[8]H. M. Fried, Brown University Report No. PUB 296 (unpublished); and Nucl. Phys. (to be published).
[9]S.-J. Chang and T.-M. Yan, Phys. Rev. Lett. 25, 1586 (1970).
[10]H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 1456 (1970).
[11]T. Gaisser and C.-I Tan, BNL Report No. PUB 18070 (unpublished) [Brown University Report No. PUB 294 (unpublished)].
[12]T. Gaisser (private communication).
[13]A. Capella, M.-S. Chen, M. Kugler, and R. D. Peccei, Phys. Rev. Lett. 31, 497 (1973).

PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

Model-independent lower bounds for massless-particle scattering*

J. B. Healy
The Institute for Advanced Study, Princeton, New Jersey 08540
(Received 28 February 1974)

We examine massless pion-pion, neutrino-neutrino, and light-by-light scattering within the context of a dispersion approach to the interaction of massless particles. We assume the existence of a fixed-t dispersion relation with a finite number of subtractions for $t \leq 0$, unitarity, crossing symmetry, and the convergence of helicity expansions in the physical region. All three processes have a zero at threshold in the forward amplitude; for massless pions it is the Adler zero and for neutrinos and photons it is a kinematic zero. We demonstrate that the forward scattering amplitudes and total cross sections have the asymptotic lower bounds $|F(s, 0)| \geq$ const and $\sigma_{\text{tot}}(s) \geq \text{const}/s^2$. We obtain the same result for scattering of neutrinos on electrons. We also generalize the upper bounds obtained by Dolgov, Zakharov, and Okun for lepton-lepton scattering to massless π-π and light-by-light scattering.

I. INTRODUCTION

It is well known that the asymptotic high-energy bounds (in particular the Froissart and Jin-Martin bounds on total cross sections) which have been proved for strong-interaction scattering processes from axiomatic field theory do not in general apply to processes involving massless particles. Such processes include weak and electromagnetic interactions, as well as the theoretically interesting interactions of massless pions. The difficulty of course is that massless particles produce singularities in elastic scattering amplitudes at $t=0$, so there is no Lehmann ellipse. It has not even been possible to prove in a rigorous way the existence of dispersion relations for those processes for which exchange of a single massless particle is ruled out by crossed-channel quantum numbers. Nonetheless, it is interesting to ask what can be learned about the high-energy behavior of cross sections, when pairs of massless particles can be exchanged, within the context of a dispersion approach to these interactions. In the last couple of years some progress in this direction has been made for the leptonic weak interactions by using mild physical assumptions to limit the strength of t-channel singularities.[1] For example, Dolgov, Zakharov, and Okun[2] have argued that the exchange of any number of neutrino pairs can produce a singular contribution to the s-channel absorptive part of the (massless) lepton-lepton elastic scattering amplitude which behaves at worst like $t^2 \ln t$. By noting that such a singularity is once-differentiable and using the MacDowell-Martin unitarity bound,[3]

$$\left.\frac{d\,\mathrm{Im}F(s,t)}{dt}\right|_{t=0} \geq \text{const}\times s\sigma_{\text{tot}}{}^2(s), \tag{1.1}$$

they obtained the upper bound on the total cross section for lepton-lepton scattering:

$$\sigma_{\text{tot}}(s) < \text{const}\times s^{1/2}. \tag{1.2}$$

Several authors[3-5] have improved this bound by making stronger assumptions. Using a fixed-s dispersion relation for s-positive and assuming the two-particle intermediate state dominates the t-channel unitarity condition, Dolgov, Gribov, Okun, and Zakharov[4] were able to show that the total cross section cannot increase asymptotically as a power of s. This result was actually obtained earlier by Rajaraman,[6] who was to our knowledge the first author to study asymptotic bounds for neutrino scattering from a model-independent point of view.

In this paper we consider elastic scattering of massless particles, specifically pions, neutrinos, and photons. Using even weaker assumptions than those employed by DZO to obtain the upper bound (1.2), we demonstrate that the forward amplitudes for these processes (we specify particular helicity amplitudes for γ-γ) must satisfy the asymptotic lower bound

$$\lim_{|s|\to\infty} |F(s,0)| \geq \text{const.} \tag{1.3}$$

We then show that the total cross sections for these processes are bounded below by

$$\sigma_{\text{tot}}(s) \geq \text{const}/s^2. \tag{1.4}$$

To be precise, the bound (1.4) holds in the neutrino case for ν-ν or ν-$\bar{\nu}$ scattering, but not necessarily for both; for the photon case it applies to the scattering of photons with either the same or opposite helicities, but not necessarily both.

We remind the reader that the rigorously proved lower bounds for the strong interactions[7,8] are much weaker:

$$\begin{aligned} &|F(s,0)| \geq \text{const}/s^2,\\ &\sigma_{\text{tot}}(s) \geq \text{const}/s^6. \end{aligned} \tag{1.5}$$

The improvements in the cases of interest here result from the existence of the Adler PCAC (partial conservation of axial-vector current) zero for massless-pion scattering and kinematical zeros for neutrino-neutrino and light-by-light scattering.

Experimentally, it is very difficult to observe the scattering of massless-on-massless particles. We examine the possibility of extending our results to lepton-lepton scattering with at least one massive incoming lepton. For ν-e scattering we obtain (for certain helicity amplitudes) the same lower bounds (1.3) and (1.4) as for ν-ν scattering. All other lepton-lepton scattering amplitudes have unphysical cuts which make our method inapplicable.

