Effect of Bose-Einstein statistics on multiplicity distributions and correlations in multiparticle production processes at high energies

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It is shown that only a small number of elementary cells in phase space are populated in a multiparticle production reaction even at high energy (> 100 GeV), leading to occupancy numbers for identical bosons in individual cells which are not small. It follows that quantum-statistical fluctuations are of significant magnitude. The effects on multiplicity distributions and integral correlation functions of π mesons produced in high-energy interactions are derived. Comparison with experimental results at incident energies above 100 GeV indicates that Bose-Einstein fluctuations may be a principal source of broadening of the multiplicity distribution and of positive short-range correlations among identical particles. The effect is nondynamical and nonkinematical in the usual sense, but is due to identical-particle symmetry, and is the counterpart of the Hanbury Brown–Twiss effect in the electromagnetic field.

The π mesons produced in interactions between hadrons at high energies constitute a partially degenerate boson gas. Thus, in principle, one expects partial second-order coherence in the boson field, or fluctuations in the numbers of mesons produced, which are a consequence of quantum-statistical interference among identical particles. (The effect is a direct counterpart of that observed by Hanbury Brown and Twiss[1] for the electromagnetic field under conditions suitable for coherence. It is known as the prototype of second-order coherence in optical coherence theory.) Whether or not these fluctuations may be observed in π-meson distributions depends on the degree of degeneracy of the boson gas and competing kinematical or dynamical effects. It is the purpose here to show that the degeneracy of the π-meson gas produced in high-energy interactions is sufficient to give rise to Bose-Einstein fluctuations in the multiplicities of mesons produced which in turn affect correlation functions significantly.

The multiplicity distributions derived are broader than the Poisson distribution; the f_2 and higher correlation functions of the distributions are positive and increase with $\ln s$. Kinematical restrictions, which oppose and dominate the quantum-statistical fluctuations at low energies (< 50 GeV), are not explicitly considered here.

Interest in multiplicity distributions and two-or-more-particle correlation functions has increased with the availability of data from NAL at incident laboratory momenta of 100 to 400 GeV/c and from the CERN ISR at energies equivalent to laboratory momenta of 300 to 2000 GeV/c. Consult Ref. 2 for several recent review articles on multiparticle production at high energies, which in turn contain extensive references to the original literature.

Although any microscopic quantum calculation in which the production matrix element is properly symmetrized in the momenta of the outgoing particles will automatically include quantum-statistical-interference effects, the difficulties of describing exactly a final state consisting of 10 or 20 particles are prohibitive. Goldhaber *et al.*[3] have symmetrized final states of $p\bar{p}\to n\pi$ for $n_+ = n_- = 2$ or 3 and $n_0 \leq 2$, and have deduced and observed correlations in azimuthal angle differing for pairs of like and unlike particles. The opening angles of like pairs are on the average less than 90°, and those of unlike pairs are greater than 90°. An interaction radius of about $\frac{3}{4}\lambda\!\!\!^{-}_\pi$ or about one fermi is required to explain the magnitude of the observed effects. This effect has been investigated by others but is still not completely understood.

Although less detailed than a microscopic quantum calculation, several approaches other than the symmetrization of the wave function may be adopted to describe the fluctuations. A quantum-statistical-mechanical argument is advanced here.

Consider an event at high energy in which several identical mesons are produced. The size of an elementary cell in phase space is given by $\Delta x^3 \Delta p^3 = h^3$, where Δx^3 is the volume in configuration space in which the particles are confined, Δp^3 is the volume in momentum space or momentum "bandwidth" of the particles, and h is Planck's constant. Assume that the interaction region in which the production occurs is limited to a small volume Δx^3. Then all identical particles within a bandwidth $\Delta p^3 = h^3/\Delta x^3$ occupy a *single elementary cell in phase space*. If Δx^3 is of the order of one cubic fermi, then Δp^3 is of the order of

$(1200\ \text{MeV}/c)^3$. Identical π mesons produced in high-energy interactions should thus exhibit partial coherence over a wide momentum band if the interaction volume in which they are produced is, for example, the cube of the range of the strong interaction.

Consider each event in which multiple bosons are produced under identical conditions as a member of a statistical ensemble. Furthermore, consider that each event is a representative sample of the phase space of the system. Assume initially that the identical mesons produced in each event occupy a single cell in phase space. Let N designate the total number of identical mesons observed in g events.

