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Quasisecular scattering in the ϕ^4 model

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The first-order renormalized Hamiltonian, calculated by the quasisecular perturbation method applied to the weakly nonlinear ϕ^4 model of quantum field theory, is used to discuss the scattering of two particles. The scattering amplitude is found by an approximation which is accurate for weak coupling. For one space dimension a resonance in the s-wave scattering occurs at threshold for both signs of the ϕ^4 coupling. For two space dimensions a resonance occurs only for the sign of coupling which also gives a bound state as found in previous work, and the resonance lies above threshold by the bound-state energy. For three dimensions the scattering amplitude shows very little structure. Comparison of the scattering amplitudes is made with the Bethe-Salpeter equation.

I. INTRODUCTION

One of the important problems in elementary particle physics is to develop methods for calculating properties of bound states and resonances based on field-theory models. These phenomena, like the self-energies of particles, are persistent effects which arise from repeated interaction over long times.[1] This is by way of contrast with the case of ordinary (nonresonant) collisions, which take place over a limited time interval. In previous work[2] it was shown that persistent effects arise from the presence of secular and quasisecular terms in a perturbative solution of the Heisenberg field equations. Secular terms are not periodic, and blow up for large times. These terms are associated with vanishing energy denominators. Quasisecular terms are associated with energy denominators which are small compared to the dimensionless coupling constant times a characteristic energy. A heuristic procedure for modifying the quasisecular terms was devised[2] via q-number frequency (mass) and amplitude (wave-function) renormalization.

The modified perturbative theory, which is called quasisecular perturbation theory, differs in several important respects from standard perturbation theory. In contrast with the latter, which is formulated within the Fock space (of the bare particles), the quasisecular perturbation theory picks out a new Hilbert space of physical states.[3] For the ϕ^4 interaction this Hilbert space differs from Fock space in the cases of two and three space dimensions. The time evolution of states in the physical Hilbert space is described by a renormalized Hamiltonian, which leads to interactions between the physical particles.[4,5] A two-particle bound state occurs for arbitrarily weak coupling[4] in the cases of one and two space dimensions, if the ϕ^4 interaction appears in the Hamiltonian with a negative coefficient. It was shown that the stability of the Hamiltonian is assured by introducing additional weak interactions,[4] which play no further role in the perturbative analysis.

In further work[5] the quasisecular perturbative method was put on a more systematic footing by deriving the same equations via the method of multiple-time-scales perturbation theory.[6] This led to a more natural derivation of the renormalized Hamiltonian.

It has been demonstrated already that the quasisecular perturbation theory can handle bound-state phenomena for weakly interacting quantum-field-theory models.[4] In this work we extend the method to handle the scattering of the physical particles and study resonances. Resonances are found to occur in the ϕ^4 model for the cases of one and two space dimensions.

The Heisenberg equation for the real scalar field $\phi(t,x)=\phi^*(t,x)$ is written as

$$(\Box + m^2)\phi = \lambda m^{3-N} : \phi^3 : , \tag{1.1}$$

where $\Box$ is the differential wave operator $\partial^2/\partial t^2 - \nabla^2$ ($\hbar = c = 1$), N is the number of space dimensions (1, 2, or 3), λ is the dimensionless coupling constant, and : : denotes normal ordering. The initial condition at $t=0$ is taken as $[\phi(0,x), \dot{\phi}(0,y)] = i\delta^{(N)}(x-y)$. In the following, vector symbols on coordinates and momenta are not indicated explicitly for $N=2,3$.

Since this paper is based on the first-order renormalized Hamiltonian acting in the physical Hilbert space, it is useful to present a heuristic derivation of it. With the assumption of periodic boundary conditions for a cubical box of volume V, the formal Hamiltonian associated with Eq. (1.1) is

$$H = H_0 + \lambda H_I , \tag{1.2}$$

$$H_0 = \sum_l \omega_l \alpha_l^* \alpha_l , \tag{1.3}$$

where l denotes the momenta allowed by periodic boundary conditions, $\omega_l{}^2 = m^2 + l^2$, and α_l, α_l^* are the annihilation and creation operators, respectively, for bare particles of momentum l and energy ω_l, satisfying canonical commutation relations. The interaction Hamiltonian is

$$H_I = -m^{3-N}(16V)^{-1}\sum_{lkpq} \delta_{l+k,p+q}(\omega_l\omega_k\omega_p\omega_q)^{-1/2} :(\alpha_l^* + \alpha_{-l})\,(\alpha_k^* + \alpha_{-k})\,(\alpha_p + \alpha_{-p}^*)\,(\alpha_q + \alpha_{-q}^*): . \tag{1.4}$$