We briefly mention the relationships among threshold zeros, crossed-channel singularities, and upper bounds on the asymptotic behavior of total cross sections. We argue that the singularity produced by exchange of massless pion pairs is no worse than that produced by exchange of neutrino pairs ($t^2 \ln t$ in the absorptive part), and that the singularity produced by exchange of photon pairs is even weaker ($t^3 \ln t$). Thus we can apply the methods of DZO directly to obtain the same bound (1.2) for π-π as for ν-ν. For light-by-light scattering we obtain the improved bound

$$\sigma_{\text{tot}}(s) < \text{const} \times s^{1/3}. \tag{1.6}$$

Finally we discuss the relation of our work to that of other authors. We point out that in a Regge model for π-π scattering the intercept of the leading Regge trajectory, i.e., the Pomeranchukon, must be bounded below by

$$\alpha_P(0) \geq 0 \tag{1.7}$$

in the limit of zero pion mass.

II. ASSUMPTIONS

For clarity we list all of our assumptions in this section. Let us first define a uniform notation for all of the massless particle interactions we will consider. For massless π^0-π^0 scattering there is of course only one elastic amplitude:

$$F^{\pi\pi}(s,t) \equiv F_R^{\pi}(s,t) = F_L^{\pi}(s,t). \tag{2.1}$$

For neutrino scattering there is one helicity amplitude for ν-ν and one for ν-$\bar{\nu}$:

$$\begin{aligned} &F^{\nu\nu}_{-1/2,-1/2;-1/2,-1/2}(s,t) \equiv F_R(s,t),\\ &F^{\nu\bar{\nu}}_{-1/2,1/2;-1/2,1/2}(s,t) \equiv F_L(s,t); \end{aligned} \tag{2.2}$$

these are related to each other under crossing. Finally for γ-γ we choose to consider the helicity amplitudes

$$\begin{aligned} &F^{\gamma\gamma}_{1,1;1,1}(s,t) \equiv F_R^{\gamma}(s,t),\\ &F^{\gamma\gamma}_{1,-1;1,-1}(s,t) \equiv F_L^{\gamma}(s,t); \end{aligned} \tag{2.3}$$

these too are related by crossing. Our scattering amplitudes are related to differential cross sections by

$$\frac{d\sigma}{dt} = \frac{|F(s,t)|^2}{s^2}. \tag{2.4}$$

Here and throughout this paper we ignore irrelevant constants. For all these processes we have

$$s+t+u=0. \tag{2.5}$$

We assume that for each process the scattering amplitudes $F_R(s,t)$ and $F_L(u,t)$ are the boundary values of a single analytic function $F(s,t)$ with the following properties:

(a) Analyticity. The forward amplitude $F(s,0)$ is analytic in the entire complex s plane except for the physical cuts $s \geq 0$ and $s \leq 0$. Along these cuts

$$\begin{aligned} &F_R(s,0) = F(s+i\epsilon,0), \quad s \geq 0\\ &F_L^*(u,0) = F(s+i\epsilon,0), \quad s \leq 0. \end{aligned} \tag{2.6a}$$

(b) Crossing symmetry. There exists a path of analytic continuation in the complex s-t space from the s-channel physical region to the u-channel physical region.

(c) Reality. It follows from (a) and (b) that

$$\begin{aligned} &F_L(u,0) = F(s-i\epsilon,0), \quad s \leq 0,\\ &F_R^*(s,0) = F(s-i\epsilon,0), \quad s \geq 0; \end{aligned} \tag{2.6b}$$

so $F(s,0)$ is a real analytic function:

$$F(s,0) = F^*(s^*,0). \tag{2.7}$$

(d) Polynomial boundedness.

$$|F(s,0)| < s^N, \quad |s| \to \infty. \tag{2.8}$$

(e) Forward dispersion relation. From the previous assumptions it follows that $F(s,0)$ must satisfy a dispersion relation with N subtractions:

$$F(s,0)=\sum_{i=0}^{N-1} C_i(\{s_j\})s^i+\frac{\prod_{j=1}^{N}(s-s_j)}{\pi}\int_0^\infty\frac{ds'}{\prod_{k=1}^{N}(s'-s_k)}\frac{\mathrm{Im}F_R(s',0)}{s'-s}$$
$$+(-1)^N\frac{\prod_{j=1}^{N}(s-s_j)}{\pi}\int_0^\infty\frac{ds'}{\prod_{k=1}^{N}(s'+s_k)}\frac{\mathrm{Im}F_L(s',0)}{s'+s}\,. \tag{2.9}$$

(f) Partial-wave and helicity expansions.

$$F_R(s,t)=\sum_0^\infty (2j+1)f^j_{\alpha,\alpha;\alpha,\alpha}(s,t)\,d^j_{00}\left(1+\frac{2t}{s}\right), \tag{2.10}$$

$$F_L(u,t)=\sum_0^\infty (2j+1)f^j_{\alpha,-\alpha;\alpha,-\alpha}(u,t)\,d^j_{2\alpha,2\alpha}\left(1+\frac{2t}{u}\right),$$

where

$$\alpha=\begin{cases}0 & \text{for pions,}\\ -\tfrac{1}{2} & \text{for neutrinos,}\\ 1 & \text{for photons.}\end{cases}$$

These expansions converge in the physical regions $-s\le t\le 0$ and $-u\le t\le 0$, respectively.

(g) Unitarity. The scattering amplitude satisfies the optical theorem in both s and u channels:

$$\mathrm{Im}F(s+i\epsilon,0)=s\sigma_R^{\mathrm{tot}}(s),$$
$$\mathrm{Im}F(-s-i\epsilon,0)=s\sigma_L^{\mathrm{tot}}(s). \tag{2.11}$$

The absorptive parts of the partial-wave amplitudes are positive-definite:

$$\mathrm{Im}f^j_{\alpha,\beta;\alpha,\beta}(s)>0. \tag{2.12}$$

(h) Zeros. The massless pion scattering amplitude must have a zero (the Adler zero) at the symmetric point $s=t=0$ in order that the forward amplitude be finite.[9] The neutrino and photon amplitudes have kinematic zeros at $s=0$ and they have no kinematical singularities.[10] The amplitudes

$$M^{\nu\nu}(s,t)=F^{\nu\nu}(s,t)/s,$$
$$M^{\gamma\gamma}(s,t)=F^{\gamma\gamma}(s,t)/s^2$$

are kinematically regular. For the purposes of deriving our lower bound it is sufficient to know that the forward amplitude has at least a simple zero at $s=0$.