If the particles obey classical or Maxwell-Boltzmann statistics and N particles are distributed among g cells, the probability of observing n of the particles in a single cell is given by

$$P_{\text{MB}}(n)=\frac{N!\,(g-1)^{N-n}}{n!\,(N-n)!\,g^N}\ . \qquad (1)$$

In the limit of $g \gg 1$ and $N \gg n$, (1) becomes the Poisson distribution,

$$P_{\text{MB}}(n)=\frac{\langle n\rangle^n}{n!}\,e^{-\langle n\rangle}\ . \qquad (2)$$

The fluctuations of this distribution are given by

$$\frac{D^2}{\langle n\rangle^2}=\frac{1}{\langle n\rangle}\ , \qquad (3)$$

where $\langle n\rangle = N/g$, the average occupation number of a cell, and $D^2=\langle (n-\langle n\rangle)^2\rangle$.

If the particles obey Bose-Einstein statistics, then the probability of observing n particles in a particular cell is given by[4]

$$P_{\text{BE}}(n)=\frac{N!\,(g-1)\,(N-n+g-2)!}{(N-n)!\,(N+g-1)!}\ . \qquad (4)$$

In the limit of $N \gg n$ and $g \gg 1$ this expression becomes the counterpart of a Poisson distribution for identical bosons,

$$P_{\text{BE}}(n)=\frac{1}{\langle n\rangle+1}\left(\frac{\langle n\rangle}{\langle n\rangle+1}\right)^n\ , \qquad (5)$$

where $\langle n\rangle = N/g$ as before. The fluctuations given by (5) are

$$\frac{D^2}{\langle n\rangle^2}=1+\frac{1}{\langle n\rangle}\ . \qquad (6)$$

Expressions (2) and (5) permit one to compare the expected single-cell Bose-Einstein distribution with the classical distribution when the average occupation number is known or determined empirically. The Bose-Einstein distribution decreases monotonically with increasing n for all values of $\langle n\rangle$ and, as one expects, enhances occupation numbers for the largest and smallest values of n while diminishing occupation numbers for values of n near the peak of the classical distribution. In the limit of very small occupation numbers, of course, the Bose-Einstein distribution approaches the classical distribution.

The argument that identical particles produced in a limited spatial volume and within a certain spread of momentum lie in a single elementary cell in phase space may be rephrased in quantum-mechanical terms. Consider the volume of limited extent in which the production is assumed to take place to define the boundaries of a quantum-mechanical system. Then all emitted particles with less than a certain linear momentum are likely to carry away no angular momentum, i.e., they occupy an S state with zero angular momentum with respect to the center of mass of the system. It again follows for identical bosons in the same quantum-mechanical state that one expects occupation numbers and fluctuations predicted by the Bose-Einstein distribution. Furthermore, the distribution of these emitted particles must be spatially isotropic, i.e., must exhibit the spherical symmetry of the S state.

It has been observed[5] in K^+-p interactions at 12 GeV/c that π mesons with momentum in the center-of-mass system below about 320 MeV/c are distributed isotropically, and that these mesons are about $\frac{1}{3}$ of the total number produced. If one assumes that Δx^3 and Δp^3 are spheres, of radius R in configuration space and of radius p in momentum space, respectively, then for $p=320$ MeV/c, $R=1.4$ F, approximately equal to the Compton wavelength of the π meson or the range of the strong interaction. This result suggests that the mesons which are isotropic in the center-of-mass system in this reaction occupy an S state with respect to the small interaction volume in which they are produced. It further suggests that all of the mesons produced in an event may be considered as occupying a small number of quantum states or elementary cells in phase space rather than a single state or cell.

It is known empirically that not all of the available phase space is occupied in the final state of a multibody production reaction at high energy. In particular the transverse momentum p_T of π mesons produced is cut off approximately exponentially with a characteristic length of about 320 MeV/c roughly independent of the energy of the incident particle from 10 GeV through ISR-equivalent laboratory energies of up to 2000 GeV. The Lorentz-invariant-phase-space (LIPS) element is d^3p/ω, where ω is the total energy of a particle. The longitudinal characteristics of multiparticle production reactions may be plotted in terms of the rapidity variable,

$$y=\tfrac{1}{2}\ln\frac{\omega+p_L}{\omega-p_L},$$

such that

$$\frac{d\sigma}{d^3p/\omega}=\frac{d\sigma}{dy\,d^2p_\perp}.$$

In a cylindrical section of momentum space, with axis parallel to the incident momentum, equal intervals of rapidity correspond to equal volumes of LIPS.

