\mu_1-\mu_2]\}\ . \qquad (2.23)$$

The replacement of $1/m!$ by $1/m$ is because there are $(m-1)!$ permutations of the final $n\pi^-$ pairs that can keep m links "connected." Summing (2.23) over m, the number of $\pi^- n$ pairs, we have for the full expansion (2.9)

$$\ln Z = \ln Z_0 + V\int\frac{d^3P}{(2\pi)^3}\ln(\exp\{-\beta[(\vec{P}^2+M^2)^{1/2}-\mu_1-\mu_2]\}+1)\ . \qquad (2.24)$$

All contributions to all the virial coefficients in (2.9) have been included in (2.24). The entire sequence of linked terms in what we called S, merely readjusted the resonance width in a self-consistent way, as shown in Eq. (2.19), giving us back just the contribution of the lowest virial term (2.10). This forms the $m=1$ case of the expression (2.23). Unlinked terms with m pieces gave (2.23) for the appropriate value of m, and the sum over m yields (2.24). There are no other contributions to any of the virial terms as $g^2 \to 0$.

The result (2.24) is exactly what one would desire intuitively, viz., that the interaction between the $\pi^- n$ system, in our model, is replaceable by a third species of N^* particles, with mass M, with Fermi statistics, and a chemical potential $\mu_{N^*} = \mu_1 + \mu_2$, as required by baryon and charge conservation. Apart from this constraint on the chemical potential (which will determine the relative "density" of N^* to those of n and π^- for a given temperature, charge, and baryon number), the N^* species is independent of the n and the π. Its states need not be antisymmetrized in any sense with those of the neutrons or symmetrical with those of the pions. All this is true only in our model of a purely narrow-resonance-dominated dynamics—an assumption we examine in the following paper.

III. A FAMILY OF RESONANCES

We now extend these ideas to more general situations. When narrow resonances scatter with other particles or other narrow resonances, we show that the resulting contributions to the partition function Z behave as if the resonances were elementary stable particles. Secondly, when resonances scatter with other particles and resonances, only to produce yet other narrow resonances, i.e., when all n-particle scattering happens through "tree graphs" as in Fig. 15, then the full $\ln Z$ is just the sum of ideal-gas contributions with one ideal gas for each separate type of resonance. Any specific process, such as Fig. 15, contributes an ideal-gas term corresponding to the super n-particle resonance from which all the initial and final particles cascade.

We prove these assertions for the simple example of 3-particle scattering in a Boltzmann gas. Let particles 1 and 2 interact only through a narrow resonance N^*_{12}, which in turn can scatter off particle 3 (Fig. 15). More precisely, let the three-particle T matrix be given by

$$\langle 1'2'3' | T(E) | 123 \rangle = (2\pi)^3 \delta^3(\vec{P}_{12} + \vec{p}_3 - \vec{P}'_{12} - \vec{p}'_3) \times \left[\frac{g}{E - e'_3 - [M^2 + (\vec{P}'_{12})^2]^{1/2} + i\Gamma} \langle P'_{12} p'_3 | \tau(E) | P_{12}, p_3 \rangle \frac{g}{E - e_3 - [(\vec{P}_{12})^2 + M^2]^{1/2} + i\Gamma} + \frac{g^2 \delta^3(\vec{p}_3 - \vec{p}'_3)}{E - e_3 - [M^2 + (\vec{P}_{12})^2]^{1/2} + i\Gamma} \right] . \quad (3.1)$$

Here, the resonance parameters M, g^2, and Γ are defined earlier (2.3), $P_{12} = (\vec{P}_{12}, E_{12}) = p_1 + p_2$ and $P'_{12} = (\vec{P}'_{12}, E'_{12}) = p'_1 + p'_2$. The second term in (3.1) corresponds to Fig. 16(b), where particle 3 does not interact at all. This clearly has also to be included in $T(E)$ if Fig. 16(a) is included. Before inserting (3.1) into the virial series, let us examine unitarity for $T(E)$. As $g \to 0$, $T(E)$ vanishes, as per the identity (2.7). So does $T^\dagger(E)$. Therefore unitarity reduces to