(i) Continuity. The real and imaginary parts of $F(s,t)$ are continuous functions of t in the s- and u-channel physical regions.

Up to this point we have simply extended the properties which have been rigorously proved for the strong interactions to massless particle interactions where they are not proved. The next assumption is somewhat stronger.

(j) Fixed-t dispersion relation. There exists a real positive t_0 such that for $-t_0\le t\le 0$ the scattering amplitude satisfies a dispersion relation with the same number of subtractions as the forward amplitude:

$$F(s,t)=\sum_{i=0}^{N-1} C_i(t,\{s_j\})s^i+\frac{\prod_{j=1}^{N}(s-s_j)}{\pi}\int_0^\infty\frac{ds'}{\prod_{k=1}^{N}(s'-s_k)}\frac{\mathrm{Im}F_R(s',t)}{s'-s}$$
$$+\frac{\prod_{j=1}^{N}(u-u_j)}{\pi}\int_0^\infty\frac{ds'}{\prod_{k=1}^{N}(s'-u_k)}\frac{\mathrm{Im}F_L(s',t)}{s'-u}\,. \tag{2.13}$$

Here we must be careful because for $0\le s'<-t$ the absorptive parts in the dispersion integrals are unphysical. Unlike the strong-interaction case, there is no Lehmann ellipse, and so this interval is not within the known analyticity domain in t of $\mathrm{Im}F(s,t)$. In fact we know that for fixed s there should be a singularity starting at $t=-s$. In order for the dispersion relation to make sense it is clear that we must require that the integrals

$$\int_0^{-t} ds'\,\frac{\mathrm{Im}F_R(s',t)}{s'-s}\,,$$
$$\int_0^{-t} ds'\,\frac{\mathrm{Im}F_L(s',t)}{s'-u} \tag{2.14}$$

exist for $-t_0\le t\le 0$ and all s in the cut plane. Although it is implicit in these statements that the dynamical singularities arising from the exchange of pairs of massless particles do not destroy the threshold zero of the forward amplitude or produce infinities on the s- or u-channel cuts, we make no assumptions about the specific nature of crossed-channel singularities. Our assumptions are thus weaker than those used by DZO to obtain the upper bound (1.2), since they required in addition that the absorptive part of the amplitude for lepton-lepton scattering be differentiable at $t=0$.

III. LOWER BOUND ON $|F(s,0)|$

It is an easy matter to prove from our assumptions an asymptotic lower bound on the magnitude of the forward amplitude.

Theorem I.

$$\lim_{|s|\to\infty} |F(s,0)| \geq \text{const}, \quad \epsilon < \arg s < \pi \tag{3.1a}$$

and there exist real sequences $\{s_n\}$, $s_n \to \infty$ as $n\to\infty$, and $\{s_m\}$, $s_m \to -\infty$ as $m\to\infty$, such that

$$|F(s_n,0)| \geq \text{const},$$
$$|F(s_m,0)| \geq \text{const}. \tag{3.1b}$$

Proof. We follow essentially the standard proof of the Jin-Martin lower bound,[7,11] the improvement being provided by the existence of the zero at $s=t=0$. As is well known, a function which satisfies a dispersion relation and whose discontinuity along the cut changes sign a finite number of times can have at most a finite number of zeros in the upper half plane. The forward amplitude, having only one such change of sign, easily satisfies these conditions. We assume that the total cross sections $\sigma_R^{\text{tot}}(s)$ and $\sigma_L^{\text{tot}}(s)$ are finite and nonvanishing for $0<s<\infty$, so the only real zero of $F(s,0)$ can be at $s=0$ and it must be of finite order; by (h) of Sec. II there must be at least a simple zero. Let $F(s,0)$ have p pairs of complex zeros at points s_i and s_i^*, and assume for the moment a simple zero at $s=0$. Define a new function $H(s)$ by

$$H(s) = \frac{F(s,0)}{s\prod_{i=1}^{p}(s-s_i)(s-s_i^*)}. \tag{3.2}$$

The function $H(s)$ has the following properties:

(i) $H(s)$ is regular in $\operatorname{Im}s>0$;

(ii) $|H(s)| < \text{const}|s|^{N-2p-1}$, $|s|\geq s_0$ large;

(iii) $H(s)$ has no zeros in $\operatorname{Im}s\geq 0$;

(iv) $\operatorname{Im}H(s+i\epsilon)>0$ for all real s.

It follows[11] that $H(s)$ is a Herglotz function, and so must satisfy the bounds

$$\frac{\text{const}}{|s|} \leq |H(s)| \leq \text{const}|s|, \quad \epsilon<\arg s<\pi-\epsilon. \tag{3.3}$$

In addition there exist real sequences $\{s_n\}$, $s_n\to\infty$ as $n\to\infty$, and $\{s_m\}$, $s_m\to-\infty$ as $m\to\infty$, on which the same result holds.[12] Therefore

$$\lim_{|s|\to\infty}|F(s,0)| \geq \frac{\text{const}}{|s|}|s|^{2p+1} \geq \text{const} \tag{3.4}$$

for $\epsilon<\arg s<\pi-\epsilon$ and on the real sequences $\{s_n\}$ and $\{s_m\}$.

If $F(s,0)$ has a higher-order zero at $s=0$, then we can always define a function $H'(s)$ as

$$H'(s) = H(s)/s^q. \tag{3.5}$$

Here q is a positive even integer chosen[11] in such a way that $H'(s)$ is a Herglotz function. The precise value of q depends on the sign of the first nonvanishing derivative of $H(s)$ at $s=0$ and on the order of the zero, but in any case we obtain a lower bound at least as good as (3.4). This proves our theorem.