In pp interactions at 300 GeV/c the total rapidity interval is about 6.4 or about ± 3.2 units with respect to the center of mass. The rapidity of a π meson produced with the same velocity as the incident proton is also 3.2 units in the center-of-mass system. The rapidity of a meson of 320 MeV/c in the center-of-mass system is about 1.5. At ISR energies a "plateau" of height independent of incident energy is observed to develop in the rapidity distribution of the π mesons produced. Thus the region of momentum space populated by π mesons produced in high-energy interactions is empirically observed to be cylindrical with axis parallel to the incident momentum. It is approximately uniformly populated longitudinally in terms of rapidity or LIPS. In terms of the cube of Planck's constant, the measure of an elementary cell in phase space, it is about one cell in diameter and a few cells long even at 300 GeV/c, if the interaction volume in which the particles are produced is a sphere of radius of the order of the range of the strong interaction. In order to establish coherence it is also necessary that the particles be produced in this volume within a time of the order of λ_π/c or about 10^{-23} sec. From this description it is not obvious precisely how many cells are involved or that their population density is constant, but qualitatively one argues that it is obvious that most of the π mesons are produced in a limited volume of phase space which corresponds to a small number of elementary cells, providing the above space-time restrictions obtain in the production process.

If n identical bosons, produced in a single event, are distributed among k elementary cells in phase space, the distribution of occupancy number for a single cell will be given by

$$P(\langle n_i\rangle, n_i)=\frac{1}{\langle n_i\rangle+1}\left(\frac{\langle n_i\rangle}{\langle n_i\rangle+1}\right)^{n_i}, \tag{7}$$

where $\langle n_i\rangle$ is the mean occupancy, and n_i the particular occupancy, of the ith cell. While the occupancy of a single cell is governed by the Bose-Einstein distribution the relative occupancies of different cells are statistically independent in the classical sense (except as governed by conservation laws or other symmetries). The general distribution of the total number n among the k cells is thus of the form

$$P(n)=\left[\sum_{n_i=0}^{\infty}\prod_{i=1}^{k}P(\langle n_i\rangle, n_i)\right]\times\delta\left(\sum_i n_i-n\right)\delta\left(\sum_i\langle n_i\rangle-\langle n\rangle\right). \tag{8}$$

The general expression (8) is rather complicated for arbitrary distributions of the $\langle n_i\rangle$, but for the simplest special case in which the n particles are distributed among the k cells with equal *a priori* probability for each cell (8) may be reduced to[6]

$${}^kP_{\rm BE}(n)=\frac{(n+k-1)!}{n!\,(k-1)!}\,\frac{\langle n_i\rangle^n}{(\langle n_i\rangle+1)^{n+k}}, \tag{9}$$

where $\langle n_i\rangle=\langle n\rangle/k$ for all cells. For $k=1$ distribution (9) becomes the single-cell Bose-Einstein distribution, (5), while for k much greater than $\langle n\rangle$ (9) approaches the Poisson distribution (2). The relative fluctuations given by (9) are

$$\frac{D^2}{\langle n\rangle^2}=\frac{1}{k}+\frac{1}{\langle n\rangle}. \tag{10}$$

Finally, the partial cross section for the production of n identical π mesons is given by

$$\sigma(n)=\sigma_{\rm inel}\,{}^kP_{\rm BE}(n), \tag{11}$$

where $\sigma_{\rm inel}$ is the measured total inelastic cross section, $\langle n\rangle$ is the measured mean multiplicity for identical mesons, and k is a small integer representing the number of elementary cells in phase space occupied by the mesons in one event. Figure 1 shows the distribution of multiplicities of negatively charged particles produced in p-p interactions at 405 GeV/c incident laboratory momentum,[7] for which $\langle n\rangle=3.50$, compared with the Poisson distribution, and with the partially degenerate boson gas distribution with $k=6$. The Poisson distribution fits the data poorly, while the boson distribution fits the data well, with a significant error for only the two-prong inelastic cross section.

