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Relativistic transition amplitudes in a class of O(4,2) infinite multiplets. II. Discrete-continuum transitions

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A general relativistic formula is derived in closed form for the photoeffect amplitude from a composite system of spin μ described by O(4, 2) infinite-multiplet wave functions. For $\mu = 0$, it reduces to the relativistic H-atom photoeffect amplitude.

In the first part[1] of our investigations of the relativistic transition amplitudes in a class of infinite multiplets, characterized by spin μ, we derived generalizations of the Coulomb scattering amplitude (the case $\mu = 0$ gives the usual relativistic Coulomb amplitude). In this second part we study the analog of the relativistic photoeffect. In contrast to the continuum-continuum transitions of paper I, we have to take into account, in the discrete-continuum transitions, the momentum transfer to the system by external interactions. This makes the problem a bit more complex.

The starting forms of the S matrix and the T matrix are the same as in the $\mu = 0$ case.[2] The S matrix for our calculation is

$$S_{fi} = -ie \int d^4x [J_\mu(x) A^\mu(x)] \ . \tag{1}$$

For the external field we take as usual

$$A_\mu(x) = (2\pi/qV)\epsilon_\mu e^{(-iqx)} \ . \tag{2}$$

Inserting (2) and the form of the initial and final states into (1), we perform the x integration, define the T matrix by

$$S_{fi} \equiv i(2\pi)^4 \delta^4(P_f - P_i - q) T_{fi} \ , \tag{3}$$

and obtain for the T-matrix elements

$$T_{fi} = e\left(\frac{M_f}{P_0^f V}\, \frac{M_i}{P_0^i V}\, \frac{2\pi}{qV}\right)^{1/2} \epsilon^\mu \langle \Psi^-_{\vec{k}_f, \vec{p}_f} | J_\mu | \Psi_{N_i, \vec{p}_i} \rangle \ . \tag{4}$$

Here $\Psi_{N_i, \vec{p}_i}$ is the initial discrete bound state and $\Psi^-_{\vec{k}_f, \vec{p}_f}$ the outgoing scattering state. They are given in terms of the group states of the μ representation of the dynamical group SO(4, 2) by[1]

$$|\Psi_{N_i, \vec{p}}\rangle = \left(\frac{\alpha m_1 m_2}{M_N(1+\alpha^2)}\right)^{1/2} e^{(i\zeta_{\vec{p}} L_{35})} e^{(i\theta_i L_{45})} |N_i\rangle \ , \tag{5}$$

where M_N is the total mass of the bound state N, and

$$|\Psi^-_{\vec{k}, \vec{p}_f}\rangle = \left(\frac{(2\pi)^3}{V}\right)^{1/2} \left(\frac{M_\lambda}{2\pi\alpha m_1 m_2} \left|\frac{\partial\lambda}{\partial k}\right|\right)^{1/2} \frac{e^{(-\pi\lambda/2)}}{2(\pi)^{1/2}} \times \Gamma(1+\mu+i\lambda) \frac{\Gamma(-i\lambda-\mu)}{\Gamma(-i\lambda+\mu)}\, e^{(i\varphi L_3)} \times e^{(i\theta L_2)} e^{(i\theta_\lambda L_{45})} e^{(-\pi L_{45}/2)} |\Psi^{-G}_{-i\lambda}\rangle \ . \tag{6}$$

The kinematics in these expressions is such that we are in the center of mass of the initial particles, i.e., $\vec{P}_i = -\vec{q}$, and in a coordinate system in which the photon comes in along the z axis with its polarization $\hat{\epsilon}$ along the x axis, and the final free-particle momentum $\vec{k}_f$ points in the direction (θ, φ). Then Eq. (4) becomes

$$T_{fi} = \frac{e}{m_1}\left(\frac{M_\lambda}{P_0^i q}\right)^{1/2} \frac{2\pi}{V^2} \left(\frac{1}{2(1+\mu^2+\alpha^2)} \left|\frac{\partial\lambda}{\partial k}\right|\right)^{1/2} \times e^{(-\pi\lambda/2)} \Gamma(1+\mu-i\lambda) \frac{\Gamma(i\lambda-\mu)}{\Gamma(i\lambda+\mu)} M_{fi} \ , \tag{7}$$

where we have introduced

$$M_{fi} = \langle \Psi^{-G}_{(-i\lambda)} | e^{-i\theta L_2} e^{-i\varphi L_3} e^{-i\theta_f L_{45}} \times e^{i\zeta_{-\vec{q}} L_{35}} e^{i\theta_i L_{45}} \Gamma^1 |N_i\rangle$$

$$\theta_f \equiv \theta_\lambda - \tfrac{1}{2} i\pi \ . \tag{8}$$

We take the current operator J_μ to be

$$J_\mu = \alpha_1 \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu \Gamma_4 \ , \tag{9}$$

i.e., the standard O(4, 2) current.[12] For our process in the kinematics chosen, only Γ^1 contributes.