Most of the interaction of Eq. (1.4) can be transformed away by introducing a formally unitary clothing transformation.[3] If S is anti-Hermitian, then $e^{\lambda S}$ is (formally) unitary, and

$$H_R = e^{-\lambda S} H e^{\lambda S} = H - \lambda[S,H] + \tfrac{1}{2}\lambda^2[S,[S,H]] + \cdots \tag{1.5}$$

is dynamically equivalent to H. By choosing

$$\begin{aligned} S = {} & m^{3-N}(4V)^{-1}\sum_{kpqr} \delta_{k+p,q+r}(\omega_k\omega_p\omega_q\omega_r)^{-1/2}\big[(\omega_k - \omega_p - \omega_q - \omega_r)^{-1}(\alpha_k^*\alpha_{-p}\alpha_q\alpha_r - \alpha_r^*\alpha_q^*\alpha_{-p}^*\alpha_k) \\ & + \tfrac{1}{4}(\omega_k + \omega_p + \omega_q + \omega_r)^{-1}(\alpha_k^*\alpha_p^*\alpha_{-q}^*\alpha_{-r}^* - \alpha_{-r}\alpha_{-q}\alpha_p\alpha_k)\big] \\ & + 3m^{3-N}(8V)^{-1}\sum_{kpqr} \delta_{k+p,q+r}(1 - D_{kpqr})\,(\omega_k\omega_p\omega_q\omega_r)^{-1/2}\,(\omega_k + \omega_p - \omega_q - \omega_r)^{-1}\,\alpha_k^*\alpha_p^*\alpha_q\alpha_r \end{aligned} \tag{1.6}$$

we see that $[S,H_0]$ contains most of the interaction in Eq. (1.4). Thus

$$H_R = H_0 + \lambda H_I - \lambda[S,H_0] + O(\lambda^2) \tag{1.7}$$

has very little interaction left in first order. In fact all the interaction would be formally transformed away were it not for the vanishing and small denominators[3,7] of the last term in Eq. (1.6). The expansion of Eq. (1.5) is valid only if λS is of order λ; hence denominators of order λ must be avoided in Eq. (1.6). The function D_{kpqr} fulfills this aim, since it is taken to be unity if $k+p=q+r$ and $|(\omega_k+\omega_p)^2 - (\omega_q+\omega_r)^2| \leq 4|\lambda|m^2$, and is zero otherwise.[2] The set on which D takes the value unity is called the quasisecular region. In Eq. (1.7), H_R has first-order interaction only in the quasisecular region. It should be noted that $e^{\lambda S}$ is a formally unitary transformation. In the cases $N=2$ and 3 it is an "improper" unitary transformation[8] (i.e., its domain is Fock space, and its range a new Hilbert space which is outside the Fock space of bare-particle states).

In Sec. II the first-order renormalized Hamiltonian H_R is discussed further. It is a well-defined operator without the divergences which are present in the formal Hamiltonian H of Eq. (1.2), which are due to the local product of field operators that makes up the interaction energy density. In Sec. III the two-particle subspace, which is invariant under H_R, is discussed and some new properties of the bound state in two space dimensions are elucidated. In Sec. IV the T-matrix equation for the two-particle scattering is set up and solved approximately for weak coupling. The properties of the scattering amplitudes for $N=1,2,3$ are the subject of Sec. V. For $N=1$ a resonance occurs for both signs of λ at threshold, and the s-wave scattering amplitude attains its unitarity bound. For $N=2$ a resonance occurs only for $\lambda>0$, and the unitarity bound is attained at a kinetic energy equal to the binding energy of the bound state. For $N=3$ the scattering amplitude shows no interesting structure for weak coupling.

II. THE FIRST-ORDER RENORMALIZED HAMILTONIAN

In previous work[5] the method of multiple time scales was applied to the ϕ^4 quantum-field-theory model. The method leads to the natural introduction of a first-order renormalized Hamiltonian which is a well-defined self-adjoint operator on the Fock space of the physical particles. The first-order renormalized Hamiltonian is given by

$$H_R = \sum_l \omega_l b_l^* b_l - 3\lambda m^{3-N}(8V)^{-1} \times \sum_{lkpr} D_{lkpr}(\omega_l\omega_k\omega_p\omega_r)^{-1/2} b_l^* b_k^* b_p b_r . \tag{2.1}$$

The operators b, b^* are the usual annihilation and creation operators which satisfy the equal-time commutation relations

$$[b_l(t), b_k(t)] = [b_l^*(t), b_k^*(t)] = 0 , \tag{2.2a}$$

$$[b_l(t), b_k^*(t)] = \delta_{l,k} . \tag{2.2b}$$

It was shown that the dynamical information

contained in the first-order Heisenberg field, as obtained by the method of multiple time scales, is unitarily equivalent to the dynamics of the renormalized Hamiltonian H_R.[5] As noted above, the transformation which maps the Fock space of bare particles into the physical Hilbert space is improperly unitary in the cases of two and three space dimensions, since in these cases the range is not the same as the domain.[3] The space of physical states does not coincide with the space of bare states (i.e., the unitary transformation mapping the bare-particle Fock space into the physical Hilbert space has a range which is outside Fock space in the cases of two and three space dimensions).