$$T^\dagger(E)\, \delta(E - H_0)\, T(E) = 0 . \quad (3.2)$$

Now take, for example (3.2) as applied to forward scattering. $T^\dagger(E)\, \delta(E - H_0)\, T(E)$ has four contributions (Fig. 17) where once again Fig. 17(d) vanishes as $g \to 0$. It is just the imaginary part of Fig. 16(b). But the remaining terms [Figs. 17(a)–17(c)] do not vanish as $g \to 0$. Following the simple algebra of narrow resonances as in the last section, it is easy to check that (3.2) merely yields, when (3.1) is substituted for $T(E)$ and the limit $g^2 \to 0$ taken,

$$0 = \text{Figs. } [17(a) + 17(b) + 17(c)] = (2\pi)^4 \delta^3(0)\, \delta(E_{12} - [(\vec{P}_{12})^2 + M^2]^{1/2}) \left(\frac{(2\pi)^4 g^2}{2\Gamma} \right) \times \Big[\langle P_{12}, p_3 | i\tau^\dagger - i\tau | P_{12}, p_3 \rangle + 2\pi \int d^3 p'_3\, d^3 P'_{12}\, \delta^3(\vec{P}_{12} + \vec{p}_3 - \vec{P}'_{12} - \vec{p}'_3)\, \delta([(\vec{P}_{12})^2 + M^2]^{1/2} + e_3 - [(\vec{P}'_{12})^2 + M^2]^{1/2} - e'_3) \times \langle P_{12} p_3 | \tau^\dagger | P'_{12} p'_3 \rangle \langle P'_{12} p'_3 | \tau | P_{12} p_3 \rangle \Big] . \quad (3.3)$$

It is not surprising that the relation $T^\dagger T = 0$ when expanded yields the familiar two-particle unitarity relation for $\tau(E)$, on the resonance mass shell $E_{12}{}^2 = (\vec{P}_{12})^2 + M^2$.

Now, let us go on to evaluate the virial contribution to $\ln Z$ due to the T matrix in (3.1). As in the last section, the $(\partial/\partial E) T(E)$ term will vanish due to identity (2.7) leaving only the $T^\dagger(E)\, (\partial/\partial E)\, T(E)$ term as in (2.9). For our scattering amplitude (3.1), this gives three contributions, which can again be represented by the same Figures 17(a), 17(b), and 17(c), with the understanding that $\partial/\partial E$ acts on the lower part of each diagram. [Note that Fig. 17(d), being disconnected, is not to be included in the virial series. The ideal-gas term due to the resonance N^*_{12} is already present in the.

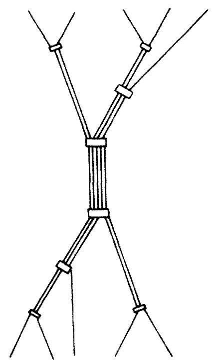

FIG. 15. Scattering of many particles through "tree graphs" made of only narrow resonances and stable particles.

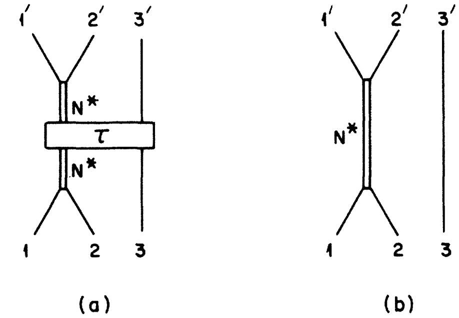

FIG. 16. The 3-particle T matrix where particles 1 and 2 can form a narrow resonance, which may (a) or may not (b) interact with 3.

lower (two-particle) virial coefficient in (2.10).]