We use again the parabolic quantum numbers to label the group states, and shall take the initial state from now on to be the ground state:

$$|\Psi^-_{-i\lambda}\rangle = [|(N_\lambda - 1 - |\mu|), 0, -|\mu|; |\mu|\rangle \pm |(N_\lambda - 1 - |\mu|), 0, |\mu|; -|\mu|\rangle] , \quad N_\lambda = -i\lambda \quad (10)$$

and

$$|\Psi_{N_i}\rangle = [|0, 0, \nu; |\mu|\rangle \pm |0, 0, \nu; -|\mu|\rangle], \quad N_i = 1 + |\mu| . \quad (11)$$

We restrict the spin-projection quantum number ν of the ground state by $|\nu - 1| \leqslant |\mu|$. Actually the limiting case $\nu = \pm|\mu|$ is required. In the oscillator representation of SO(4, 2) we have explicitly ($\mu \geqslant 0$)

$$|0,0,\nu;|\mu|\rangle \equiv |0, 0, \nu\rangle = [(\mu+\nu)!(\mu-\nu)!]^{1/2} a_1^{\dagger(\mu+\nu)} a_2^{\dagger(\mu-\nu)} |0\rangle .$$

The matrix element M_{fi} can be split into two parts, one part for $\mu \geqslant 0$, and the other for $\mu \leqslant 0$, and these two parts do not interfere, because the group generators do not connect the two representations. Hence

$$M_{fi} = M_{fi}^{\mu=|\mu|} + M^{\mu=-|\mu|} . \quad (12)$$

It will therefore be sufficient to give the calculation for $\mu = |\mu|$. The case $\mu = -|\mu|$ is similar. From now on, therefore, μ means $|\mu|$.

We first evaluate the action of Γ^1. Because

$$\Gamma^1 = \tfrac{1}{2} i (a_1^\dagger b_1^\dagger - a_2^\dagger b_2^\dagger + a_1 b_1 - a_2 b_2) , \quad (13)$$

we find

$$\Gamma^1 |0, 0, \nu\rangle = \tfrac{1}{2} i [(\mu+\nu+1)^{1/2} |1, 0, \nu+1\rangle - (\mu-\nu+1)^{1/2} |0, 1, \nu-1\rangle] . \quad (14)$$

The matrix element M_{fi} can be split into a rotation and a Lorentz-transformation part. In fact, because L_{35} and L_{45} commute with J_3 we can write

$$M_{fi} = \tfrac{1}{2} i \sum_{n_1' n_2'} [\langle\psi^-|e^{(-i\theta L_2)} e^{(-i\varphi L_3)} |n_1' n_2', \nu+1\rangle (\mu+\nu+1)^{1/2} \langle n_1' n_2', \nu+1| e^{(-i\theta_f L_{45})} e^{(i\zeta_{-q} L_{35})} e^{(i\theta_i L_{45})} |1, 0, \nu+1\rangle$$
$$- \langle\psi^-|e^{(-i\theta L_2)} e^{(-i\varphi L_3)} |n_1' n_2', \nu-1\rangle (\mu-\nu+1)^{1/2} \langle n_1' n_2', \nu-1| e^{(-i\theta_f L_{45})} e^{(i\zeta_{-q} L_{35})} e^{(i\theta_i L_{45})} |0, 1, \nu-1\rangle] . \quad (15)$$

We shall treat only the term with $(\nu+1)$; the $(\nu-1)$ term is similar and we shall state the result for the $(\nu-1)$ case. Thus, we wish to evaluate

$$A \equiv \sum_{n_1' n_2'} \tfrac{1}{2} i R_{n_1' n_2'}(\theta, \varphi) T_{n_1' n_2'}(\theta_f, \zeta, \theta_i) , \quad (16)$$

where the rotation matrix $R_{n_1' n_2'}(\theta, \varphi)$ and the transition matrix $T_{n_1' n_2'}$ are given by

$$R_{n_1' n_2'} \equiv \langle\Psi^-|e^{(-i\theta L_2)} e^{(-i\varphi L_3)} |n_1' n_2', \nu+1\rangle ,$$
$$T_{n_1' n_2'} \equiv (\mu+\nu+1)^{1/2} \times \langle n_1' n_2', \nu+1| e^{(-i\theta_f L_{45})} e^{(i\zeta L_{35})} \times e^{(-i\theta_i L_{45})} |1, 0, \nu+1\rangle . \quad (17)$$

