Relativistic transition amplitudes in a class of O(4,2) infinite multiplets. I. Continuum-continuum transitions

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A class of unitary irreducible O(4, 2) representations characterized by the lowest spin μ, $\mu=0$, $\pm\frac{1}{2}$, $\pm1,\ldots$, and the corresponding infinite-component wave equations are considered. The continuum-continuum transition amplitudes are evaluated by the method of analytic continuation generalizing the relativistic Coulomb case ($\mu=0$). A general exact Regge-type amplitude is derived, which contains the relativistic scattering amplitudes of O(4)-symmetric Coulomb interactions of two dyons in the symmetric model and amplitudes in certain hadron models as special cases.

I. INTRODUCTION

The purpose of this work is to derive group-theoretically a class of explicit relativistic scattering amplitudes for a variety of systems. The method is the following. The states of the system in the center-of-mass frame are described by a representation of the dynamical group containing the Lorentz group and labeled by certain quantum numbers. The general states are constructed by means of Lorentz boosts. The calculation of bound-state properties within this framework has been extensively studied We are now interested in the scattering processes. From a configuration-space representation of the Lie algebra we deduce analytically continued values of the quantum numbers (from their bound-state values) for scattering states. The S-matrix elements can then be evaluated as the matrix elements of a rotation (from the initial to final three-momentum) taken between the scattering states, i.e., as matrix elements of a group element.

We carry out this program for a class of unitary irreducible representations of the group SO(4, 2). The results generalize the Coulomb scattering amplitude in two directions: first, in the direction of allowing a tower of states with spin beginning with a lowest spin μ; second, in the direction of relativistic kinematics. The general scattering formula we derive is in terms of μ and the "principal quantum number" n whose relation to the energy variable s (or momentum q) depends on the kinematics and dynamics of the interaction. Inelastic amplitudes can also be treated by this method. Some of the physical problems covered by this theory are relativistic and nonrelativistic scattering of dyons in the symmetric model, O(4)-symmetric Klein-Gordon or Dirac type Hamiltonians with spin, and relativistic infinite-component wave equations that have been used in hadron models.

II. DYNAMICAL GROUP AND PHYSICAL SCATTERING STATES

A. The group states

The group SO(4, 2) has a well-known class of most degenerate irreducible unitary representations characterized by a quantum number μ, $\mu=0$, $\pm\frac{1}{2}$, ±1, $\pm\frac{3}{2},\ldots$, in which the states are labeled completely by three quantum numbers only. (In the general case, apart from the three Casimir operators, we need six labels altogether.) These can be chosen to be the eigenvalues of the elements of Cartan subalgebra[1]: L_{12}, L_{34}, and L_{56}. Hence the states can be denoted by $|n_1n_2m\rangle$,

$$\begin{aligned}
L_{12}|n_1n_2m\rangle &= m|n_1n_2m\rangle\,,\\
L_{34}|n_1n_2m\rangle &= [n_1-n_2+\tfrac{1}{2}(|m-\mu|-|m+\mu|)]\\
&\quad\times|n_1n_2m\rangle\,,\\
L_{56}|n_1n_2m\rangle &= [n_1+n_2+1+\tfrac{1}{2}(|m-\mu|+|m+\mu|)]\\
&\quad\times|n_1n_2m\rangle\\
&\equiv n|n_1n_2m\rangle\,.
\end{aligned}\tag{2.1}$$

The case where $\mu=0$ corresponds to the so-called parabolic group states of the H-atom problem.

We make use of two explicit realizations of the states $|n_1n_2m\rangle$. One is in terms of the parabolic coordinates:

$$\begin{aligned}
\psi^G_{n_1n_2m} &= N^G_{n_1n_2m}e^{im\varphi}e^{-\xi/2}\xi^{|m-\mu|/2}\\
&\quad\times{}_1F_1(-n_1,|m-\mu|+1;\xi)e^{-\eta/2}\\
&\quad\times\eta^{|m+\mu|/2}{}_1F_1(-n_2,|m+\mu|+1;\eta)\,,
\end{aligned}\tag{2.2}$$

where

$$N^G_{n_1n_2m}=\frac{A}{\sqrt{\pi}}\left[\frac{\Gamma(|m-\mu|+n_1+1)\Gamma(|m+\mu|+n_2+1)}{\Gamma(n_1+1)\Gamma(n_2+1)}\right]^{1/2}\times\frac{1}{\Gamma(|m-\mu|+1)\Gamma(|m+\mu|+1)},$$
$$A=(-1)^{n_1+(m+\mu+|m+\mu|)/2}. \quad (2.3)$$

In this realization the generators L_{ab} of SO(4, 2) are the following differential operators[2]:

$$J_i=\tfrac{1}{2}\epsilon_{ijk}L_{jk}=\epsilon_{ijk}r_j\pi_k-\mu\hat{r}_i,$$
$$A_i=L_{i4}=\tfrac{1}{2}r_i\pi^2-\pi_i(\vec{r}\cdot\vec{\pi})+\frac{\mu}{r}J_i+\frac{\mu^2}{2r^2}r_i+\tfrac{1}{2}r_i,$$
$$M_i=L_{i5}=\tfrac{1}{2}r_i\pi^2-\pi_i(\vec{r}\cdot\vec{\pi})+\frac{\mu}{r}J_i+\frac{\mu^2}{2r^2}r_i-\tfrac{1}{2}r_i,$$
$$\Gamma_i=L_{i6}=r\pi_i, \quad (2.4)$$
$$\Gamma_0=L_{56}=\tfrac{1}{2}\left(r\pi^2+r+\frac{\mu^2}{r}\right),$$
$$\Gamma_4=L_{46}=\tfrac{1}{2}\left(r\pi^2-r+\frac{\mu^2}{r}\right),$$
$$T=L_{45}=\vec{r}\cdot\vec{\pi}-i,$$

with

$$[\pi_i,\pi_j]=i\mu\xi_{ijk}r_k/r^3. \quad (2.5)$$

These operators are self-adjoint with respect to the scalar product

$$(\psi^G_{n_1n_2m}\psi^G_{n_1'n_2'm'})\equiv\int\psi^{G*}_{n_1n_2m}\psi^G_{n_1'n_2'm'}\frac{1}{r}\,dV=\delta_{n_1n_1'}\delta_{n_2n_2'}\delta_{mm'}. \quad (2.6)$$