The emergence of a plateau in the single-particle rapidity distribution, and the maintenance of the limitation on transverse momentum for most of the particles produced at ISR energies, may be interpreted as approximately uniform density of population of LIPS, independent of center-of-mass energy. Without reference to the underlying dynamics responsible for the observed behavior this provides the empirical interpretation of the increase of $\langle n\rangle$ with $\ln s$ as the longitudinal growth of the populated cylindrical portion of momentum space with $\ln s$ while maintaining approximately

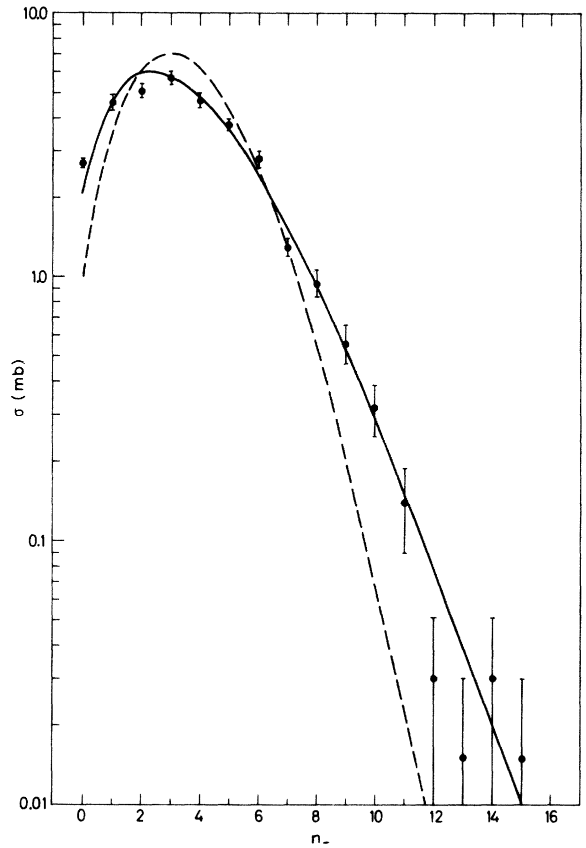

FIG. 1. Multiplicity distribution for the production of negatively charged particles in pp interactions at 405 GeV/c. The data are the measured partial cross sections in mb. The dashed curve is a Poisson distribution using the measured total inelastic cross section and the measured mean multiplicity. The solid curve is the partially degenerate boson gas distribution using the same inelastic cross section, the same mean multiplicity, and a partial degeneracy parameter $k=6$.

constant density of population. That is, $\langle n\rangle \propto k$, and both increase as $\ln s$ while $\langle n\rangle/k$ remains constant.

The inverse of the relative dispersion $\langle n_c\rangle/D_c$ has been surmised to approach a constant value of approximately 2.0 at high energies,[8] where $\langle n_c\rangle$ and D_c refer to the mean multiplicity and dispersion respectively for all charged particles. More recently Wroblewski[9] has parametrized a new relationship, $\sqrt{3}\,D_c=\langle n_c\rangle-1$, which fits the data accurately from 10 to 10^4 GeV/c. In the Bose-Einstein interference effect the more significant quantity is $\langle n_-\rangle/D_-$, where $\langle n_-\rangle$ and D_- refer strictly to the mean number and dispersion of multiplicities of π^- (or alternatively π^+) mesons only, but approximately to all negative tracks, since π^- mesons constitute most of the negative particles produced. It follows from distribution (9) that

$$\frac{\langle n_-\rangle}{D_-}=\left(\frac{\langle n_-\rangle}{1+\langle n_-\rangle/k}\right)^{1/2} . \qquad (12)$$

The quantity $\langle n_-\rangle/D_-$ then would increase slowly as $\langle n_-\rangle^{1/2}$ or $(\ln s)^{1/2}$ if $\langle n\rangle/k$ remains constant as mentioned above. The quantity $\langle n_c\rangle/D_c$ is, of course, directly related to $\langle n_-\rangle/D_-$ since $n_c=2n_-+2$ (for pp interactions). In this view the apparent approach from above of $\langle n_c\rangle/D_c$ to approximately a constant value at NAL energies is a fortuitous combination of kinematical effects, which predominantly narrow the multiplicity distribution at lower energies, the Bose-Einstein fluctuation effect, and the fact that $\langle n_c\rangle$ includes the charges of the two protons in the final state with no fluctuation. At high energies there should be a very slow increase, as $(\ln s)^{1/2}$, of $\langle n_-\rangle/D_-$. A plot of $\langle n_-\rangle/D_-$ vs $(\ln s)^{1/2}$ is shown in Fig. 2. Above about 50 GeV/c a linear relationship provides a good fit. If $\langle n_-\rangle/D_-$ is approximately proportional