The amplitude $T(E)$ in (3.1) depends on E through $\tau(E)$ as well as the resonance propagators. When $\partial/\partial E$ operates on $[\delta(E-H_0)\,T(E)]$ as required in the virial expansion, it will give terms where the N_{12}^* resonance propagator *is* differentiated and other terms where the resonance propagator is *not* differentiated. Let us begin with the former set of terms. This contribution from Figs. 17(a), 17(b), and 17(c) to the virial series can be seen to be proportional to

$$\sum_{123}\frac{\delta(E_{12}{}^2-[(\vec{\mathbf{P}}_{12})^2+M^2])}{i\Gamma}\Big[i\langle P_{12}p_3|\,\tau^\dagger-\tau\,|P_{12}p_3\rangle+2\pi\int d^3P_{12}'\int d^3p_3'\,\delta^3(\vec{\mathbf{P}}_{12}+\vec{\mathbf{p}}_3-\vec{\mathbf{P}}_{12}'-\vec{\mathbf{p}}_3')$$
$$\times\delta([(\vec{\mathbf{P}}_{12})^2+M^2]^{1/2}+e_3-[(\vec{\mathbf{P}}_{12}')^2+M^2]^{1/2}-e_3')$$
$$\times\langle P_{12}p_3|\,\tau^\dagger|P_{12}'p_3'\rangle\langle P_{12}'p_3'|\,\tau\,|P_{12}p_3\rangle\Big]\,. \qquad (3.4)$$

This is essentially the unitarity constraint in (3.3) except for the extra factor $1/i\Gamma$ which occurs because the $\partial/\partial E$ factor operates on the resonance propagator. From (3.3) then, the contribution (3.4) vanishes by unitarity, even though any single term in it would diverge as $\Gamma\to 0$.

We are left with contributions where $\partial/\partial E$ differentiates the product $\delta(E-H_0)\,\tau(E)$, leaving the resonance propagators alone. These clearly exist only for Figs. 17(b) and 17(c). These give for the virial series (2.5)

$$\text{Fig. 17(b)}=\frac{\delta^3(0)}{2\pi i}\int dE\sum_{123}\exp\Big[-\beta\Big(E-\sum_i\mu_i\Big)\Big]\frac{(2\pi)^4g^2}{(x-i\Gamma)(x+i\Gamma)}$$
$$\times\sum_{1'2'}\delta^4(P_{12}'-P_{12})\,\frac{(2\pi)^4g^2}{y+i\Gamma}\,\frac{\partial}{\partial E}[\delta(E-E_{12}-e_3)\langle P_{12}'p_3|\,\tau(E)\,|P_{12}p_3\rangle]\ ,$$

where

$$x=E_{12}-[(\vec{\mathbf{P}}_{12})^2+M^2]^{1/2}\,,\qquad y=E_{12}'-[(\vec{\mathbf{P}}_{12}')^2+M^2]^{1/2}\,,$$

and μ_i are the three chemical potentials. As $g^2\to 0$, this gives

$$\text{Fig. 17(b)}=\frac{\delta^3(0)}{2\pi i}\int dE\exp\Big[-\beta\Big(E-\sum_i\mu_i\Big)\Big]\sum_{\vec{\mathbf{P}}_{12}}\sum_{\vec{\mathbf{p}}_3}(-2\pi i)\frac{\partial}{\partial E}[\delta(E-E_{12}-e_3)\,\langle P_{12},p_3|\,\tau(E)\,|P_{12},p_3\rangle]\ ,\qquad (3.5)$$

where $E_{12}{}^2$ is constrained to be $(\vec{\mathbf{P}}_{12})^2+M^2$. Similarly, Fig. 17(c) will yield, when $\partial/\partial E$ acts on $\delta(E-H_0)\,\tau(E)$,

$$\text{Fig. 17(c)}=\frac{\delta^3(0)}{2\pi i}\int dE\exp\Big[-\beta\Big(E-\sum_i\mu_i\Big)\Big]\sum_{\vec{\mathbf{P}}_{12}}\sum_{\vec{\mathbf{p}}_3}2\pi\,\delta(E-E_{12}-e_3)\sum_{\vec{\mathbf{P}}_{12}'}\sum_{\vec{\mathbf{p}}_3'}\langle P_{12}p_3|\,\tau^\dagger(E)\,|P_{12}'p_3'\rangle$$
$$\times\frac{\partial}{\partial E}\,[2\pi\delta(E_{12}+e_3-E_{12}'-e_3')\,\delta^3(\vec{\mathbf{P}}_{12}+\vec{\mathbf{p}}_3-\vec{\mathbf{P}}_{12}'-\vec{\mathbf{p}}_3')\langle P_{12}'p_3'|\,\tau(E)\,|P_{12}p_3\rangle]\ ,\qquad (3.6)$$