The transition matrix $T_{n_1' n_2'}$ can now be evaluated by algebraic techniques using the direct product O(2, 1) × O(2, 1) as shown in the Appendix. The result is

$$T_{n_1' n_2'} = (\mu+\nu+1)^{1/2} D^{(\mu-\nu)/2}_{n_1'+(\mu-\nu)/2,\, 1+(\mu-\nu)/2} \times D^{(\mu+\nu)/2+1}_{n_2'+(\mu+\nu)/2+1,\, (\mu+\nu)/2+1} , \quad (18)$$

where the hyperbolic rotation matrices are given by

$$D^k_{n+k,k}(W) = \left(\frac{(n+2k-1)!}{(2k-1)!n!}\right)^{1/2} (\bar\alpha)^{-(n+2k)} \beta^n ,$$
$$D^k_{n+k,k+1}(W) = (n)^{1/2} \left(\frac{(n+2k-1)!}{(2k)!(n-1)!}\right)^{1/2} \times (\bar\alpha)^{-(n+2k+1)} \beta^{n-1} - (2k)^{1/2} \left(\frac{(n+2k-1)!}{(2k-1)!n!}\right)^{1/2} \times (\bar\alpha)^{-(n+2k+1)} (\beta\bar\beta) \beta^{n-1} . \quad (19)$$

The arguments W_1 and W_2 are identical to those given in paper I. We have, therefore,

$$T_{n_1' n_2'} = [(\mu+\nu+1) n_1']^{1/2} \left(\frac{(n_1'+\mu-\nu-1)!(n_2'+\mu+\nu+1)!}{(n_1'-1)!n_2'!(\mu+\nu+1)!(\mu-\nu)!}\right)^{1/2} (\bar\alpha_1)^{-(n_1'+\mu-\nu+1)} \beta_1^{n_1'-1} (\bar\alpha_2)^{-(n_2'+\mu+\nu+2)} \beta_2^{n_2'}$$
$$- [(\mu-\nu)(\mu+\nu+1)]^{1/2} \left(\frac{(n_1'+\mu-\nu-1)!(n_2'+\mu+\nu+1)!}{(\mu-\nu-1)!(\mu+\nu+1)!n_1'!n_2'!}\right)^{1/2} (\bar\alpha_1)^{-(n_1'+\mu-\nu+1)} \beta_1^{n_1'-1} (\beta_1\bar\beta_1) (\bar\alpha_2)^{-(n_2'+\mu+\nu+2)} \beta_2^{n_2'} . \quad (20)$$

We next evaluate the rotation-matrix elements $R_{n_1' n_2'}$, again algebraically, by using the decomposition, this time O(3) × O(3). Namely, we write the states as

$$\begin{aligned} &|n_1' n_2', \nu+1\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle , \\ &j_1 = \tfrac{1}{2}(n_1' + n_2') + \mu = j + \mu , \\ &j_2 = \tfrac{1}{2}(n_1' + n_2') = j , \\ &m_1 = \tfrac{1}{2}(n_2' - n_1') + \nu = m + \nu , \\ &m_2 = \tfrac{1}{2}(n_1' - n_2') = -m , \end{aligned} \tag{21}$$

and

$$|(N-1-\mu), 0, -\mu\rangle = |j_1', m_1'\rangle \otimes |j_2', m_2'\rangle ,$$

where

$$\begin{aligned} &j_1' = \tfrac{1}{2}(N-1-\mu) + \mu = j + \mu , \\ &j_2' = \tfrac{1}{2}(N-1-\mu) , \\ &m_1' = -j - \mu , \\ &m_2' = j . \end{aligned} \tag{22}$$