The parabolic coordinates in (2.2) and (2.6) are, as usual, related to the Cartesian coordinates by

$$\xi=r+z,\quad \eta=r-z,\quad \tan\varphi=y/x. \quad (2.7)$$

The second realization is in terms of two pairs of creation and annihilation operators $a_i, b_i, a_i^\dagger, b_i^\dagger$, $i=1, 2$, $[a_i,a_j^\dagger]=\delta_{ij}$, $[b_i,b_j^\dagger]=\delta_{ij}$:

$$|n_1n_2m\rangle=C_{n_1n_2m}a_1^{\dagger n_2+(|m+\mu|+m+\mu)/2}a_2^{\dagger n_1+(|m-\mu|-m+\mu)/2}\times b_1^{\dagger n_1+(|m-\mu|+m-\mu)/2}b_2^{\dagger n_2+(|m+\mu|-m-\mu)/2}|0\rangle, \quad (2.8)$$

where

$$C_{n_1n_2m}{}^{-2}=[n_2+\tfrac{1}{2}(|m+\mu|+m+\mu)]!\times[n_1+\tfrac{1}{2}(|m-\mu|-m+\mu)]!\times[n_1+\tfrac{1}{2}(|m-\mu|+m-\mu)]!\times[n_2+\tfrac{1}{2}(|m+\mu|-m-\mu)]!. \quad (2.9)$$

In this second realization the generators are given by bilinear combinations of the creation and annihilation operators, and are given in Ref. 1, for example.

B. Schrödinger states

We shall now compare the states (2.2) with the following Schrödinger states in parabolic coordinates:

$$\psi^S_{n_1n_2m}=N_{n_1n_2m}e^{im\varphi}\left(\frac{\xi}{n}\right)^{|m-\mu|/2}e^{-(\xi/n)/2}\times{}_1F_1(-n_1,|m-\mu|+1;\xi/n)\times(\eta/n)^{|m+\mu|/2}e^{-(\eta/n)/2}\times{}_1F_1(-n_2,|m+\mu|+1;\eta/n),$$
$$N_{n_1m_2m}=\frac{1}{n^2}N^G_{n_1n_2m}, \quad (2.10)$$

with $N_{n_1n_2m}$ given in Eq. (2.3). The wave functions (2.10) are generalizations of the Coulomb wave functions ($\mu=0$) and can be realized as the eigenfunctions of the Hamiltonian

$$H=\frac{\pi^2}{2m}-\frac{\alpha}{r}\,\frac{\mu^2}{2mr^2}, \quad (2.11)$$

where

$$\vec{\pi}=\vec{p}-\mu\vec{D}(\vec{r}),\quad \vec{D}(\vec{r})=\frac{\vec{r}\times\hat{n}\,\vec{r}\cdot\hat{n}}{r(r^2-(\vec{r}\cdot\hat{n})^2)}, \quad (2.12)$$

representing the system of two dyons with electric and magnetic charges $q_1=(e_1,g_1)$ and $q_2=(e_2,g_2)$ with an extra scalar potential $\mu^2/2mr^2$, where $\alpha=e_1e_2+g_1g_2$ and $\mu=e_1g_2-e_2g_1$ (Refs. 2, 3) in suitable units. The difference between the group states (2.2) and the Schrödinger states (2.11) is essentially an n-dependent dilatation of the coordinates ξ and η by $1/n$ and a different scalar product. The explicit connection between these two states is now well understood and is given by the dilatation operator of SO(4, 2),

$$\psi^S_{n_1n_2m}=\frac{1}{n}e^{i\theta L_{45}}\psi^G_{n_1n_2m},\quad \theta=-\ln n. \quad (2.13)$$

In addition, matrix elements of an arbitrary operator A between physical states can be written, in terms of the group states, as

$$\int dV\,\psi^{S*}_{n_1n_2m}A\psi^S_{n_1n_2m}=\frac{1}{n^2}\,{}^t\langle n_1n_2m|(\Gamma_0-\Gamma_4)A|n_1'n_2'm'\rangle^t\equiv\frac{1}{n^2}\langle n_1n_2m|e^{-i\theta L_{45}}(\Gamma_0-\Gamma_4)\times Ae^{i\theta L_{45}}|n_1'n_2'm'\rangle, \quad (2.14)$$

where we used the fact that

$$r=\Gamma_0-\Gamma_4$$

from (2.4) and the scalar product (2.6). The states $|n_1n_2m\rangle^t$ are usually referred to in the literature as the "tilted states," i.e., tilted by $e^{i\theta L_{45}}$.

III. CALCULATION OF CONTINUUM-CONTINUUM TRANSITION AMPLITUDES

A. Scattering states

We shall define the "scattering states" for our representations by analytic continuation in the quantum numbers n_1, n_2, and m and in angle θ. For this purpose we go to the realization on the space of Schrödinger wave functions (2.11). We take for the "in" and "out" states $m=+\mu$ and $-\mu$, respectively. This is because the components of $\vec{J}=\vec{r}\times\vec{\pi}-\mu\hat{r}$ in the direction of $-\hat{r}$ and $\hat{r}$ are respectively $+\mu$ and $-\mu$. Thus, the scattering states are states of fixed helicity $\pm\mu$, determined by the representation of the group. With these values of m we expand ψ^S in (2.10) into the asymptotic region and impose on it the condition of "in" and "out" states with proper normalization:

$$\psi^{\pm}\sim\left(e^{ikz}+f^{\pm}(\theta)\frac{e^{\pm ikr}}{r}\right)e^{\pm i\mu\phi}.$$

This procedure gives the following values of the quantum numbers:

"in" states:

$$n_1^{(+)}=-1,\quad n_2^{(+)}=n-|\mu|,\quad m^{(+)}=\mu;$$

"out" states:

$$n_1^{(-)}=n-1-|\mu|,\quad n_2^{(-)}=0,\quad m^{(-)}=-\mu. \qquad (3.1)$$

We shall take the values in (3.1) to define "scattering states" moving asymptotically parallel to the z axis in general. We assume from now on that μ is positive; the calculations are the same for μ negative.