IV. LOWER BOUND ON TOTAL CROSS SECTIONS

We will now prove from our lower bound on the forward amplitude and the assumptions listed in Sec. II the following lower bound on total cross sections:

Theorem II. There exists a real sequence $\{s_n\}$, $s_n\to\infty$ as $n\to\infty$, such that on that sequence either $\sigma_R^{\text{tot}}(s)$ or $\sigma_L^{\text{tot}}(s)$ satisfies the lower bound

$$\sigma_{\text{tot}}(s_n) \geq \text{const}/s_n^{\,2}. \tag{4.1}$$

Proof. We assume the contrary, namely, that both $\sigma_R^{\text{tot}}(s)$ and $\sigma_L^{\text{tot}}(s)$ satisfy the upper bound

$$\sigma_{\text{tot}}(s) < \text{const}/s^2, \quad s\geq s_0 \tag{4.2}$$

with s_0 large. This will lead to a contradiction.

For clarity we will first outline the argument. Theorem I tells us that the forward amplitude must satisfy a dispersion relation with at least one subtraction. We will use the fixed-t dispersion relation (2.13) and the assumed upper bound (4.2) on the total cross section to show that for some range of t, $-t_1\leq t\leq 0$, the magnitude of the nonforward amplitude $F(s,t)$ is bounded below by a constant asymptotically. Then the total elastic cross section is bounded below by const/s^2, and by unitarity this violates (4.2).

Define the functions $G(s,t)$, $I_1(s,t)$, and $I_2(s,t)$ by

$$I_1(s,t) = \frac{1}{\pi}\int_0^\infty ds' \,\frac{\operatorname{Im}F_R(s',t)}{s'-s}, \tag{4.3a}$$

$$I_2(s,t) = \frac{1}{\pi}\int_0^\infty ds' \,\frac{\operatorname{Im}F_L(s',t)}{s'-u}, \tag{4.3b}$$

$$G(s,t) = I_1(s,t) + I_2(s,t). \tag{4.3c}$$

Both integrals converge by assumptions (j) and (4.2), and unitarity, which implies

$$|\operatorname{Im}F(s,t)| \leq \operatorname{Im}F(s,0), \quad s\geq -t. \tag{4.4}$$

Since the scattering amplitude has no singularities in the s plane other than the cuts $s\geq 0$ and $u\geq 0$, the function

$$F_p(s,t) \equiv F(s,t) - G(s,t) \tag{4.5}$$

is entire and polynomially bounded, and so must be a polynomial in s:

$$F_p(s,t) = \sum_{i=0}^{N-1} C_i(t)\,s^i. \tag{4.6a}$$

For simplicity we take $F_p(s,t)$ to be a constant in s:

$$F_p(s,t) = C_0(t). \tag{4.6b}$$

Higher powers of s would only improve our result.

Then

$$F(s,t)=C_0(t)+I_1(s,t)+I_2(s,t). \tag{4.7}$$

By unitarity and (4.2) the integrals

$$\begin{aligned}&\int_{-t}^{\infty} ds' \frac{|\mathrm{Im}F_R(s',t)|}{s'} \leq \int_{-t}^{\infty} ds' \frac{\mathrm{Im}F_R(s',0)}{s'},\\ &\int_{-t}^{\infty} ds' \frac{|\mathrm{Im}F_L(s',t)|}{s'} \leq \int_{-t}^{\infty} ds' \frac{\mathrm{Im}F_L(s',0)}{s'}\end{aligned} \tag{4.8}$$

converge, and it follows[8] that

$$\lim_{|s|\to\infty}\left|\int_{-t}^{\infty} ds' \frac{\mathrm{Im}F_R(s',t)}{s'-s}\right|=0, \quad \arg s\neq 0, \tag{4.9a}$$

$$\lim_{|s|\to\infty}\left|\int_{-t}^{\infty} ds' \frac{\mathrm{Im}F_L(s',t)}{s'-u}\right|=0, \quad \arg s\neq \pi. \tag{4.9b}$$

In addition, using assumption (j) we have

$$\begin{aligned}&\lim_{|s|\to\infty}\left|\int_0^{-t} ds' \frac{\mathrm{Im}F_R(s',t)}{s'-s}\right| = \lim_{|s|\to\infty}\frac{1}{|s|}\left|\int_0^{-t} ds' \,\mathrm{Im}F_R(s',t)\right|=0,\\ &\lim_{|s|\to\infty}\left|\int_0^{-t} ds' \frac{\mathrm{Im}F_L(s',t)}{s'-u}\right| = \lim_{|s|\to\infty}\frac{1}{|s|}\left|\int_0^{-t} ds' \,\mathrm{Im}F_L(s',t)\right|=0\end{aligned} \tag{4.10}$$

in all directions. Thus only the subtraction term survives in the asymptotic limit:

$$\lim_{|s|\to\infty}|F(s,t)|=C_0(t), \quad \arg s\neq 0,\pi. \tag{4.11}$$

From Theorem I and the fact that $F(0,0)=0$ it follows that

$$C_0(0)\neq 0. \tag{4.12}$$

To establish that $C_0(t)\neq 0$ for some finite interval of t near the forward direction, it is only necessary to show that $C_0(t)$ is a continuous function of t. The real and imaginary parts of the scattering amplitude $F(s,t)$ have been assumed to be continuous functions of t in the s- and u-channel physical regions. In order that the dispersion relation (4.7) be compatible with this requirement, $C_0(t)$ must indeed be continuous in $-t_0\leq t\leq 0$. To be precise we can use the arguments leading to (4.19) below to show that the measure of the set of points $\{s\}$, $s\geq s_0$, on which a discontinuity in $C_0(t)$ at some $t=t'$ could be canceled by a discontinuity in the dispersion integral is arbitrarily small. Therefore, there exists a real positive t_1, $t_0\geq t_1>0$ such that

$$C_0(t)\neq 0, \quad -t_1\leq t\leq 0. \tag{4.13}$$

We next show, following Cornille,[8] that along the real axis for $s\geq s_0$, s_0 large, (4.11) holds for at least half the interval $[-t_1,0]$, except possibly on a set of arbitrarily small measure. For large real $s\to\infty$ we already know that $|I_2(s,t)|\to 0$, and the only piece of $I_1(s,t)$ which may not vanish is

$$\hat{I}_1(s,t)\equiv\frac{1}{\pi}\int_{t_2}^{\infty} ds' \frac{\mathrm{Im}F_R(s',t)}{s'-s}$$

for fixed $t_2>t_1$. Let E be the set of points on which $|\mathrm{Re}\hat{I}_1(s,t)|$ can be greater than $\{|C_0(t)|-\epsilon\}$:

$$E=\{(s,t)|s\geq s_0, t\in[-t_1,0], |\mathrm{Re}\hat{I}_1(s,t)|\geq|C_0(t)|-\epsilon\}. \tag{4.14}$$