We solve for n_1' and n_2' in terms of j and m, and use the usual d functions of the rotation group, and obtain

$$\begin{aligned} A = \tfrac{1}{2} i \sum_m \exp[-i\varphi(\nu+1)](\bar\alpha_1\bar\alpha_2)^{-\mu} \left(\frac{\bar\alpha_1}{\bar\alpha_2}\right)^{\nu} \left(\frac{\beta_1\beta_2}{\bar\alpha_1\bar\alpha_2}\right)^{j} \frac{1}{\bar\alpha_1\bar\alpha_2} \left(\frac{\bar\alpha_1\beta_2}{\bar\alpha_2\beta_1}\right)^{m} d^{j+\mu}_{-j-\mu, m+\nu+1}(-\theta) d^{j}_{j,m}(-\theta) \\ \times \Bigg\{ \frac{1}{\bar\alpha_2\beta_1}[(j-m)(j+m+1)]^{1/2} \left[\frac{(j+\mu+m+\nu+1)!(j+\mu-m-\nu-1)!}{(j+m+1)!(j-m-1)!(\mu+\nu)!(\mu-\nu)!}\right]^{1/2} \\ - \frac{\beta_1\bar\beta_1}{\bar\alpha_2\beta_1}[(\mu-\nu)(\mu+\nu+1)]^{1/2} \left[\frac{(j+\mu+m+\nu+1)!(j+\mu-m-\nu-1)!}{(j+m)!(j-m)!(\mu+\nu+1)!(\mu-\nu-1)!}\right]^{1/2} \Bigg\} . \end{aligned} \tag{23}$$

We shall now apply the general coupling of the d functions[3]:

$$d^{j_1}_{m_1' m_1}(\theta) d^{j_2}_{m_2' m_2}(\theta) = \sum_{j=j_1-j_2}^{j_1+j_2} \langle j_1 m_1', j_2 m_2' | j, m_1'+m_2'\rangle d^{j}_{m_1'+m_2', m_1+m_2}(\theta) \langle m_1'+m_2', j | j_1 m_1, j_2 m_2\rangle . \tag{24}$$

In our case $j_1 = -m_1'$ and $j_2 = -m_2'$, and the sum over j reduces to the one term $j = j_1 + j_2$. Using explicit values of the Clebsch-Gordan coefficients we get

$$d^{j_1+j_2}_{-(j_1+j_2), m_1+m_2}(\theta) \left[\frac{(j_1+j_2+m_1+m_2)!(j_1+j_2-(m_1+m_2))!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}\right]^{1/2} = \left(\frac{[2(j_1+j_2)]!}{(2j_1)!(2j_2)!}\right)^{1/2} d^{j_1}_{-j_1, m_1}(\theta) d^{j_2}_{-j_2, m_2}(\theta) . \tag{25}$$

In the expression (23) for A we have precisely the Clebsch-Gordan coefficients (25). If in the first term we identify

$$j_1 = j , \quad j_2 = \mu , \qquad m_1 = m+1, \quad m_2 = \nu ,$$

and in the second term

$$j_1 = j , \quad j_2 = \mu , \qquad m_1 = m , \quad m_2 = \nu + 1 ,$$

the quantity A becomes

$$\begin{aligned} A = \tfrac{1}{2} i \left(\frac{[2(j+\mu)]}{(2j)!(2\mu)!}\right)^{1/2} \left(\frac{\beta_1\beta_2}{\bar\alpha_1\bar\alpha_2}\right)^{j} (\bar\alpha_1\bar\alpha_2)^{-\mu-1} \left(\frac{\bar\alpha_1}{\bar\alpha_2}\right)^{\lambda} e^{(-i\varphi\lambda)} \\ \times \sum_m \Bigg\{ \left(\frac{\bar\alpha_1\beta_2}{\bar\alpha_2\beta_1}\right)^{m} \Bigg[\frac{e^{-i\varphi}}{(\bar\alpha_2\beta_1)} [(j-m)(j+m+1)]^{1/2} d^{j}_{-j, m+1}(-\theta) d^{j}_{j,m}(-\theta) d^{\mu}_{-\mu,\nu}(-\theta) \\ - \frac{\beta_1\bar\beta_1}{(\bar\alpha_2\beta_1)} e^{-i\varphi}[(\mu-\nu)(\mu+\nu+1)]^{1/2} d^{\mu}_{-\mu, \nu+1}(-\theta) d^{j}_{-j,m}(-\theta) d^{j}_{j,-m}(-\theta) \Bigg] \Bigg\} . \end{aligned} \tag{26}$$