Our aim is to develop a scattering theory in terms of the quantum numbers directly, sidestepping the use of relative coordinates and their asymptotic forms. It is then natural to take the same universal group-theoretical definition of scattering states given in (3.1) in all cases; it is the simplest definition which in the nonrelativistic case coincides with the usual definition of scattering states. The reason for trying to construct a scattering matrix directly in terms of the quantum numbers is that a coordinate-space representation of the wave functions may not always exist, as in some relativistic cases.

The continuum-group states are eigenstates of L_{46} with continuous eigenvalues λ. In contrast, the bound-group states were eigenstates of L_{56}, with discrete eigenvalues n. We can pass from bound to the continuum states by a hyperbolic rotation with the tilt operator L_{45} followed by an analytic continuation of the quantum number n to $-i\lambda$. The "in" and "out" group scattering states are then

$$\psi^{G\pm}_{\lambda(\theta,\varphi)}=R_3(\varphi)R_2(\theta)C^{\pm}_{\lambda}e^{-\pi L_{45}/2}\psi^{G}_{n_1n_2m}, \qquad (3.2)$$

where n_1, n_2, and m are to be continued to the values given in (3.1). The standard helicity rotations $R_3(\varphi)R_2(\theta)$ bring the direction of the asymptotic plane wave from a direction parallel to the z axis to the direction defined by (θ,φ). The normalization coefficients $C^{\pm}_{\lambda}$ are

$$C^{+}_{\lambda}=\frac{(-1)^{\mu}}{(8\pi^2)^{1/2}}e^{\pi\lambda/2}\Gamma(1+\mu-i\lambda)\left[\frac{\Gamma(-i\lambda-\mu+1)}{\Gamma(-i\lambda+\mu+1)}\right]^{1/2},$$

$$C^{-}_{\lambda}=\frac{1}{(8\pi^2)^{1/2}}e^{-\pi\lambda/2}\Gamma(1+\mu+i\lambda)\left[\frac{\Gamma(-i\lambda-\mu)}{\Gamma(-i\lambda+\mu)}\right]^{1/2}.$$

The improper tilt $e^{-\pi L_{45}/2}$ is to be understood as the limit of $e^{-i\phi L_{45}}$ as $\phi\rightarrow -i\frac{1}{2}\pi$. The states so defined have the normalization

$$\langle\psi^{G\pm}_{\lambda'(\theta,\varphi)}|\psi^{G\pm}_{\lambda(0,0)}\rangle=\delta(\lambda-\lambda')\delta(1-\cos\theta)\delta(\varphi). \qquad (3.3)$$

The physical states are related to these group states by a tilt, boost, and normalization (as was given for the Schrödinger states), depending on the particular application and model to be considered.

B. The scattering-matrix elements

The scattering-matrix elements are given by

$$S_{fi}=\langle\psi^{(-)}_{\vec{k}_f}|\psi^{(+)}_{\vec{k}_i}\rangle, \qquad (3.4)$$

where $\vec{k}_f$ and $\vec{k}_i$ denote the direction of propagation of the "out" and "in" states, respectively. Because we have an exact specification of the states occurring in (3.4), we can actually evaluate the amplitudes as an overlap of two wave functions. We shall use the method of analytic continuation.[4] It consists of first evaluating (3.4) for the bound states and then continuing the result analytically in the quantum numbers to the values given in (3.1). (3.1).

In the center-of-momentum frame the S-matrix elements S_{fi} are given by

$$S_{fi}=Ge^{-i\mu\Phi}\langle n_1'n_2'm'|J_0R^{\dagger}(\vec{k}_f,\vec{k}_i)|n_1n_2m\rangle. \qquad (3.5)$$

Here G is a model-dependent factor and J_0 the current operator of the model. The matrix elements are taken between physical states defined in (2.14). Because the initial and final states have the same energy, we can move the tilt operations together and then evaluate the matrix elements of

J_0. Putting all these factors and G together into a new term C, we are left with a simple matrix element of a rotation between the group states,[5]

$$S_{fi} = Ce^{-i\mu\Phi}\langle\psi^{G-}_{\lambda'(0,0)}|R_2^\dagger(\theta)|\psi^{G+}_{\lambda(0,0)}\rangle . \quad (3.6)$$

Here the two λ's have been kept different because we shall go to the limit of the improper tilt afterwards. Thus, in all cases considered, the S-matrix elements are proportional to the matrix elements of a model-independent rotation. This factor common to all models is

$$\langle\psi^{G-}_{\lambda'(0,0)}|R_2^\dagger(\theta)|\psi^{G+}_{\lambda(0,0)}\rangle e^{-i\mu\Phi} = (8\pi^2)^{-1}e^{-i\mu\Phi}e^{\pi(\lambda-\lambda')/2}(-1)^{-\mu}\Gamma(1+\mu-i\lambda)\Gamma(1+\mu-i\lambda')\left[\frac{\Gamma(-i\lambda-\mu+1)\Gamma(i\lambda'-\mu)}{\Gamma(-i\lambda+\mu+1)\Gamma(i\lambda'+\mu)}\right]^{1/2}$$

$$\times \lim_{\substack{\phi\to -i\pi \\ n'\to i\lambda';\, n\to -i\lambda}} [\langle n'-1-\mu, 0, -\mu|R_2(-\theta)e^{-i\phi L_{45}}|-1, n-\mu, \mu\rangle] . \quad (3.7)$$

We take $n'\neq n$ and then go to the limit $n'=n$ in order to extract the δ function in energy that must appear in S_{fi}. We have chosen our coordinate system in such a way that the rotation $R(\vec{k}_f, \vec{k}_i)$ does not involve any φ dependence (the so-called standard orientation), then $R(\vec{k}_f, \vec{k}_i)$ becomes a rotation through the y axis by an angle θ, the scattering angle. Thus the S-matrix elements are reduced essentially to the matrix elements of a rotation. They are proportional to the following quantity I:

$$F_{nn'} \equiv \langle n'-1-\mu, 0, -\mu|R_2(-\theta)e^{i\theta_{nn'}L_{45}}|-1, n-\mu, \mu\rangle , \quad (3.8)$$

with $\theta_{nn'} = \theta_n - \theta_{n'} \equiv \phi$.