Let E_s and E_t be the sections of E for fixed s and fixed t, respectively. We can get an upper bound on the Lebesgue measure of E, $\mu(E)$, if we can find an upper bound on the Lebesgue measure of E_t for all $t\in[-t_1,0]$. To do this notice first that, for large $|s|$ outside the s-channel cut,

$$|s^{1-\epsilon}\hat{I}_1(s,t)|<\text{const}. \tag{4.15}$$

This is a consequence of the assumed upper bound (4.2) on the total cross section. We can therefore write an unsubtracted dispersion relation for the function $(-s)^{1/2}[\hat{I}_1(s,t)]^2$:

$$(-s)^{1/2}[\hat{I}_1(s,t)]^2=\frac{1}{\pi}\int_{t_2}^{\infty} ds' \frac{(s')^{1/2}\{[\mathrm{Re}\hat{I}_1(s',t)]^2-[\mathrm{Im}\hat{I}_1(s',t)]^2\}}{s'-s}. \tag{4.16}$$

Multiplying by $(-s)$ and taking $s\to-\infty$, we get

$$\lim_{|s|\to-\infty}(-s)^{3/2}[\hat{I}_1(s,t)]^2=\frac{1}{\pi}\int_{t_2}^{\infty} ds'(s')^{1/2}\{[\mathrm{Re}\hat{I}_1(s',t)]^2-[\mathrm{Im}\hat{I}_1(s',t)]^2\}=0. \tag{4.17}$$

Again using (4.2) we see that we can choose an s_0 large enough to make the integral

$$\int_{s_0}^{\infty} ds'(s')^{1/2}[\mathrm{Im}\hat{l}_1(s',t)]^2$$

arbitrarily small. Then since for any finite s_0 the integral

$$\int_{t_2}^{s_0} ds'(s')^{1/2}\{[\mathrm{Re}\hat{l}_1(s',t)]^2-[\mathrm{Im}\hat{l}_1(s',t)]^2\}$$

is finite, in order to satisfy (4.17) we must have

$$\int_{s_0}^{\infty} ds'(s')^{1/2}[\mathrm{Re}\hat{l}_1(s',t)]^2<\text{const.} \tag{4.18}$$

Therefore the Lebesgue measure of E_t is bounded above by

$$\mu(E_t)<\frac{\text{const}}{(s_0)^{1/2}[C_0(t)-\epsilon]^2}\,. \tag{4.19}$$

Since the denominator in the right-hand side of (4.19) is integrable for $t\in[-t_1,0]$, we obtain the upper bound on the Lebesgue measure of the set E:

$$\mu(E)=\int_{-t_1}^{0} dt\,\mu(E_t)<\frac{\text{const}}{(s_0)^{1/2}}\,. \tag{4.20}$$

Now let α be the set of values of $s, s\geq s_0$, for which $\mu(E_s)$ is greater than $t_1/2$, say. We have

$$\mu(E)>\int_\alpha ds\,\mu(E_s)>\mu(\alpha)\tfrac{1}{2}t_1. \tag{4.21}$$

Combining (4.20) and (4.21) we arrive at

$$\mu(\alpha)<\frac{\text{const}}{(s_0)^{1/2}t_1}\,. \tag{4.22a}$$

The same reasoning can be applied for $s\to-\infty$. If β is the set of values of s on the negative real axis, defined analogously with the set α, then

$$\mu(\beta)<\frac{\text{const}}{(s_0)^{1/2}t_1}\,. \tag{4.22b}$$

Clearly $\mu(\alpha)$ and $\mu(\beta)$ can be made arbitrarily small by choosing s_0 large enough.

Finally we can use the unitarity condition on the complements of the sets α and β to obtain for both $\sigma_R^{\text{tot}}(s)$ and $\sigma_L^{\text{tot}}(s)$

$$\sigma_{\text{tot}}(s)\geq\sigma_{\text{el}}(s)\geq\int_{-t_1}^{0} dt\,\frac{|F(s,t)|^2}{s^2}\geq\frac{\text{const}}{s^2} \tag{4.23}$$

for $s\geq s_0$, except on a set of very small measure. This contradicts our assumption (4.2), so the theorem is proved.

V. MASSIVE LEPTONS

While it is theoretically interesting to have a lower bound on the neutrino-neutrino total cross section, from an experimental point of view it would be more desirable to have a bound on a lepton-lepton cross section with at least one massive incoming lepton. Unfortunately, most lepton-lepton elastic scattering amplitudes have unphysical thresholds in the s- or u-channel, so the discontinuity of the forward amplitude across the cut does not satisfy the positivity properties we need to get a lower bound on the forward amplitude. This rules out e-e, μ-μ, e-μ, and ν-μ scattering.