Summation. The summation over m in (26) can actually be carried out. The key lies in recognizing that the factor $(\bar\alpha_1\beta_2/\bar\alpha_2\beta_1)^m$ is an m-dependent phase and that the index m is the eigenvalue of J_3. We have from the Appendix

$$\alpha_1 = \bar\alpha_2, \quad \beta_1 = -\bar\beta_1 , \tag{27}$$

so that

$$\frac{\bar\alpha_1\beta_2}{\bar\alpha_2\beta_1} = -\frac{(\alpha_1\beta_1)^*}{\alpha_1\beta_1} = \exp[-2i(\eta - \tfrac{1}{2}\pi)] , \tag{28}$$

where we have defined

$$\alpha_1\beta_1 \equiv |\alpha_1\beta_1|e^{i\eta} = |\alpha_1\beta_1|(\cos\eta + i\sin\eta) \equiv R_+ + iI_+ \,. \qquad (29)$$

Thus the m-dependent part in the first term of the sum, for example, can be written as

$$\sum_m e^{-2i(\eta-\pi/2)m}[(j-m)(j+m+1)]^{1/2} \times d^j_{-j,m+1}(-\theta)d^j_{j,m}(-\theta) = \sum_m e^{-2i(\eta-\pi/2)m}\langle -j,j|e^{-i\theta J_2}J_+|jm\rangle \times \langle jj|e^{-i\theta J_2}|j,-m\rangle \,. \qquad (30)$$

The phase can be placed into the rotation-matrix elements. Furthermore we make use of the identity

$$d^j_{m'm}(\theta) = (-1)^{m'm} d^j_{-m,-m'}(-\theta)\,, \qquad (31)$$

and the sum becomes

$$\sum_m e^{i\pi j}e^{-2i\eta m}\langle -jj|e^{-i\theta J_2}J_+|jm\rangle\langle mj|e^{i\theta J_2}|j,-j\rangle = e^{i\pi j}e^{2i\eta}\sum_m \langle -jj|e^{-i\theta J_2}e^{-2i\eta J_3}J_+|jm\rangle \times\langle mj|e^{i\theta J_2}|j,-j\rangle = e^{i\pi j}e^{2i\eta}\langle -jj|e^{-i\theta J_2}e^{-2i\eta J_3}J_+e^{i\theta J_2}|j,-j\rangle\,, \qquad (32)$$

where we have replaced a sum over a complete set by unity. The rotation-matrix elements can then be evaluated easily. We find

$$e^{i\pi j}e^{2i\eta}e^{-i\eta}j\sin\theta(\cos\eta + i\sin\eta\cos\theta)^{2j-1} = e^{i\pi j}e^{i\eta}j\sin\theta\left(\frac{R_+ + iI_+\cos\theta}{|\alpha_1\beta_1|}\right)^{2j-1}\,. \qquad (33)$$

The second sum A can be done in the same way. The complete expression for $M^{\mu=|\mu|}_{fi}$ is then

$$M^{|\mu|}_{fi} = \tfrac{1}{2}i\left(\frac{[2(j+\mu)]!}{(2j)!(2\mu)!}\right)^{1/2} e^{-i\nu\varphi}\left(\frac{\bar\alpha_1}{\bar\alpha_2}\right)^{\nu}\left(\frac{R_+ + iI_+\cos\theta}{\alpha_1\bar\alpha_1}\right)^{2j+1+\mu} + \left\{\frac{2j\sin\theta\cos\varphi}{(R_+ + iI_+\cos\theta)^{2+\mu}}d^{\mu}_{-\mu,\nu}(-\theta) - \left[\left(\frac{\beta_1\bar\beta_1}{\bar\alpha_2\beta_1}\right)\frac{e^{-i\varphi}[(\mu-\nu)(\mu+\nu+1)]^{1/2}}{(R_+ + iI_+\cos\theta)^{1+\mu}}d^{\mu}_{-\mu,\nu+1}(-\theta) - \left(\frac{\beta_2\bar\beta_2}{\bar\alpha_1\beta_2}\right)\frac{e^{i\varphi}[(\mu+\nu)(\mu-\nu+1)]^{1/2}}{(R_+ + iI_+\cos\theta)^{1+\mu}}d^{\mu}_{-\mu,\nu-1}(-\theta)\right]\right\}. \qquad (34)$$