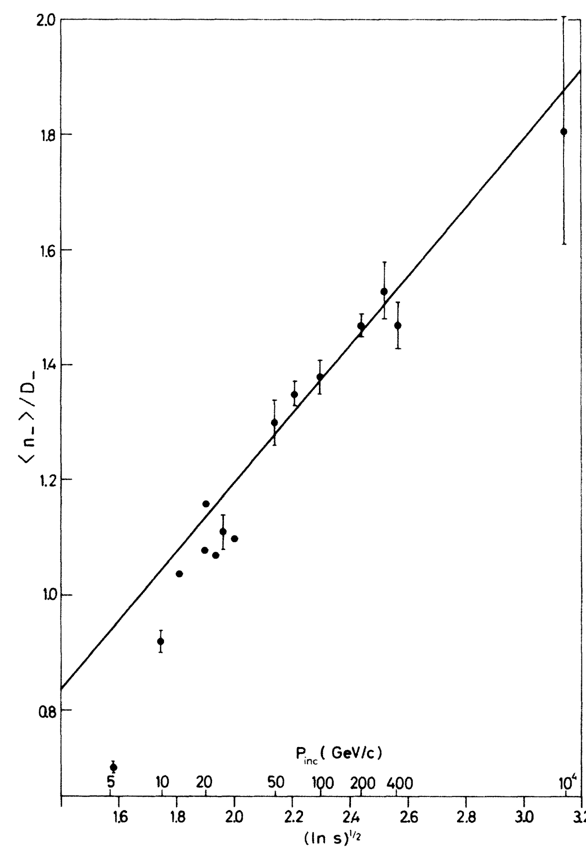

FIG. 2. Inverse of relative dispersion vs $(\ln s)^{1/2}$ for negative particles produced in pp interactions at incident laboratory momenta from 5.5 to 10^4 GeV/c. See Ref. 7 for sources of data. The straight line shows a reasonable linear fit above about 50 GeV/c. Deviations below 50 GeV/c are attributable to kinematical factors.

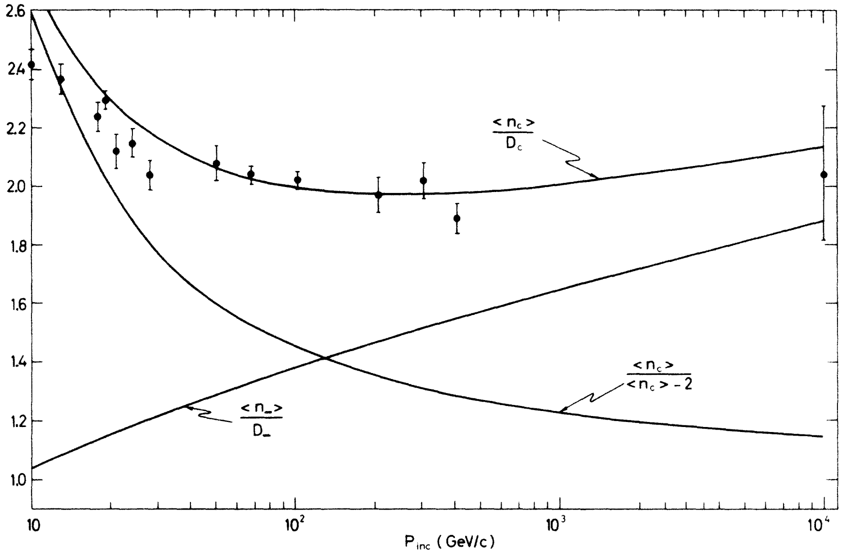

FIG. 3. The behavior of $\langle n_c\rangle/D_c$ as a function of incident laboratory momentum from 10 to 10^4 GeV/c. The curve $\langle n_-\rangle/D_-$ shows the contribution of the Bose-Einstein fluctuation effect, while the curve $\langle n_c\rangle/(\langle n_c\rangle-2)$ shows the contribution of the charges of the two original protons. These combine to produce a broad flat minimum near a value of 2.00. Deviations below about 30 GeV/c are attributable to kinematical effects.