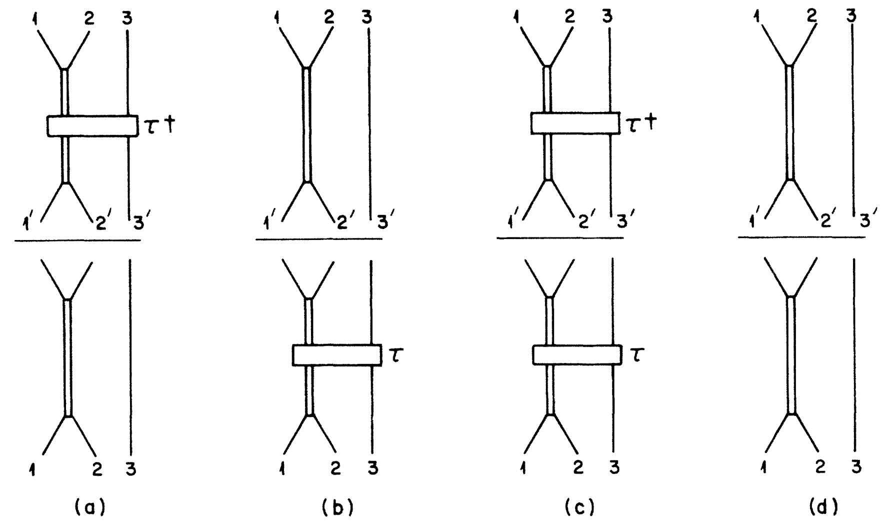

FIG. 17. The four graphs representing $T^{\dagger}T$ corresponding to the two terms in Fig. 16 for $T(E)$. These same graphs will be used to represent $T^{\dagger}(\partial/\partial E)T$ as well.

where again $(E'_{12})^2=(\vec{P}'_{12})^2+M^2$, $E_{12}{}^2=(\vec{P}_{12})^2+M^2$.

In deriving (3.6), the resonance propagators have been combined to form δ functions in the variables y and x, which restrict the pairs 12 and $1'2'$ to a total mass M. The results (3.5) and (3.6) are just what the scattering of particle 3 with a stable particle of mass M, momentum $\vec{P}_{12}$, and chemical potential $\mu_1+\mu_2$ would yield. Now take the special case when the N^*_{12}-particle 3 scattering is itself dominated by a narrow resonance N^*_{123} (Fig. 18). That is, let

$$\langle P_{12}p_3|\tau(E)|P'_{12}p'_3\rangle=\frac{(2\pi)^3G^2}{E-[\mathfrak{M}^2+(\vec{P}_{12}+\vec{p}_3)^2]^{1/2}+i\gamma}, \tag{3.7}$$

with

$$2\gamma=\sum_{\vec{P}_{12}}\sum_{\vec{p}_3}\delta^3(\vec{P}_{12}+\vec{p}_3)\times\delta([(\vec{P}_{12})^2+M^2]^{1/2}+e_3-\mathfrak{M})\,(2\pi)^4G^2.$$

Then, as $G^2\to 0$, the virial contributions in (3.5) and (3.6) follow the same pattern as the lowest virial term in Fig. 1 and Eq. (2.10). The contribution in (3.5) vanishes due to identity (2.7), and in (3.6) only the term where the derivative $\partial/\partial E$ acts on $\tau(E)$ survives. The surviving contribution to (3.6) is just

$$\frac{\delta^3(0)}{2\pi i}\sum_{\vec{P}_{12},\vec{p}_3}\exp\left[-\beta\left([(\vec{P}_{12})^2+M^2]^{1/2}+e_3-\sum\mu_i\right)\right]\times\sum_{\vec{P}'_{12},\vec{p}'_3}\delta^4(P_{12}+p_3-p'_3-P'_{12})\frac{[(2\pi)^4G^2]^2(-1)}{(Z+i\gamma)(Z-i\gamma)^2} \tag{3.8}$$

where

$$Z\equiv[(\vec{P}_{12})^2+M^2]^{1/2}+e_3-[\mathfrak{M}^2+(\vec{P}_{12}+\vec{p}_3)^2]^{1/2}.$$