The expression for $M^{-|\mu|}_{fi}$ is obtained from this expression by changing the indices 1 and 2 and by replacing the spin projection $-\mu$ of the d^j function by $+\mu$:

$$d^{\mu}_{-\mu,\nu'}(-\theta) \to d^{\mu}_{\mu,\nu'}(-\theta)\,.$$

The final expression for the matrix elements is

$$M^{\pm\mu}_{fi} = \tfrac{1}{2}i\left(\frac{\Gamma(i\lambda+\mu)}{\Gamma(i\lambda-\mu)(2\mu)!}\right)^{1/2} e^{\pi\lambda}\left(\frac{4i\alpha m_1 k}{[(1+\mu)^2+\alpha^2]^{1/2}}\right)^{2+\mu} e^{-i\varphi\nu}\left(\frac{E_1\pm k}{m_1}\right)^{\nu} e^{\mp i\gamma\nu}e^{\lambda\tau}\left(\frac{m_1^2 - t}{|c|}\right)^{i\lambda}$$
$$\times\left[\frac{(i\lambda-1-\mu)\sin\theta\cos\varphi}{(m_1^2-t)^{2+\mu}}d^{\mu}_{\mp\mu,\nu}(-\theta) \mp \tfrac{1}{8}i\frac{[(1+\mu)^2+\alpha^2]^{1/2}}{\alpha m_1^2 kq}\frac{|c|e^{-i\tau}}{(m_1^2-t)^{1+\mu}}\left(\frac{m_1}{E_1\mp k}e^{-i\varphi}[(\mu-\nu)(\mu+\nu+1)]^{1/2}d^{\mu}_{\mp\mu,\nu+1}(-\theta) + \frac{m_1}{E_1\pm k}e^{i\varphi}[(\mu+\nu)(\mu-\nu+1)]^{1/2}d^{\mu}_{\mp\mu,\nu-1}(-\theta)\right)\right]. \qquad (35)$$

In the Appendix we have also listed the values of the α's and the β's for both the relativistic and the nonrelativistic cases. In the limit as $\mu\to 0$ we recover the previous values.

Nonrelativistic limit. The nonrelativistic limit of the matrix elements is given by

$$M^{\pm\mu}_{fi} = \tfrac{1}{2}i\left(\frac{\Gamma(i\lambda+\mu)}{\Gamma(i\lambda-\mu)(2\mu)!}\right)^{1/2} e^{\pi\lambda}\left(\frac{4ik}{(1+\mu)a_0}\right)^{2+\mu} e^{-i\nu\varphi}(|a|e^{-i\gamma})^{\pm\nu}e^{\tau\lambda}\left(\frac{c+b}{|c|}\right)^{i\lambda}$$
$$\times\left\{\frac{(i\lambda-1-\mu)\sin\theta\cos\varphi}{(c+b)^{2+\mu}}d^{\mu}_{\mp\mu,\nu}(-\theta) \mp\tfrac{1}{4}i\frac{(1+\mu)a_0}{k}\frac{|c|e^{-i\tau}}{(c+b)^{1+\mu}}\left[\frac{e^{-i\varphi}[(\mu-\nu)(\mu+\nu+1)]^{1/2}}{(qm/m_1\mp k)^2+1/(1+\mu)^2a_0^2}d^{\mu}_{\mp\mu,\nu+1}(-\theta) + \frac{e^{i\varphi}[(\mu+\nu)(\mu-\nu+1)]^{1/2}}{(qm/m_1\pm k)^2+1/(1+\mu)^2a_0^2}d^{\mu}_{\mp\mu,\nu-1}(-\theta)\right]\right\}. \qquad (36)$$

The essential angle dependence is the factor

$$(m_1{}^2 - t) \xrightarrow{\text{NR}} (c+b) = \left(q\,\frac{m}{m_1}\right)^2 + k^2 + \frac{1}{a_0{}^2(1+\mu)^2} - 2\left(q\,\frac{m}{m_1}\right)^2 k\cos\theta .$$

The matrix element thus has two parts, one part coming from the $\vec{p}\cdot\hat{\epsilon}$ term as in the spinless case

$$\frac{\sin\theta\cos\varphi}{(m_1{}^2 - t)^{2+\mu}} \sim \frac{\hat{\epsilon}\cdot\vec{k}}{(c+b)^{2+\mu}} ,$$

and a second term characteristic of the algebraic current in a spin-μ theory.