to $(\ln s)^{1/2}$ as in Fig. 2, then $\langle n_c\rangle/D_c$ will pass through a very broad minimum, varying from 2.00 by less than 2% from 50 GeV/c to 2000 GeV/c. That is, since

$$\frac{\langle n_c\rangle}{D_c}=\frac{\langle n_c\rangle}{\langle n_c\rangle-2}\times\frac{\langle n_-\rangle}{D_-},$$

the slow decrease of the first factor and the slow increase of the second factor with energy combine to produce a broad minimum. Figure 3 shows graphically how these two factors combine to produce a flat minimum together with experimental values of $\langle n_c\rangle/D_c$. Both $\langle n_-\rangle/D_-$ and, consequently, $\langle n_c\rangle/D_c$ deviate from the curves shown at energies less than about 30 GeV/c, presumably because of kinematical effects at low energies which are incompletely incorporated. Kinematical effects at low energies are partially included empirically through the use of experimental values for the factor $\langle n_c\rangle/(\langle n_c\rangle-2)$. Although Wroblewski's parametrization of $\langle n_c\rangle/D_c$ fits the data better at low energies, Fig. 3 shows that an eventual slow rise of $\langle n_c\rangle/D_c$ at high energies is not excluded by the present data.

Mueller[10] has shown that the binomial moments of the multiplicity distribution are useful in the description and analysis of multiparticle final states. These are given by

$$\begin{aligned} g_1&=\langle n\rangle ,\\ g_2&=\langle n(n-1)\rangle ,\\ g_3&=\langle n(n-1)(n-2)\rangle ,\\ &\ldots . \end{aligned} \tag{13}$$

Related to the binomial moments are the correlation functions

$$\begin{aligned} f_1&=g_1=\langle n\rangle ,\\ f_2&=g_2-g_1^2=\langle n(n-1)\rangle-\langle n\rangle^2 ,\\ f_3&=g_3-3g_2g_1+2g_1^3 ,\\ &\ldots . \end{aligned} \tag{14}$$

A Poisson distribution has the property that f_2 and all higher-order correlation functions are equal to zero. Conversely, if f_2 and higher moments of the distribution of final-state particles in a given reaction are zero, then the particles can be said to be produced independently in some classical sense. However, it can be shown that conservation of momentum and energy require some of the f_m to be nonzero and in fact negative in a multiparticle production reaction. Further, the correlation function f_2 is the integral of the normalized two-particle density distribution or differential two-particle correlation function. That is, if $f_2>0$, then the particles tend to cluster compared with a classically random distribution. Conversely, if f_2 is negative the particles are distributed more uniformly than a classically random distribution, or the distribution is narrower than Poisson or the particles are anticorrelated.

At low energies of interaction, it is observed that produced particles occupy almost all of the available phase space, i.e., they are found in appreciable numbers near all boundaries of the phase space. Conservation laws then enforce anticorrelation, i.e., if a particle is located near the boundary of available phase space (near maximum allowable momentum), then a second particle cannot be located near it, or be closely correlated in momentum. Therefore multiplicity distributions are narrower than Poisson and f_2 is negative at low interaction energies. On the other hand, Bose-Einstein fluctuations alone, disregarding conservation laws, require positive values for f_2 and all higher moments, i.e., all distributions are broader than or equal to Poisson. Values of f_m derived from the partially degenerate Bose-Einstein distribution (9) are

$$
\begin{aligned}
f_2 &= \frac{1}{k}\langle n\rangle^2 , \\
f_3 &= \frac{2}{k^2}\langle n\rangle^3 , \\
&\ldots , \\
f_m &= \frac{(m-1)}{k^{m-1}}\langle n\rangle^m .
\end{aligned} \tag{15}
$$

For $k \to \infty$ and fixed $\langle n\rangle$ all of these higher correlation functions vanish as distribution (9) reverts to the Poisson. For $k=1$ the correlation functions assume the correct values for the limiting or single-cell Bose-Einstein distribution (5).

At low energies f_2 is kinematically forced to be negative for particle-production reactions as mentioned above. At higher energies, as the boundaries of available phase space become less populated, because of the empirically observed limitation on transverse momentum, the constraint on f_2 is relaxed and it more clearly reflects the Bose-Einstein effect. The kinematical constraint remains but is shifted toward higher moments at higher energies. Conservation laws always require that some of the higher correlation functions be negative, e.g., at least that the multiplicity distribution be cut off at some finite value of n. The value of f_2 for the 405-GeV/c pp multiplicity distribution for negative particles only is 2.1 ± 0.2, and the value of $\langle n_-\rangle$ is 3.50.[7] Expression (15) for f_2 then gives a most likely value of $k=6$. The Bose-Einstein fit to the multiplicity distribution shown in Fig. 1 uses $k=6$ accordingly. Using the same values of k and $\langle n_-\rangle$ the Bose-Einstein distribution predicts values of 2.4 for f_3 and 2.1 for f_4, whereas the experimentally observed values are 0.1 ± 0.9 and 1 ± 4 respectively.