For ν-e we can obtain essentially the same results as for ν-ν by choosing the right helicity amplitudes. We take

$$F_R(s,t)\equiv\frac{1}{\cos^2(\frac{1}{2}\theta_s)}F^{\nu e}_{-1/2,1/2;-1/2,1/2}(s,t), \tag{5.1a}$$

$$F_L(u,t)\equiv\frac{1}{\cos^2(\frac{1}{2}\theta_s)}\sum_{\alpha=-1/2}^{1/2}\sum_{\beta=-1/2}^{1/2} d^{1/2}_{-1/2,\alpha}(\chi)F^{\bar{\nu}e}_{1/2,\alpha;1/2,\beta}(u,t)\times d^{1/2}_{\beta,-1/2}(\chi)\,. \tag{5.1b}$$

Here

$$\begin{aligned}\cos\theta_s&=1+\frac{2st}{(s-m_e^2)^2}\,,\\ s+t+u&=2m_e^2\,,\end{aligned} \tag{5.1c}$$

and the crossing angle χ is given by[10]

$$\begin{aligned}\cos\chi&=\frac{(s+m_e^2)(u+m_e^2)-4m_e^4}{(s-m_e^2)(u-m_e^2)}\\ &=1-\frac{2m_e^2t}{(s-m_e^2)(u-m_e^2)}\,,\\ \sin\chi&=\frac{-2m_e[t(su-m_e^4)]^{1/2}}{(s-m_e^2)(u-m_e^2)}\,.\end{aligned} \tag{5.2}$$

We have defined $F_R(s,t)$ and $F_L(u,t)$ in such a way that they are the analytic continuations of each other under crossing and can be taken to be the boundary values of a single analytic function $F(s,t)$, just as for the ν-ν case. The amplitude $F(s,t)$ has no kinematical singularities (the half-angle factor has been removed) and has a kinematic zero at $s=m_e^2$.[10]

In the forward direction $\cos\theta_s=1$ and $\cos\chi=1$, so $\chi=0$ and we obtain

$$\begin{aligned}\lim_{\epsilon\to0}F(s+i\epsilon,0)&=F^{\nu e}_{-1/2,1/2;-1/2,1/2}(s,0)\,,\\ \lim_{\epsilon\to0}F(s-i\epsilon,0)&=F^{\bar{\nu}e}_{1/2,1/2;1/2,1/2}(u,0)\end{aligned} \tag{5.3}$$

on the right- and left-hand cuts, respectively. We can thus carry through the same arguments used in Sec. II to show that

$$\lim_{|s|\to\infty}|F(s,0)|\geq\text{const} \tag{5.4}$$

in complex directions and on real sequences $\{s_n\}$, $s_n \to \infty$ as $n \to \infty$, and $\{s_m\}$, $s_m \to -\infty$ as $m \to \infty$.

To get the lower bound on the total cross section is just a little more difficult than before. The only problem is that for $t \neq 0$ the amplitude on the left-hand cut is a linear combination of four helicity amplitudes and we can no longer use the unitarity condition in the simple form (4.4). We must now replace the assumed inequality (4.2) with the three inequalities

$$s\sigma^{\nu e}_{-1/2,1/2}(s) = \mathrm{Im}\, F^{\nu e}_{-1/2,1/2;\,-1/2,1/2}(s,0) < \frac{\text{const}}{s}\,, \tag{5.5a}$$

$$s\sigma^{\bar\nu e}_{1/2,1/2}(s) = \mathrm{Im}\, F^{\bar\nu e}_{1/2,1/2;\,1/2,1/2}(s,0) < \frac{\text{const}}{s}\,, \tag{5.5b}$$

$$s\sigma^{\bar\nu e}_{1/2,-1/2}(s) = \mathrm{Im}\, F^{\bar\nu e}_{1/2,-1/2;\,1/2,-1/2}(s,0) < \frac{\text{const}}{s}\,. \tag{5.5c}$$

By noting that for large s the rotation matrices approach

$$\begin{aligned} d^{1/2}_{1/2,1/2}(\chi) &= \cos\tfrac{1}{2}\chi \to 1, \\ d^{1/2}_{-1/2,1/2}(\chi) &= \sin\tfrac{1}{2}\chi \to \frac{m_e\sqrt{-t}}{s}\,, \end{aligned} \tag{5.6}$$

we see that if on the left-hand cut $|\mathrm{Im}\, F(s,t)| \geq \text{const}/|s|$, then one of the following possibilities must obtain:

$$|\mathrm{Im}\, F^{\bar\nu e}_{1/2,1/2;\,1/2,1/2}(s,t)| \geq \text{const}/|s|\,, \tag{5.7a}$$

$$|\mathrm{Im}\, F^{\bar\nu e}_{1/2,-1/2;\,1/2,-1/2}(s,t)| \geq \text{const}|s|\,, \tag{5.7b}$$

$$|\mathrm{Im}\, F^{\bar\nu e}_{1/2,1/2;\,1/2,-1/2}(s,t)| \geq \text{const}\,, \tag{5.7c}$$

$$|\mathrm{Im}\, F^{\bar\nu e}_{1/2,-1/2;\,1/2,1/2}(s,t)| \geq \text{const}. \tag{5.7d}$$

By unitarity the inequalities (5.7a) and (5.7b) clearly violate (5.5b) and (5.5c), respectively. In addition, if (5.7c) or (5.7d) were to hold on a set of non-negligible measure, then (5.5b) or (5.5c) would again be violated because

$$\begin{aligned} \sigma^{\bar\nu e}_{1/2,\pm 1/2}(s) &\geq \int_{-t_0}^{0} dt\, \frac{|F^{\bar\nu e}_{1/2,\pm 1/2;\,1/2,\mp 1/2}(s,t)|^2}{s^2} \\ &> \int_{-t_0}^{0} dt\, \frac{|\mathrm{Im}\, F^{\bar\nu e}_{1/2,\pm 1/2;\,1/2,\mp 1/2}(s,t)|^2}{s^2}\,. \end{aligned} \tag{5.8}$$

Therefore for large enough real s, $|s| \geq s_0$, and $-t_0 \leq t \leq 0$ the assumptions (5.5) imply

$$|\mathrm{Im}\, F(s,t)| < \text{const}/|s|\,. \tag{5.9}$$

The rest of the proof proceeds just as in Sec. III, and we find that at least one of the three total cross sections

$$\sigma^{\nu e}_{-1/2,1/2}(s),\ \sigma^{\bar\nu e}_{1/2,1/2}(s),\ \text{or}\ \sigma^{\bar\nu e}_{1/2,-1/2}(s) \tag{5.10}$$

must obey the lower bound

$$\sigma_{\text{tot}}(s) \geq \text{const}/s^2\,. \tag{5.11}$$

By exactly the same reasoning we can establish that at least one of the following total cross sections must also obey the lower bound (5.11):

$$\sigma^{\bar\nu e}_{1/2,-1/2}(s),\ \sigma^{\nu e}_{-1/2,-1/2}(s),\ \text{or}\ \sigma^{\nu e}_{-1/2,1/2}(s). \tag{5.12}$$

These results apply equally well for electron or muon neutrinos and for electrons or positrons.