APPENDIX

Consider the transition-matrix elements

$$T_{n_1' n_2'} = (\mu+\nu+1)^{1/2} \times \langle n_1' n_2', \nu+1 | e^{-i\theta_f L_{45}} e^{i\zeta_{-q} L_{35}} e^{i\theta_i L_{45}} | 1, 0, \nu+1\rangle .$$

The O(2, 1) generators for raising and lowering n_1 and n_2 are

$$L_{45} = -N_1^{(2)} + N_2^{(2)} ,\quad L_{35} = -(N_1^{(1)} + N_2^{(1)}) ,\quad L_{34} = N_1^{(3)} - N_2^{(3)} , \tag{A2}$$

and the quantum numbers of the two O(2, 1) groups are given by

$$N_1 = n_1 + \tfrac{1}{2}(|\mu-\nu|+1) ,\quad N_2 = n_2 + \tfrac{1}{2}(|\mu+\nu|+1) ,\quad k_1 = \tfrac{1}{2}(|\mu-\nu|+1) ,\quad k_2 = \tfrac{1}{2}(|\mu+\nu|+1) . \tag{A3}$$

Thus

$$T_{n_1' n_2'} = (\mu+\nu+1)^{1/2} \langle n_1' + \tfrac{1}{2}(\mu-\nu), \tfrac{1}{2}(\mu-\nu) | \exp[i\zeta_{-q}(-N_1^{(1)}\cosh\theta_f + N_1^{(3)}\sinh\theta_f)]\, e^{i\theta_- N_1^{(1)}} | \tfrac{1}{2}(\mu-\nu), 1+\tfrac{1}{2}(\mu-\nu)\rangle$$
$$\times \langle n_2' + \tfrac{1}{2}(\mu+\nu)+1, \tfrac{1}{2}(\mu+\nu+1) | \exp[i\zeta_{-q}(-N_2^{(1)}\cosh\theta_f - N_2^{(3)}\sin\theta_f)]\, e^{-i\theta_- N_2^{(2)}} | \tfrac{1}{2}(\mu+\nu)+1, \tfrac{1}{2}(\mu+\nu)+1\rangle . \tag{A4}$$

Introducing the hyperbolic rotation-matrix elements $D^k_{N'N}(W)$ we obtain the expression (18) of the text. The arguments W_1 and W_2 are found by using the 2-dimensional representation of the operators in (A2):

$$l^1 = \tfrac{1}{2}i\sigma_2 ,\quad l^2 = -\tfrac{1}{2}i\sigma_1 ,\quad l^3 = \tfrac{1}{2}\sigma_3 , \tag{A5}$$

so that

$$W_1 = \exp[i\zeta_{-q}(-\tfrac{1}{2}i\sigma_2\cosh\theta_f - \tfrac{1}{2}i\sigma_1\sinh\theta_f)] \times e^{+i\theta_-\sigma_3/2} \equiv \begin{pmatrix} \alpha_1 & \beta_1 \\ \bar{\beta}_1 & \bar{\alpha}_1 \end{pmatrix} , \tag{A6}$$

$$W_2 = \exp[i\zeta_{-q}(-\tfrac{1}{2}i\sigma_2\cosh\theta_f + \tfrac{1}{2}i\sigma_1\sinh\theta_f)]\, e^{-i\theta_-\sigma_3/2} \equiv \begin{pmatrix} \alpha_2 & \beta_2 \\ \bar{\beta}_2 & \bar{\alpha}_2 \end{pmatrix},$$

and

$$\alpha_1 = \cosh(\tfrac{1}{2}\zeta)\cosh(\tfrac{1}{2}\theta_-) + i\sinh(\tfrac{1}{2}\zeta)\sinh(\tfrac{1}{2}\theta_+) ,$$
$$\beta_1 = \cosh(\tfrac{1}{2}\zeta)\sinh(\tfrac{1}{2}\theta_-) - i\sinh(\tfrac{1}{2}\zeta)\cosh(\tfrac{1}{2}\theta_+) ,$$
$$\theta_+ = \theta_f + \theta_i ,\quad \theta_- = \theta_f - \theta_i \tag{A7}$$

and

$$\alpha_2 = \bar{\alpha}_1 ,\quad \beta_2 = -\bar{\beta}_1 . \tag{A8}$$

We give now a list of the most important relations:

$$\alpha_1\beta_1 \equiv R_+ + iI_+ ,\quad \alpha_1\beta_2 \equiv R_- - iI_- ,$$
$$R_+ = -\frac{1}{2\alpha}\, i\,\frac{q}{k}\,\frac{E_1}{m_1}[(1+\mu)^2 + \alpha^2]^{1/2} ,\quad I_+ = \tfrac{1}{2}\,\frac{q}{\alpha m_1}[(1+\mu)^2 + \alpha^2]^{1/2} ,$$
$$R_- = \frac{1}{2\alpha}\, i\,\frac{q}{k}\,\frac{1}{m_1}\,\frac{m_i{}^2 + m_1{}^2 - m_2{}^2}{2m_i}[(1+\mu)^2 + \alpha^2]^{1/2} ,\quad I_- = \tfrac{1}{2}i\frac{q}{k}\,\frac{m_2}{m_1} ; \tag{A9}$$