A plot of f_2^- (negative tracks only) is shown as a function of ln$s$ in Fig. 4. At low energies f_2 is definitely negative but at energies above about 50 GeV/c a linear relationship to ln$s$ is reasonable, assuming that $\langle n_-\rangle/k$ remains constant as discussed above.

The positive values of f_2 observed at sufficiently high energies reflect the largely positive two-particle differential correlations also observed. The correlation length or the range of the correlation cannot be deduced from f_2 alone. However, the correlation produced by the Bose-Einstein fluctuations may be qualitatively expected to be of a short-range nature. As argued above, if the configuration volume in which the π mesons are initially confined is a sphere of radius approximately 1.4 fermi, then the partial volume of the elementary cell in momentum space will be that of a sphere of radius approximately 320 MeV/c because of the magnitude of Planck's constant. Since the particles are empirically largely confined to a transverse momentum of about 320 MeV/c, the longitudinal extent of a single elementary cell will also be about 320 MeV/c. For π mesons this corresponds to a rapidity of about 1.5 units. One expects the quantum-statistical interference to occur among particles occupying a region in phase space comparable in volume to a single elementary cell. Thus the correlations should be local and of short range. For π mesons the correlation length longi-

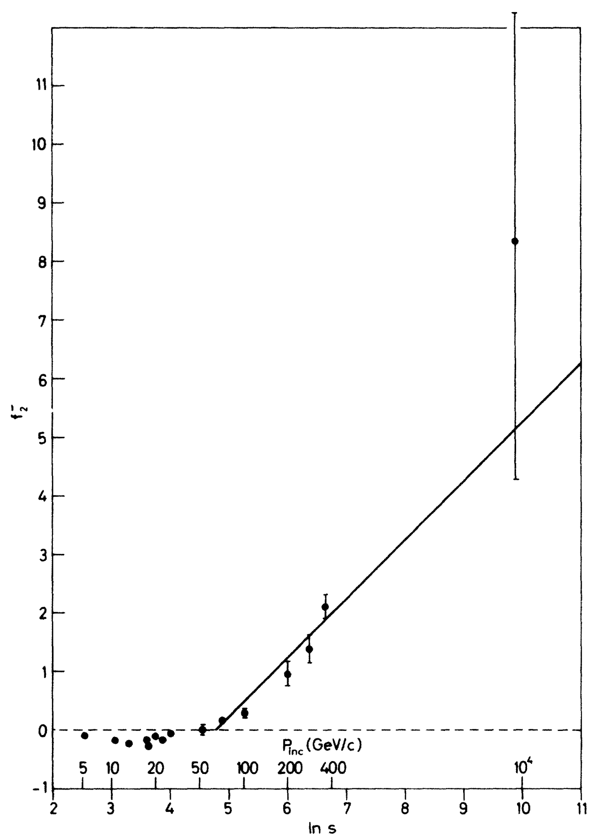

FIG. 4. The correlation function f_2^- vs ln$s$ for negatively charged particles produced in pp interactions from 5.5 to 10^4 GeV/c. See Ref. 7 for sources of data. Below about 50 GeV/c f_2^- is constrained to negative values by conservation of energy and momentum. Above about 50 GeV/c a linear dependence on ln$s$ is possible.

tudinally should be approximately 1.5 units of rapidity. This is roughly what is observed in detailed two-particle correlation studies at the ISR.[11] If a smaller value of the interaction radius in configuration space is used as suggested by the results of Goldhaber *et al.*,[3] then the predicted correlation length in rapidity becomes greater.

Koba[2] has shown that for a purely short-range-correlation model all of the correlation integrals f_m would vary as $\ln s$ at sufficiently high energies. The functions f_m [Eqs. (15)] derived from the Bose-Einstein distributions (9) all vary as $\ln s$ if $\langle n\rangle/k$ is constant, as discussed above. Thus the interpretation of Bose-Einstein fluctuations as a short-range-correlation effect is consistent with Koba's treatment.