C. Evaluation of $F_{nn'}$

Inserting a complete set of states in (3.8), we obtain

$$F_{nn'} = \sum_a R_a(\theta)T_a(\theta_{nn'}) , \quad (3.9)$$

where $R_a(\theta)$ and $T_a(\theta_{nn'})$ are defined by the following relations:

$$R_a(\theta) = \langle n'-1-\mu, 0, -\mu|R_2(-\theta)|a, -a+n'-1-\mu, \mu\rangle \quad (3.10)$$

$$T_a(\theta_{nn'}) = \langle a, -a+n'-1-\mu, \mu|e^{i\theta_{nn'}L_{45}}|-1, n-\mu, \mu\rangle .$$

The rotation matrix elements $R_a(\theta)$ are given by (see the Appendix)

$$R_a(\theta) = A_1(-1)^{n'-a-1}(\beta_R\bar{\beta}_R)^{n'-a-1}(1-\beta_R\bar{\beta}_R)^a , \quad (3.11a)$$

where

$$A_1 \equiv \left[\frac{\Gamma(n'+\mu)\Gamma(n'-\mu)}{\Gamma(n'+a+\mu)\Gamma(n'-a-\mu)}\right]^{1/2}(-1)^{-\mu} \quad (3.11b)$$

and $\beta_R = -\sin(\tfrac{1}{2}\theta)$.

For the transition-matrix elements $T_a(\theta_{nn'})$ we obtain (see the Appendix)

$$T_a(\theta_{nn'}) = A_2(-1)^{n'+a}(\alpha_T)^{-2}\left(\frac{\bar{\beta}_T}{\alpha_T}\right)^{n+n'} \times {}_2F_1\left(n+1+\mu, -n'+a+1+\mu, 1+2\mu, \frac{1}{\alpha_T\bar{\alpha}_T}\right) , \quad (3.12)$$

with

$$A_2 = \frac{(-1)^{-\mu}}{\Gamma(1+2\mu)}\left[\frac{\Gamma(n+1+\mu)\Gamma(n'-a+\mu)}{\Gamma(n+1-\mu)\Gamma(n'-a-\mu)}\right]^{1/2} \times (\alpha_T\bar{\alpha}_T)^{-\mu}$$

and

$$\alpha_T = \cosh\theta_{nn'} ,$$

$$\beta_T = \sinh\theta_{nn'} .$$

We substitute Eqs. (3.11) and (3.12) into Eq. (3.9) and use the identity[6]

$$\sum_a \frac{l!}{(l-a)!}\frac{1}{a!}x^{a-l}(1-x)^a {}_2F_1(-l+a, \alpha, \beta; y) = {}_2F_1(-l, \alpha, \beta; xy) \quad (3.13)$$

and the relation[7]

$${}_2F_1(a, b, c; z) = (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b, c; z) \quad (3.14)$$

to obtain the following general expression:

$$F_{nn'} = F'_{nn'}\frac{(-1)^{2\mu}}{\Gamma(1+2\mu)}[\sin^2(\tfrac{1}{2}\theta)]^\mu \times\left[\frac{\Gamma(n'+\mu)\Gamma(n+\mu+1)}{\Gamma(n'-\mu)\Gamma(n+1-\mu)}\right]^{1/2}e^{-i(n-n')\pi/2} , \quad (3.15)$$

where $F'_{nn'}$ is given by

$$F'_{nn'} = (\cosh^2\theta_{nn'})^{-\mu-1}\left(\frac{i\sinh\theta_{nn'}}{\cosh\theta_{nn'}}\right)^{n-n'} \times\left(1-\frac{\sin^2(\frac{1}{2}\theta)}{\cosh^2\theta_{nn'}}\right)^{-(n-n'+1)} \times {}_2F_1\left(n'+\mu, -n+\mu, 1+2\mu; \frac{\sin^2(\frac{1}{2}\theta)}{\cosh^2\theta_{nn'}}\right). \quad (3.16)$$

Next we evaluate the limit of F (more precisely the limit of the singular term F') whe· n goes to n'. In order to extract the usual δ f·nction in energy, we use the following transfo·· ıation formula[7]:

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a, -1-c+a, 1+b+a; 1/z) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b, 1-c+b, 1-a+b; 1/z). \quad (3.17)$$

This yields

$$F_{nn'} = A_3\frac{(\beta\alpha)^{n-n'}}{\alpha^{2\mu}}[\alpha^2-\sin^2(\tfrac{1}{2}\theta)]^{-(n-n'+1)} \times\left\{\frac{\Gamma(2\mu+1)\Gamma(-n-n')}{\Gamma(\mu-n)\Gamma(\mu+1-n')}\left(-\frac{\sin^2(\frac{1}{2}\theta)}{\alpha^2}\right)^{-\mu-n'}F(\mu+n', -\mu+n', 1+n-n'; \alpha^2/\sin^2(\tfrac{1}{2}\theta)) + \frac{\Gamma(2\mu+1)\Gamma(n+n')}{\Gamma(\mu+n')\Gamma(\mu+1+n)}\left(-\frac{\sin^2(\frac{1}{2}\theta)}{\alpha^2}\right)^{-\mu+n}F(\mu-n', -\mu-n, 1-n+n'; \alpha^2/\sin^2(\tfrac{1}{2}\theta))\right\},$$

$$A_3 = \frac{(-1)^{-2\mu-1}}{(2\mu)!}\sin^2(\tfrac{1}{2}\theta)\left[\frac{\Gamma(n'+\mu)\Gamma(n+\mu-1)}{\Gamma(n'-\mu)\Gamma(n-\mu-1)}\right]. \quad (3.18)$$

We are now ready to take the limit $\theta\to -i\pi$ and perform the analytic continuation $n\to -i\lambda$, $n'\to +i\lambda$. The limit $\theta\to -i\pi$ is equivalent to the following limit on α and β:

$$\lim_{\theta\to -i\pi}\{\ \} = \lim_{\alpha\to 0,\ \beta\to -i}\{\ \}. \quad (3.19)$$