$$\alpha_1\alpha_2 = \tfrac{1}{4}i\,\frac{[(1+\mu)^2+\alpha^2]^{1/2}}{\alpha m_1 k}\left[\frac{2m_i{}^2(M_f{}^3 + M_i{}^2) - (M_i{}^2 + m_2{}^2 - m_1{}^2)(M_f{}^2 + m_2{}^2 - m_1{}^2)}{2m_2 M_f} - \frac{2i\alpha m_1 k}{[(1+\mu)^2+\alpha^2]^{1/2}}\right]$$
$$\equiv \tfrac{1}{4}i\,\frac{[(1+\mu)^2+\alpha^2]^{1/2}}{\alpha m_1 k}\,|c|e^{i\tau} , \tag{A10}$$

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{2M_i(E_1+k)} \left[(M_i{}^2 + m_1{}^2 - m_2{}^2) + i\frac{2\alpha m_1 m_2}{[(1+\mu)^2+\alpha^2]^{1/2}} \right] \equiv \frac{|\alpha|}{(E_1+k)}\, e^{i\gamma} = \frac{m_1}{E_1+k}\, e^{i\gamma} \,. \tag{A11}$$

In the nonrelativistic limit we have

$$R_+ = -\frac{1}{4k}\, i(1+\mu)a_0 \left[k^2 + \frac{1}{a_0{}^2(1+\mu)^2} + \left(q\frac{M}{m_1} \right)^2 \right], \quad I_+ = \frac{1}{4k}\, i(1+\mu)a_0 \left(2iq\frac{M}{m_1}\, k \right),$$

$$R_- = \frac{1}{4k}\, i(1+\mu)a_0 \left[\left(q\frac{M}{m_1} \right) - k^2 - \frac{1}{(1+\mu)^2 a_0{}^2} \right], \quad I_- = \frac{1}{4k}\, i(1+\mu)a_0 \left(\frac{2qM/m_1}{a_0(1+\mu)} \right); \tag{A12}$$

$$\alpha_1\alpha_2 = \frac{1}{4k}\, i(1+\mu)a_0 \left[\left(q\frac{M}{m_1} \right)^2 - k^2 + \frac{1}{(1+\mu)^2 a_0{}^2} - \frac{2ik}{(1+\mu)a_0} \right] \equiv \frac{1}{4k}\, i(1+\mu)a_0 |c| e^{i\tau}, \tag{A13}$$

$$\frac{\alpha_1}{\alpha_2} = \frac{(qM/m_1)^2 - 1/a_0{}^2(1+\mu)^2 - k^2 - 2iqM/m_1(1+\mu)a_0}{(qM/m_1)^2 + 1/a_0{}^2(1+\mu)^2 + k^2 - 2q(M/m_1)k} \equiv |a| e^{i\gamma}; \tag{A14}$$

$$\frac{R_+ + iI_+\cos\theta}{\alpha_1\bar{\alpha}_1} = e^{\pm i\pi}\left[\frac{(qM/m_1)^2 + k^2 + 1/a_0{}^2(1+\mu)^2 - 2q(M/m_1)k\cos\theta}{(qM/m_1)^2 - k^2 + 1/a_0{}^2(1+\mu)^2 - 2ik/(1+\mu)^2 a_0{}^2} \right],$$

$$\beta_1\bar{\beta}_1 = \frac{i(1+\mu)a_0}{4k}\left[\left(q\frac{M}{m_1} \right)^2 - \left(k - \frac{i}{(1+\mu)a_0} \right)^2 \right], \quad \alpha_1\beta_1 = -i\frac{(1+\mu)a_0}{4k}\left[\left(k - q\frac{M}{m_1} \right)^2 + \frac{1}{a_0{}^2(1+\mu)^2} \right],$$

$$\alpha_2\beta_2 = \frac{i(1+\mu)a_0}{k}\left[\left(k + q\frac{M}{m_1} \right)^2 + \frac{1}{a_0{}^2(1+\mu)^2} \right], \quad \frac{1}{a_0} = \alpha\frac{m_1 m_2}{m_1+m_2} \,.$$

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