Bose-Einstein interference is not a dynamical effect, nor is it a kinematical effect in the usual sense. It is a quantum-mechanical effect of identical-particle symmetry. It does not explain all of the observations of multiplicity distributions and correlations. At low energies (< 50 GeV/c) the multiplicity distributions and correlation functions are apparently dominated by the conservation laws for momentum, energy, and charge. Local charge conservation in rapidity may be important to explain the strong correlations observed between positively and negatively charged particles. At very high energies dynamical effects may dominate. However, the empirical evidence is such that the expected effects of identical-particle symmetry are of the same order of magnitude and have the same trends as the observations at NAL and ISR energies related to multiplicity distributions and two-particle correlations. Thus quantum-statistical fluctuations in addition to conservation laws and other symmetries must be considered if intrinsically dynamical properties are to be extracted unambiguously from the experimental observations.

Occupancy of any particular geometric configuration in phase space by the π mesons is *not* required for the occurrence of the Bose-Einstein fluctuations. All that is required is that one or more elementary cells in phase space, the size of a cell being dictated by the magnitude of Planck's constant, have occupancy numbers for identical particles which are not small compared to unity. Several distinct areas of phase space could be populated and the fluctuations of the overall distribution would still depend only on the particular cell occupation numbers as in expression (8). For example, the fluctuations would still occur if, rather than a cylindrical uniformly populated section of phase space, two or more distinct regions were populated as suggested by diffraction dissociation, central pionization, clustering, or thermodynamic multiple-fireball models. In addition to some cells having occupancy numbers not much less than one, in the present application it is important that the configuration-space portion of the elementary cell volume is very small because of the short range of the strong interaction, so that the corresponding momentum-space portion of the cell is very extended. It is this latter condition which gives rise to appreciable occupancy numbers and permits the observation of the fluctuations in high-energy multiparticle hadronic production reactions.

Expressions similar to (9) and (10) may be derived for partially degenerate systems of identical fermions in which occupancies of single states are restricted to 0–1, and fluctuations are suppressed rather than enhanced compared with the classical prediction (3). These may have significance in interactions at high energies in which two or more identical fermions are produced.

In summary it is argued that quantum-statistical interference must in principle occur for identical particles produced in high-energy interactions. The predicted magnitude of the effect depends only on the magnitude of Planck's constant, the range of the strong interaction, and the observed limitation on transverse momentum of particles produced. The shapes of the multiplicity distributions and trends with energy of integrated correlation functions are derived. These agree with experimental data above about 50 GeV/c within the accuracy of the data. At energies lower than 50 GeV/c deviations of predictions from observations occur which may reasonably be attributed to kinematical effects. Quantum-statistical interference may principally account for the short-range positive correlations observed among identical particles at high energies. Strong correlations among unlike particles cannot be explained by this effect and require additional assumptions such as local isotopic spin conservation in rapidity.

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Nucleon-nucleon scattering near 50 MeV. III. Analysis of new Davis *np* differential cross-section data*

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New Davis *np* differential cross-section data at 50 MeV are phase-shift-analyzed together with the other world *pp* and *np* data in the laboratory scattering energy range 47.5 to 60.9 MeV. Various combinations of the *np* $d\sigma/d\Omega$ data, taken by groups at Davis, Oak Ridge, and Harwell, are included in the analysis and are found to affect mainly just the phase parameter $\delta(^1P_1)$. We argue that the Harwell only, or Harwell + Oak Ridge + Davis data analyses call for a strong long-range potential such as might come from ABC (Abashian-Booth-Crowe) exchange, while the Davis only or Davis + Oak Ridge data analyses are compatible with ordinary (non-ABC) meson-theoretical models. We urge that more precise *np* absolute $d\sigma/d\Omega$ data be taken, to 1 or 2% accuracy, especially at far forward angles.

Recently the Davis group has reported on new 50-MeV neutron-proton differential cross-section data taken at both forward and backward scattering angles.[1] These new data are highly interesting in view of a recent phase-shift analysis of the world *pp* and *np* data falling in the scattering energy range of 47.5 to 60.9 MeV. This analysis, carried out by Arndt, Binstock, and Bryan[2] (henceforth referred to as paper I), included the new Davis backward *np* $d\sigma/d\Omega$ data (which were then available in preliminary form), but did not include the new forward scattering data. The phase shifts that resulted from this analysis were in good agreement with meson-theoretical models[3] at 50 MeV except for the phase parameter $\delta(^1P_1)$, which, being $-3.5° \pm 1.0°$, was five or more standard deviations above the meson-theoretically expected range of $-8.5°$ to $-11°$. (We comment below on