We see that $\lim_{\alpha\to 0}F_{nn'}$ is zero unless $n=-n'$ or $\lambda=\lambda'$. Taking out the nonsingular parts, we find

$$\lim F_{nn'} = A_3(-i)^{-2i\lambda}[-\sin^2(\tfrac{1}{2}\theta)]^{i\lambda-1} \times\frac{\Gamma(2\mu+1)}{\Gamma(\mu+i\lambda)\Gamma(\mu+1-i\lambda)} \times\lim_{\alpha\to 0}[\Gamma(i(\lambda-\lambda'))\alpha^{-i(\lambda-\lambda')} + \Gamma(-i(\lambda-\lambda'))\alpha^{i(\lambda-\lambda')}]. \quad (3.20)$$

We compare this expression with the following relation for the δ function[8]:

$$-2\pi\delta(x) = \lim_{\alpha\to 0}[\alpha^{ix}\Gamma(-ix)+\alpha^{-ix}\Gamma(ix)], \quad (3.21)$$

and obtain

$$\lim F_{nn'} = (-1)^{-\mu-1}[\sin^2(\tfrac{1}{2}\theta)]^{i\lambda-1}2\pi\delta(\lambda-\lambda') \times[\Gamma(i\lambda+\mu)\Gamma(i\lambda-\mu) \times\Gamma(-i\lambda+\mu+1)\Gamma(-i\lambda-\mu+1)]^{-1/2}. \quad (3.22)$$

D. The result

We now go back to the S-matrix elements S_{fi}. We convert the δ function over $(\lambda-\lambda')$ into a δ function over the energy and obtain

$$S_{fi} = C2\pi\delta(p_0'-p_0)\frac{1}{8\pi^2}e^{-i\mu\Phi}\left|\frac{\partial p_0}{\partial\lambda}\right|(-1)^{-2\mu-1} \times[\sin^2(\tfrac{1}{2}\theta)]^{i\lambda-1}(i\lambda+\mu)\frac{\Gamma(-i\lambda+\mu+1)}{\Gamma(i\lambda+\mu+1)}. \quad (3.23)$$

This is our general result. The basic Coulomb-type structure of the amplitude can be recognized. Indeed, for $\mu=0$ we recover the relativistic Coulomb-type amplitudes for the spinless case. We shall see that these amplitudes are also of the Regge type. The basic ingredients upon which this formula is based are the representation of the dynamical group describing the physical states of the relativistic composite system, and second, the analytically continued values of the parabolic quantum numbers (3, 1) representing the scattering states of the system.

It remains now to evaluate the model-dependent coefficient C. In Sec. III E we consider explicitly several applications to relativistic and nonrelativistic cases with spin μ, evaluate the scattering amplitude, and determine the corresponding "trajectory functions."

E. Relativistic and nonrelativistic kinematics: examples

The asymptotic quantum numbers of the "in" and "out" states given in Eq. (3.1) depend only on the principal quantum number n for a given μ representation of SO(4, 2). Our fundamental formula (3.23) depends only on n. The relation of the quantum number n to the energy (or center-or-mass momentum k) depends on the mass spectrum of the system, or on the internal dynamics. The basic dynamical group SO(4, 2) is common to the Schrödinger-type Coulomb problem, or to the Dirac or Klein-Gordon type, or even to the fully relativistic Coulomb infinite-component wave equation. The only difference lies, for the present purpose, in the functional dependence

$$n = n(k),$$

which we now evaluate for a number of cases.

1. *"Schrödinger case"*

By this we mean the Hamiltonian (2.11). The total spin of the system is μ (integer or half-odd integer), but the kinematics is nonrelativistic. In this case, from the algebraic solution of H we know that[6] the eigenvalue of L_{46} is $\lambda = 1/k$, hence

$$n = -i/k, \quad n' = -i/k'$$

and

$$\theta_{nn'} = \theta_n - \theta_{n'} = \ln n' - \ln n = \ln\frac{n'}{n} = \ln\frac{k}{k'} . \tag{3.24}$$

It is instructive to derive the limits from the beginning.[9] Equations (3.15) and (3.16) become

$$I = I' \frac{(-1)^{2\mu}}{\Gamma(2\mu+1)} [\sin^2(\tfrac{1}{2}\theta)]^{\mu} \times\left[\frac{\Gamma(-n'+\mu)\Gamma(n+\mu+1)}{\Gamma(-n'-\mu)\Gamma(n+1-\mu)}\right]^{1/2} \times e^{-i(n+n')\pi/2}(4kk')^{\mu+1}, \tag{3.25}$$

$$I' = (k'-k)^{-2\mu}\left(\frac{k'+k}{k-k'}\right)^{n+n'} \times \frac{1}{(k'-k)^2}\left[1+\frac{4kk'}{(k-k')^2}\sin^2(\tfrac{1}{2}\theta)\right]^{-(n+n'+1)} \times {}_2F_1\left(-n'+\mu, -n+\mu, 1+2\mu; \frac{-4kk'}{(k-k')^2}\sin^2(\tfrac{1}{2}\theta)\right). \tag{3.26}$$

Further, Eq. (3.18) now reads, with $y = k - k'$,

$$I' \xrightarrow[|z|\to 0]{y\to 0} (2k)^{2n'}[4k^2\sin^2(\tfrac{1}{2}\theta)]^{-(n'+1+\mu)} \times\frac{\Gamma(2\mu+1)}{\Gamma(-n'+\mu)\Gamma(n'+\mu+1)} . \tag{3.27}$$

Again using Eq. (3.19), Eq. (3.20) becomes

$$I' = -\left(\frac{1}{4k^2}\right)^{1+\mu}[\sin^2(\tfrac{1}{2}\theta)]^{-(n+\mu+1)} \times\frac{\Gamma(2\mu+1)}{\Gamma(-n+\mu)\Gamma(1+\mu+n)}\frac{2\pi}{n^3}\delta(E_i - E_f). \tag{3.28}$$