VI. THRESHOLD ZEROS, CROSSED-CHANNEL SINGULARITIES, AND UPPER BOUNDS ON TOTAL CROSS SECTIONS

By studying diagrams for lepton-lepton scattering, DZO have argued that the exchange of neutrino pairs can produce a singular contribution to the s-channel absorptive part which behaves at worst like $t^2 \ln t$. Their argument uses the kinematic vanishing of the (massless) lepton-lepton scattering amplitude at $s = t = 0$; given the existence of the Adler zero, it can easily be seen to apply to massless pion scattering as well. For light-by-light scattering there is a double zero in all helicity amplitudes at $s = t = 0$; the discussion of DZO leads us to conclude that the exchange of photon pairs can produce a t-channel singularity which behaves at worst like $t^3 \ln t$. It is in fact well known that the exchange of photon pairs produces a weaker singularity than the exchange of neutrino pairs. In coordinate space the exchange of photon pairs corresponds to the $1/r^7$ Van der Waals potential, whereas the exchange of neutrino pairs corresponds to a $1/r^5$ potential.[13,6]

To get an upper bound on the lepton-lepton total cross section DZO noted that a singularity of the form $t^2 \ln t$ is once-differentiable at $t = 0$. Assuming that the absorptive part of the full amplitude is no more singular than the contributions of individual diagrams and that for $t \leq 0$ the amplitude satisfies a dispersion relation with a finite number of subtractions, then the first derivative of the dispersion integral must exist. This can be consistent with the MacDowell-Martin unitarity bound,[3]

$$\left.\frac{d^n\, \mathrm{Im}\, F(s,t)}{dt^n}\right|_{t=0} \geq \text{const} \times s\sigma_{\text{tot}}{}^{n+1}(s)\,, \tag{6.1}$$

only if

$$\sigma^{\nu\nu}_{\text{tot}}(s) < \text{const} \times s^{1/2}\,. \tag{6.2}$$

We see that the same argument applies equally well to massless π-π scattering, and so the upper bound (6.2) should hold in that case as well.

For light-by-light scattering we expect the absorptive part to be no more singular than $t^3 \ln t$, and hence twice differentiable at $t=0$. By the same reasoning as before we see that the dispersion integral should now be twice differentiable, which is compatible with the MacDowell-Martin bound (6.1) only if

$$\sigma_{tot}^{\gamma\gamma}(s) < \text{const} \times s^{1/3} . \tag{6.3}$$

It is reasonable to expect that the bounds (6.2) and (6.3) should be able to be improved. In a sense the singularities $t^2 \ln t$ and $t^3 \ln t$ are almost twice differentiable and three times differentiable, respectively, and so a good guess would be that (6.2) and (6.3) could be improved to at least

$$\sigma_{tot}^{\nu\nu}(s) \leq \text{const} \times s^{1/3} , \tag{6.4a}$$

$$\sigma_{tot}^{\gamma\gamma} \leq \text{const} \times s^{1/4} . \tag{6.4b}$$

Indeed, DZO were able to obtain (6.4a) by a slightly less rigorous argument than that leading to (6.2). Choosing a simple exponential form for the t dependence of the absorptive part in the diffraction peak region, they calculated the singularity of the amplitude produced by the shrinkage of the diffraction peak which, by unitarity, must accompany a power-law growth of the total cross section. Requiring that this singularity be no worse than that produced by the exchange of neutrino pairs, they obtained the bound (6.4a) on the growth of the total cross section. We remark that the same argument applied to light-by-light scattering gives the result (6.4b).

It was also pointed out by DZO that in order for the bound (6.2) to be consistent with unitarity the number of subtractions in the fixed-t dispersion relation for lepton-lepton scattering with $t \leq 0$ must be less than or equal to 2. We now see that the light-by-light and massless π-π elastic amplitudes must also obey twice-subtracted dispersion relations for $t \leq 0$.

VII. DISCUSSION

We discuss here the relation between our lower bounds and those of other authors. Auerbach, Pennington, and Rosenzweig [14] have already noted that the Adler zero is inconsistent with an unsubtracted forward dispersion relation for massless π-π scattering. However, their conclusion that $\text{Im}\, F(s,0) \geq \text{const}$ is incorrect, at least without further assumptions.[15] If the high-energy behavior of pion-pion scattering is dominated by the exchange of Regge trajectories, then in the limit $m_\pi^2 \to 0$ the intercept of the leading trajectory (presumably the Pomeranchukon) must satisfy the lower bound:

$$\alpha_P(0) \geq 0 . \tag{7.1}$$

With the usual Regge signature factor we obtain the lower bound on the total cross section for scattering of massless pions:

$$\sigma_{tot}(s) \geq \text{const}/s . \tag{7.2}$$

This bound has in fact been obtained as a strict inequality for lepton-lepton scattering by Anselm and Gribov,[5] who examined the high-energy behavior of leptonic weak interactions using the complex angular momentum theory. Within that framework they demonstrated that unitarity requires the existence of a j-plane singularity located to the right of $j=0$ for $t=0$ and the positivity of the contribution of this singularity to the absorptive parts of the s-channel partial-wave amplitudes.