Integrating (3.8) over all variables except $\vec{P}_{123}=\vec{P}_{12}+\vec{p}_3$ and $E_{123}=[(\vec{P}_{12})^2+M^2]^{1/2}+e_3$, we obtain when G^2 and γ tend to zero, the result

$$\delta^3(0)\int d^3P_{123}\int dE_{123}\exp\left[-\beta\left(E_{123}-\sum_i\mu_i\right)\right]\delta(Z)=V\int\frac{d^3P_{123}}{(2\pi)^3}\exp\left(-\beta[(\vec{P}_{123})^2+\mathfrak{M}^2]^{1/2}+\sum_i\beta\mu_i\right). \tag{3.9}$$

Thus, the scattering through the "super" resonance N^*_{123} again produces an "ideal-gas" term corresponding to mass $\mathfrak{M}$ and chemical potential $\mu_1+\mu_2+\mu_3$, *in addition* to the ideal-gas term corresponding to the two-particle resonance N^*_{12} present in the lowest virial coefficient.

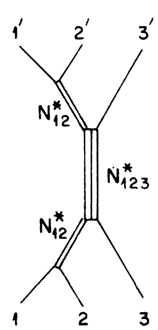

FIG. 18. The scattering of the resonance N^*_{12} with particle 3 through another narrow resonance N^*_{123}.

These results have been derived only for three-particle scattering of a Boltzmann gas. This was just to keep the algebra simple. It is clear from the proof that the generalization of Eqs. (3.5) and (3.6) where a narrow resonance scatters just like a stable particle, and of (3.9) where a super resonance formed out of lesser resonances again gives an ideal-gas term, to more than three Boltzmann particles will be straightforward. One must merely keep track of all connected and disconnected pieces of the S matrix, and the width of any given resonance will have to include the phase space of all the channels connected to it. Generalization to Bose or Fermi systems will of course be complicated. In Sec. II, where scattering took place only by forming one type of two-particle resonance, we saw that certain exchange diagrams exist in all virial coefficients. The only "direct" diagram in that case was Fig. 1(b), which would have been the sole contributor had the particles obeyed Boltzmann statistics. In the boson-fermion case dealt with in Sec. II, the exchange diagrams required a considerable amount of diagrammatic analysis and narrow-resonance algebra. But the net result was simple and physically reasonable. The higher exchange terms merely altered the width self-consistency to account for the fact that the resonance decays in a medium, and provided the resonance "particle" with approximate statistics. The many-particle dynamics of Sec. III, when applied to Bose and Fermi systems, will again yield a family of exchange diagrams in higher virial coefficients. But we hope the reader will be convinced that their effect will once again be to merely provide the appropriate statistics to all the resonances.

IV. CONCLUSION

We showed in Sec. II that a narrow elastic resonance in a $\pi^- n$ channel behaves like an elementary particle with Fermi statistics. It is clear that this result is more general. If the initial stable particles were both bosons or both fermions, the corresponding changes in sign of exchange diagrams would ensure that the resonance is a boson. Inelastic resonances will again give the same result. Suppose a resonance had two decay channels. (This cannot happen in the examples of Sec. II where "n" and "π^-" had separately conserved quantum numbers, but it can if π^+ were included.) Let the N^* decay into $n\pi^-$ and $n\pi^-\pi^+\pi^-$. Then the N^* ideal-gas term will arise by including both the $n\pi^-$ and the $n\pi^-\pi^+\pi^-$ states in the initial and intermediate states of Fig. 1. Both these states have the same chemical potential, and are both to be included in the width of the N^*. Thus, our result that narrow resonances act as a separate species generalizes to all choices of quantum numbers, for elastic and inelastic resonances.

In both the previous sections, we always started with idealized many-particle scattering amplitudes—those involving only narrow resonances. Thus the elastic two-particle amplitude in Sec. II had only a pole at the resonance energy and nothing else. This is not to imply that any real system has such simple dynamics. In fact, as we discuss in the following paper, such a simple amplitude is forbidden in potential theory.