Hence we have the elastic scattering amplitude

$$f(\vec{k}_f, \vec{k}_i) = (-1)^{2\mu}\frac{\exp[-n\ln\sin^2(\tfrac{1}{2}\theta)]}{2k\sin^2(\tfrac{1}{2}\theta)} \times(-n+\mu)\frac{\Gamma(n+\mu+1)}{\Gamma(-n+\mu+1)}e^{-i\mu\Phi}. \tag{3.29}$$

In this form the amplitude agrees with the result of Zwanziger,[3] obtained by an entirely different method, and, for $\mu = 0$, it reduces to the Coulomb scattering amplitude. The poles of the amplitude (3.29) agree with the corresponding bound-state spectrum of the system.[2]

2. *Relativistic examples*

(i) O(4)-symmetric "Klein-Gordon"-type Hamiltonian with spin μ (Ref. 10). The Hamiltonian

$$H^2 = \pi^2 + m^2 + 2H\left(\frac{-\alpha}{r}\right) + \frac{\mu^2}{r^2} \tag{3.30}$$

can be solved exactly by the O(2, 1) algebra Γ_0, Γ_4, T of Eq. (2.4). The spectrum is given by

$$E = m\left(1+\frac{\alpha^2}{N^2}\right)^{-1/2} = m\left(1-\frac{\alpha^2}{N^2+\alpha^2}\right)^{1/2} . \tag{3.31}$$

The factors occurring in the general formula (3.23) become in this case

$$q = \alpha m/\lambda, \quad n = -i\lambda,$$

and

$$\frac{1}{q^2}\left|\frac{\partial\lambda}{\partial q}\right|\left|\frac{\partial p}{\partial\lambda}\right| = \frac{1}{q^2}\left|\frac{\partial\lambda}{\partial q}\right|\left|\frac{\partial E}{\partial\lambda}\right| = \frac{v}{q^2} . \tag{3.32}$$

Using the kinematics appropriate for a particle moving relativistically in the field of another, we obtain

$$f^{\mathrm{KG}}(\theta) = \frac{(-1)^{2\mu+1}e^{-i\mu\Phi}}{v\sin^2(\tfrac{1}{2}\theta)}(i\lambda+\mu)\frac{\Gamma(1+\mu-i\lambda)}{\Gamma(1+\mu+i\lambda)} \times\exp[i\lambda\ln\sin^2(\tfrac{1}{2}\theta)]. \tag{3.33}$$

(ii) Dirac-type Hamiltonian with O(4) symmetry.[11] The radial wave equation is given by

$$\left[p_r^2 + \frac{1}{r^2}J^2 + (m^2 - E^2) - \frac{\alpha E}{r} - \frac{\alpha m}{r} - (\rho^2 m - i\rho_1 E)\frac{\vec{\sigma}\cdot\hat{r}}{r}\right]\psi = 0, \tag{3.34}$$

with the spectrum

$$E = m\left(1 - \frac{\frac{1}{2}\alpha^2}{n^2 + \frac{1}{4}\alpha^2}\right)$$

$$= m\frac{n^2 - \frac{1}{4}\alpha^2}{n^2 + \frac{1}{4}\alpha^2} \quad . \tag{3.35}$$

Hence

$$n = \tfrac{1}{2}\alpha\left(\frac{m+E}{m-E}\right)^{1/2} = -i\lambda \tag{3.35'}$$

and

$$\frac{1}{q^2}\left|\frac{\partial\lambda}{\partial q}\right|\left|\frac{\partial E}{\partial\lambda}\right| = \frac{1}{q^2}\left|\frac{\partial E}{\partial q}\right|$$
$$= \frac{1}{q^2}\,\frac{q}{E}$$
$$= \frac{v}{q^2} \ . \tag{3.36}$$

The scattering amplitude is identical to $f^{\rm KG}$ in (3.33) except for the different relation between q and λ. This case is an approximation to the Dirac Hamiltonian with spin to order α^4; it has much nicer symmetry properties and contains recoil corrections.

(iii) Infinite-component wave equations for hadrons.[12] A class of O(4, 2) infinite-component wave equations in momentum space is of the form

$$(J_\mu P^\mu + \beta\Gamma_4 + \gamma)\tilde{\psi}(P) = 0 ,$$

with (3.37)

$$J_\mu = \alpha_1\Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu\Gamma_4 + i\alpha_4 L_{\mu\nu}q^\nu ,$$

where α_i, β, γ are parameters, Γ_μ, Γ_4, $L_{\mu\nu}$ are the SO(4, 2) generators [see Eq. (2.4)]; P_μ is the total momentum, and q^ν the momentum difference in matrix elements between two states. These wave equations are exactly soluble. The principal quantum number n, eigenvalue of L_{56}, is related to the mass spectrum by

$$n = -\frac{\alpha_2 M^2 + \gamma}{[\alpha_1{}^2 M^2 - (\alpha_3 M^2 - \beta)^2]^{1/2}} \quad . \tag{3.38}$$

Because $s = M^2$, and introducing the magnitude of momentum in the center-of-mass frame q by

$$4q^2 s = [s - (m_1{}^2 + m_2{}^2)]^2 - 4m_1{}^2 m_2{}^2 ,$$

we have

$$\frac{1}{q^2}\left|\frac{\partial\lambda}{\partial q}\right|\left|\frac{\partial P_0}{\partial\lambda}\right| = \frac{1}{q^2}\left|\frac{\partial P_0}{\partial q}\right|$$
$$= \frac{1}{2q^2\sqrt{s}}\left|\frac{\partial s}{\partial q}\right|$$
$$= \frac{1}{q\sqrt{s}}\left|\frac{\partial s}{\partial(q^2)}\right|$$
$$= 4s^{3/2}\{q[s^2 - (m_1{}^2 - m_2{}^2)^2]^{1/2}\} . \tag{3.39}$$