It would be remarkable if the bound (7.2) could be obtained from more general considerations. In a theory with only massless particles one could argue on dimensional grounds that the asymptotic behavior of total cross sections must be essentially $\sigma_{tot}(s) \sim \text{const}/s$, since there are no masses around to scale the energy. So the lower bound (7.2) would be saturated.

Let us summarize the results we have obtained without appealing to any model. For massless pion-pion, neutrino-neutrino, neutrino-electron, and light-by-light scattering the forward dispersion relation (for the appropriate helicity amplitudes) requires at least one subtraction. The total cross sections for these processes are bounded below by

$$\sigma_{tot}(s) \geq \text{const}/s^2 . \tag{7.3}$$

This is already much stronger than the Jin-Martin result for the strong interactions.

The upper bound of DZO, namely

$$\sigma_{tot}(s) < \text{const} \times s^{1/2} , \tag{7.4}$$

applies to massless π-π as well as to lepton-lepton scattering. Using the same method we have found for light-by-light scattering the improved bound

$$\sigma_{tot}(s) < \text{const} \times s^{1/3} . \tag{7.5}$$

Of course (7.4) and (7.5) are not as strong as the Froissart bound.

We hope experimental information on some of these cross sections will be available in the not too distant future.

ACKNOWLEDGMENTS

I am grateful to R. Rajaraman for arousing my interest in the problem of bounds for massless particle interactions and to H. Cornille and N. N. Khuri for several enlightening discussions. It is also a pleasure to acknowledge valuable conversations with S. Adler, R. Dashen, D. Freedman, and A. Neveu. Finally I wish to thank Dr. C. Kaysen for his hospitality at the Institute for Advanced Study.

*Research sponsored by the U. S. Atomic Energy Commission under Grant No. AT(11-1)-2220.
[1]V. I. Zakharov, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972*, edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), summarizes the recent developments in this field.
[2]A. D. Dolgov, V. I. Zakharov, and L. B. Okun', Yad. Fiz. 15, 808 (1971) [Sov. J. Nucl. Phys. 15, 451 (1972)]; Phys. Lett. 37B, 298 (1971). These papers will be referred to as DZO.
[3]S. W. MacDowell and A. Martin, Phys. Rev. 135, B960 (1964).
[4]A. D. Dolgov, V. N. Gribov, L. B. Okun, and V. I. Zakharov, Nucl. Phys. B59, 611 (1973).
[5]A. A. Anselm and V. N. Gribov, Nucl. Phys. B61, 253 (1973).
[6]R. Rajaraman, Phys. Rev. 178, 2211 (1969).
[7]Y. S. Jin and A. Martin, Phys. Rev. 135, B1369 (1964).
[8]H. Cornille, Nuovo Cimento 4A, 549 (1971).
[9]R. Dashen and D. J. Gross, Phys. Rev. D 8, 559 (1973).
[10]J. P. Ader, M. Capdeville, and H. Navelet, Nuovo Cimento 56A, 315 (1968). This paper contains a thorough discussion of the zeros, singularities, and crossing relations for helicity amplitudes involving massless particles.
[11]See, for example, N. N. Khuri, in *Theories of Strong Interactions at High Energies*, edited by R. Peierls (BNL, Upton, New York, 1969).
[12]The argument is due to A. Martin and is given in a footnote of a paper by T. Kinoshita, Phys. Rev. 154, 1438 (1967).
[13]G. Feinberg and J. Sucher, Phys. Rev. 139, B1619 (1965); Phys. Rev. 166, 1638 (1968).
[14]S. P. Auerbach, M. R. Pennington, and C. Rosenzweig, LBL report, 1973 (unpublished).
[15]We have just received from M. R. Pennington an erratum which corrects this error.

PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

Effect of internal symmetry on hadronic binding forces*

Richard H. Capps
Physics Department, Purdue University, West Lafayette, Indiana 47907
(Received 29 April 1974)

The possibility of molecular-type binding in a many-multiplet quark model is considered. This is defined as the binding of three or more constituents (quarks and antiquarks) that results from a spatial configuration in which attracting pairs of particles are relatively close together. It is shown that the strength of such forces is a decreasing function of the size of the regular representation of the assumed internal-symmetry group. It is suggested that the actual hadronic internal symmetry may be the simplest symmetry that does not lead to many bound states of many hadrons.

I. INTRODUCTION

We consider the question of why SU(3) is the internal-symmetry group of hadronic interactions. The hypothesis that nature prefers simplicity can be used as an argument against more complicated groups, but points up the fact that SU(3) is not the simplest conceivable symmetry. One hopes that a consistency criterion will be found that rules out other internal symmetries, especially those simpler than SU(3). Most discussions of the possibility of such a criterion occur in reference to bootstrap models, but the question is equally valid if hadrons are composed of quarks.

In this paper we propose a criterion that provides a distinction between internal symmetries, and might rule out symmetries simpler than SU(3). Our hypothesis is that the interaction constants involving basic particles must not be so large as to lead to large numbers of many-particle states bound by strong interactions.

The motivation for this hypothesis comes from comparing various known forces between particles. We consider first the simple symmetry of the electrostatic interactions, mediated by one vector field, the photon field. The electrostatic interaction leads to many weakly bound systems, i.e., systems in which the volume is approximately proportional to the number of constituents. Examples are the hydrogen diatomic molecule and a chunk of iron, although the strong interactions within the iron nuclei play an essential role in the latter case.

Within the realm of hadronic interactions, atomic nuclei are weakly bound, many-particle states. However, nuclear forces are unusual in that their effective strength depends on the fact that the pion is much lighter than all other mesons. Thus, strong nuclear forces are the result of SU(3)-symmetry breaking. Our hypothesis is that nature must forbid a stronger mechanism for hadron binding, a mechanism that does not depend on symmetry breaking. Lacking a theory that includes strong interactions, we do not know that