To the extent that a real system has wide resonances, nonresonant scattering in addition, or no resonances at all, its thermodynamics will have to be calculated the usual hard way. This is obvious and not germane to our interest here. Instead, we wanted to verify the notion that *to the extent that* scattering in some case of interest is dominated by narrow resonances, it could be replaced by additional ideal-gas terms with the quantum numbers of the resonances. This is one concrete way of stating the vague equivalence of narrow resonances with stable particles. We found that in fact such a replacement can be made in precisely the way expected intuitively.

When a resonance is wide, as compared with kT (this is so for the Δ resonance in neutron stars), then two types of corrections need to be made. A trivial correction is that expressions such as $\Gamma/[(E-i\Gamma)(E+i\Gamma)]$ cannot be replaced by $\pi\delta(E)$ and will have to be integrated more accurately. More seriously, the vast families of diagrams and contributions we discarded will now contribute. An example is a process such as that shown in Fig. 3, which can be considered as repeated resonance particle scattering. For a wide resonance (large coupling g^2), such scattering will become important and will have to be corrected for. This is of course very difficult in practice, but estimates are being made in neutron-star calculations involving the 3-3 resonance Δ to correct for Δ-n and Δ-Δ scattering.[5]

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Effective elementarity of resonances and bound states in statistical mechanics*

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Operational criteria are presented for determining those bound states and resonances which can approximately be included in the complete set of states in the S-matrix formulation of statistical mechanics. The criteria depend only on the energy dependence of S-matrix elements, as compared to the energy scales determined by the temperature and density. They are thus expressible free of nonrelativistic potential-theory language, and are hopefully valid for relativistic hadron systems as well. As an application, it is shown that the Δ resonance can be effectively treated as an elementary species under the temperature and density conditions encountered in neutron stars, while nuclei such as the deuteron can be ignored to lowest approximation. Possible conflicts with the Pauli principle, as invoked between the "constituents" of composite resonances and bound states on the one hand and free particles on the other, are resolved.

I. INTRODUCTION

In the quantum statistical mechanics of a nonrelativistic system, the grand partition function Z is defined by

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu)}, \qquad (1)$$

where H is the total Hamiltonian. The value of μ, the total chemical potential operator for any given state depends on all its internal quantum numbers, and the trace is taken over a complete set (say, the plane waves) of states of all possible numbers of "elementary" particles. Once Z is known, the thermodynamic behavior of the system can be deduced.

The prescription in Eq. (1) is satisfactory for most nonrelativistic systems of interest. For example, in atomic systems at $\beta^{-1} \equiv kT \simeq$ ionization energy, free states of nuclei, electrons, and photons form a satisfactory complete set, with H well defined in that basis. In the MeV energy range (nuclear physics), protons and neutrons instead of nuclei act as elementary particles. While H is not so well known here, thanks to the complicated nuclear two-body force and uncertain many-body forces, the choice of neutrons and protons as elementary particles with their respective statistics is well supported by the structure of large nuclei.

Unfortunately, in relativistic systems, particularly ones involving hadrons, there is no clear knowledge of which particles are elementary or of what the Hamiltonian should be in a basis constructed from them. Nor is it certain that either "elementarity" or the Hamiltonian operator will remain as viable concepts, if and when a fully satisfactory dynamical theory of such systems becomes available.

It is therefore better, from the point of view of extending quantum statistics to relativistic systems, to use an alternate prescription for Z in the place of (1). The S-matrix formalism of statistical mechanics[1] provides a candidate. Therein,

$$\ln Z = \sum_i \ln Z_0^{(i)} + \frac{1}{2\pi i}\sum_{\{n_i\}}\int dE\, e^{-\beta(E-\mu)}\left[\mathrm{Tr}_{\{n_i\}} A S^{-1}(E)\frac{\partial}{\partial E}S(E)\right]_c. \qquad (2)$$

The label i runs over all species of *stable* particles. The trace is taken over all plane-wave states of every set $\{n_i\}$ with n_i particles from the ith stable species. The other symbols in (2) are defined in Ref. 1 and the preceding paper.[2]

Equation (2) is derived[1] from potential theory,