Thus the S-matrix elements are given by

$$S_{fi} = (2\pi)^4\delta^4(P - P')\frac{\pi}{V^2}\frac{e^{-i\mu\Phi}(-1)^{2\mu+1}}{[\sin^2(\frac{1}{2}\theta)]^{-1}}$$
$$\times \exp\{i\lambda\ln[\sin^{-2}(\tfrac{1}{2}\theta)]\}(i\lambda + \mu)$$
$$\times\frac{\Gamma(1+\mu-i\lambda)}{\Gamma(1+\mu+i\lambda)}\frac{1}{q^2}\left|\frac{\partial\lambda}{\partial q}\right|\left|\frac{\partial P_0}{\partial\lambda}\right|$$
$$= (2\pi)^4\delta^4(P-P')\frac{\pi}{V^2}\frac{e^{-i\mu\Phi}(-1)^{2\mu+1}}{t}$$
$$\times\exp[i\lambda\ln(-t/4q^2)](i\lambda + \mu)$$
$$\times\frac{\Gamma(1+\mu-i\lambda)}{\Gamma(1+\mu+i\lambda)}\frac{(-16s^{3/2})}{[s^2 - (m_1{}^2 - m_2{}^2)^2]^{1/2}} , \tag{3.40}$$

where we have introduced the invariant square of the momentum transfer

$$t = -4q^2\sin^2(\tfrac{1}{2}\theta) .$$

The same S matrix results from a relativistic generalization of the H-like models apart from a change in the relationship between s and n or λ, and spin $|\mu|$. In the latter case,

$$s = m_1{}^2 + m_2{}^2 + 2m_1 m_2(1 + \alpha^2/n^2) .$$

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APPENDIX

In order to evaluate the matrix elements $R_a(\theta)$ in Eq. (3.10) we decompose the basis $|n_1 n_2 m\rangle$ as a direct product of two O(3) bases, namely, $|n_1 n_2 m\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$. Therefore $R_a(\theta)$ can be written as

$$R_a(\theta) = \langle j_1, m_1|R_2^{(j_1)}(-\theta)|j_1, m_1'\rangle$$
$$\times\langle j_2, m_2|R_2^{(j_2)}(-\theta)|j_2, m_2'\rangle .$$

In our particular case we have $j_1 = \frac{1}{2}(-n' - 1 + \mu)$, $m_1 = -j_1$, $m_1' = j_1 - a$; $j_2 = j_1 - \mu$, $m_2 = j_1 - \mu$, and $m_2' = -j_1 - a - \mu$. Furthermore,[6,8]

$$D^{(j)}_{m_1,m_1'}(\theta) = \left[\frac{(j+m_1')!(j-m_1)!}{(j-m_1')!(j+m_1)!}\right]^{1/2}\frac{1}{(m_1'-m_1)!}(\alpha)^{m_1+m_1'}(-\beta)^{m_1'-m_1}{}_2F_1(-j-m_1, j-m_1+1, m_1'-m+1;\ \beta\bar{\beta}),$$

$$m_1' > m_2 \tag{A1}$$

and

$$D^{(j)}_{m_1,m_1'}(\theta)=\left[\frac{(j+m_1)!\,(j-m_1')!}{(j-m_1)!\,(j+m_1')!}\right]^{1/2}\frac{1}{(m_1-m_1')!}\,\overline{\alpha}^{\,m_1+m_1'}\overline{\beta}^{\,m_1-m_1'}\,{}_2F_1(-j+m_1,j+m_1+1,m_1-m_1'+1;\,\beta\overline{\beta}),\qquad m_1>m_1' \quad \text{(A2)}$$

in which $\alpha=\cos(\frac{1}{2}\theta)$ and $\beta=-\sin(\frac{1}{2}\theta)$. Then using (A1) and (A2) we obtain the result given in Eqs. (3.11).

The transition-matrix elements $T_a(\theta_{nn'})$ in Eq. (3.10) can be computed in a similar way using the fact that the basis $|n_1n_2m\rangle$ is decomposable into a product of two O(2, 1) bases. It is easy to prove that $T_a(\theta_{nn'})$ is given by

$$T_a(\theta_{nn'})=D^{1/2}_{a+1/2,-1/2}(-\theta_{nn'})\times D^{1/2+\mu}_{-n'-a-1/2,n+1/2}(\theta_{nn'})\,. \quad \text{(A3)}$$

Using the O(2, 1) matrix elements [6,8]

$$D^{(k)}_{m,n}(W)=\frac{(-1)^{k-m}}{\Gamma(2k)}\left[\frac{\Gamma(k+n)\Gamma(k+m)}{\Gamma(n-k-1)\Gamma(m-k+1)}\right]^{1/2}\times(\overline{\alpha})^{m-k}(\alpha)^{-(n+k)}(-\beta)^{n-m}\times{}_2F_1\left(n+k,\,k-n,\,2k;\,\frac{1}{\alpha\overline{\alpha}}\right), \quad \text{(A4)}$$

one obtains for $D^{1/2}_{a+1/2,-1/2}(-\theta_{nn'})$

$$D^{1/2}_{a+1/2,-1/2}(-\theta_{nn'})=(-1)^a(\alpha_1)^a(-\beta_1)^{-(a+1)}, \quad \text{(A5)}$$

and for $D^{1/2+\mu}_{-n'-a-1/2,n+1/2}(\theta_{nn'})$

$$D^{1/2+\mu}_{-n'-a-1/2,n+1/2}(\theta_{nn'})=\frac{(-1)^{n'+a+\mu+1}}{\Gamma(1+2\mu)}\left[\frac{\Gamma(n+1+\mu)\Gamma(-n'-a+\mu)}{\Gamma(n+1-\mu)\Gamma(-n'-a-\mu)}\right]^{1/2}\times(\overline{\alpha}_2)^{-(n'+a+\mu+1)}(\alpha_2)^{-(n+1+\mu)}(-\beta)^{n+n'+a+1}\,{}_2F_1\left(n+\mu+1,n'+a+\mu+1,1+2\mu;\,\frac{1}{\alpha_2\overline{\alpha}_2}\right), \quad \text{(A6)}$$

where $\alpha_1=\alpha_2$ and $\beta_1=-\beta_2$. These results inserted in (A3) yield Eq. (3.12).

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[1]We denote a basis of the Lie algebra of SO(4, 2) by $L_{ab}=-L_{ba}$ $(a,b=1,\ldots,6)$ with signature $----++$. The generators L_{ij} represent the spin, L_{i4} the analog of the Runge-Lenz vector, L_{i5} the generators of Lorentz boosts, $L_{\mu 6}=\Gamma_\mu$ the current operator. We can either diagonalize $\vec{J}^2$, J_z, and Γ_0, giving the states $|njm\rangle$, or the three commuting Lie-algebra elements L_{12}, L_{34}, and L_{56}, giving the states in (2.1). For more detail on the mathematical properties of this class of representations and their reductions, see A. O. Barut and A. Böhm, J. Math. Phys. 11, 2938 (1970).

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$$R^{-1}(\varphi',\theta',-\varphi')R(\alpha,\beta,\gamma)R(\varphi,\theta,-\varphi)=e^{-i\Phi J_3}.$$

Therefore we have for the "standard orientation" the following values for the angles involved: $(\varphi',\theta',-\varphi')=(0,\theta,0)$, $(\alpha,\beta,\gamma)=(0,\theta,0)$, and $(\varphi,\theta,-\varphi)=(\varphi,0,-\varphi)$; thus $\Phi=2\varphi$.

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$$\left|\frac{\partial\lambda}{\partial q}\right|\left|\frac{\partial p_0}{\partial\lambda}\right|=\left|\frac{\partial p_0}{\partial q}\right|=q\,.$$

This then gives exactly the result (3.29).

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Relativistic transition amplitudes in a class of O(4,2) infinite multiplets. II. Discrete-continuum transitions

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A general relativistic formula is derived in closed form for the photoeffect amplitude from a composite system of spin μ described by O(4, 2) infinite-multiplet wave functions. For $\mu = 0$, it reduces to the relativistic H-atom photoeffect amplitude.

In the first part[1] of our investigations of the relativistic transition amplitudes in a class of infinite multiplets, characterized by spin μ, we derived generalizations of the Coulomb scattering amplitude (the case $\mu = 0$ gives the usual relativistic Coulomb amplitude). In this second part we study the analog of the relativistic photoeffect. In contrast to the continuum-continuum transitions of paper I, we have to take into account, in the discrete-continuum transitions, the momentum transfer to the system by external interactions. This makes the problem a bit more complex.

The starting forms of the S matrix and the T matrix are the same as in the $\mu = 0$ case.[2] The S matrix for our calculation is

$$S_{fi} = -ie \int d^4x [J_\mu(x) A^\mu(x)] \ . \tag{1}$$

For the external field we take as usual

$$A_\mu(x) = (2\pi/qV)\epsilon_\mu e^{(-iqx)} \ . \tag{2}$$

Inserting (2) and the form of the initial and final states into (1), we perform the x integration, define the T matrix by

$$S_{fi} \equiv i(2\pi)^4 \delta^4(P_f - P_i - q) T_{fi} \ , \tag{3}$$

and obtain for the T-matrix elements

$$T_{fi} = e\left(\frac{M_f}{P_0^f V} \frac{M_i}{P_0^i V} \frac{2\pi}{qV}\right)^{1/2} \epsilon^\mu \langle \Psi^-_{\vec{k}_f, \vec{P}_f} | J_\mu | \Psi_{N_i, \vec{P}_i} \rangle \ . \tag{4}$$

Here $\Psi_{N_i, \vec{P}_i}$ is the initial discrete bound state and $\Psi^-_{\vec{k}_f, \vec{P}_f}$ the outgoing scattering state. They are given in terms of the group states of the μ representation of the dynamical group SO(4, 2) by[1]

$$|\Psi_{N_i, \vec{P}}\rangle = \left(\frac{\alpha m_1 m_2}{M_N(1+\alpha^2)}\right)^{1/2} e^{(i\zeta_{\vec{P}} L_{35})} e^{(i\theta_i L_{45})} |N_i\rangle \ , \tag{5}$$

where M_N is the total mass of the bound state N, and

$$|\Psi^-_{\vec{k}, \vec{P}_f}\rangle = \left(\frac{(2\pi)^3}{V}\right)^{1/2} \left(\frac{M_\lambda}{2\pi\alpha m_1 m_2} \left|\frac{\partial\lambda}{\partial k}\right|\right)^{1/2} \frac{e^{(-\pi\lambda/2)}}{2(\pi)^{1/2}} \times \Gamma(1+\mu+i\lambda) \frac{\Gamma(-i\lambda-\mu)}{\Gamma(-i\lambda+\mu)} e^{(i\varphi L_3)} \times e^{(i\theta L_2)} e^{(i\theta_\lambda L_{45})} e^{(-\pi L_{45}/2)} |\Psi^{-G}_{-i\lambda}\rangle \ . \tag{6}$$

The kinematics in these expressions is such that we are in the center of mass of the initial particles, i.e., $\vec{P}_i = -\vec{q}$, and in a coordinate system in which the photon comes in along the z axis with its polarization $\hat{\epsilon}$ along the x axis, and the final free-particle momentum $\vec{k}_f$ points in the direction (θ, φ). Then Eq. (4) becomes

$$T_{fi} = \frac{e}{m_1}\left(\frac{M_\lambda}{P_0^i q}\right)^{1/2} \frac{2\pi}{V^2} \left(\frac{1}{2(1+\mu^2+\alpha^2)} \left|\frac{\partial\lambda}{\partial k}\right|\right)^{1/2} \times e^{(-\pi\lambda/2)} \Gamma(1+\mu-i\lambda) \frac{\Gamma(i\lambda-\mu)}{\Gamma(i\lambda+\mu)} M_{fi} \ , \tag{7}$$

where we have introduced

$$M_{fi} = \langle \Psi^{-G}_{(-i\lambda)} | e^{-i\theta L_2} e^{-i\varphi L_3} e^{-i\theta_f L_{45}} \times e^{i\zeta_{-\vec{q}} L_{35}} e^{i\theta_i L_{45}} \Gamma^1 |N_i\rangle$$

$$\theta_f \equiv \theta_\lambda - \tfrac{1}{2} i\pi \ . \tag{8}$$

We take the current operator J_μ to be

$$J_\mu = \alpha_1 \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu \Gamma_4 \ , \tag{9}$$