H_R is a particularly simple operator since it commutes with the particle number operator $n=\sum_l b_l^* b_l$. The vacuum or no-particle state is taken as the unique (up to a phase) state $|0\rangle$ satisfying $b_l|0\rangle=0$ for all l. The one-particle states are of the form $|l\rangle=b_l^*|0\rangle$ and lie at the unperturbed energy ω_l. For the two-particle states the interaction part of H_R, the last term of Eq. (2.1), affects the dynamics.

Since H_R leaves invariant subspaces of a definite particle number, it is the direct sum of Schrödinger Hamiltonians over subspaces of particle number $n=0,1,2,3,\ldots$. The n-particle subspace can be represented by symmetric wave functions $\Psi(p_1,p_2,\ldots,p_n)$ which are square-summable (normalizable) over the various particle momenta $p_1,p_2,\ldots,p_n$. The Hamiltonian on the n-particle subspace is

$$H_R^{(n)}=\sum_{i=1}^{n}\omega_{p_i}+\sum_{i<j=1}^{n}V_{ij} \tag{2.3}$$

and consists of kinetic energies together with a nonlocal potential acting between pairs of particles. The interaction V_{12} has the form

$$(V_{12}\Psi)(p_1,p_2,\ldots,p_n)=-3\lambda m^{3-N}(4V)^{-1}\sum_{k_1k_2}D_{p_1p_2k_1k_2}(\omega_{p_1}\omega_{p_2}\omega_{k_1}\omega_{k_2})^{-1/2}\times\Psi(k_1,k_2,p_3,\ldots,p_n). \tag{2.4}$$

The interaction is a bounded, real, and symmetric operator,[5] and the domain of the self-adjoint operator $H_R^{(n)}$ is just the domain of the kinetic energy $H_0^{(n)}=\sum_{i=1}^{n}\omega_{p_i}$.

The function D_{lkpr}, which restricts the region of interaction in momentum space, has been defined in a relativistically invariant manner. The condition

$$|(\omega_l+\omega_k)^2-(\omega_p+\omega_r)^2|\leq 4|\lambda|m^2$$

can be written in terms of space-time vectors $L=(\omega_l,l)$, using the invariant inner product $L\cdot K=\omega_l\omega_k-l\cdot k$ and $L^2=L\cdot L$, as $|(L+K)^2-(P+R)^2|\leq 4|\lambda|m^2$. Thus the invariant mass squared of the two particles created by the last term of Eq. (2.1) [which is $(L+K)^2$] differs from the invariant mass squared of the two particles annihilated by less than $4|\lambda|m^2$.

III. TWO-PARTICLE BOUND STATES

The renormalized Hamiltonian in the two-particle subspace is simply

$$H_R^{(2)}=\omega_{p_1}+\omega_{p_2}+V_{12}. \tag{3.1}$$

In the limit of infinite volume the potential of Eq. (2.4) becomes an integral operator,

$$(V_{12}\Psi)(p_1,p_2)=-\tfrac{3}{4}\lambda m^{3-N}(2\pi)^{-N}\times\int D(p_1,p_2,k_1,k_2)(\omega_{p_1}\omega_{p_2}\omega_{k_1}\omega_{k_2})^{-1/2}\times\Psi(k_1,k_2)d^Nk_1d^Nk_2. \tag{3.2}$$

Here $D(p_1,p_2,k_1,k_2)$ contains a Dirac delta $\delta^{(N)}(p_1+p_2-k_1-k_2)$ instead of the previous Kronecker delta, but the (invariant) mass restrictions are the same.

Some properties of the bound states of $H_R^{(2)}$ were discussed in previous work.[4] A variational calculation showed the existence of a bound state for arbitrarily small positive λ in the cases of one and two space dimensions ($N=1,2$). In the case of three space dimensions no bound state appears until $\lambda\approx 8.5$, which is undoubtedly beyond the range of validity of the perturbative method. The bound state is an eigenstate of the part of $H_R^{(2)}$ which describes the relative motion of the two particles (i.e., the two-particle energy in the center-of-mass coordinate system). The (covariant) mass operator

$$M=[(H_R^{(2)})^2-(p_1+p_2)^2]^{1/2} \tag{3.3}$$

becomes, in the center-of-mass coordinate system,

$$M=2\omega_p+V=H_0+V, \tag{3.4a}$$

$$(V\Psi)(p)=-\tfrac{3}{4}\lambda m^{3-N}(2\pi)^{-N}\times\int D(p,k)\omega_p^{-1}\omega_k^{-1}\Psi(k)d^Nk. \tag{3.4b}$$

Here $\Psi(p)$ is the wave function of the motion in the center-of-mass coordinate system, and the new symbol $D(p,k)$ denotes the function which takes the value unity if $|\omega_k^2-\omega_p^2|=|k^2-p^2|\leq|\lambda|m^2$ and is zero otherwise.

Bound states occur when an eigenvalue of M, which

is a value of the internal energy of the two particles, lies below $2m$, the sum of the rest energies of the two particles. The eigenstates are normalized by $(2\pi)^{-N}\int |\Psi(k)|^2 d^N k = 1$.

The potential of Eq. (3.4) depends only on the magnitudes of p,k and not on their directions as vectors, and therefore only has an s-wave component. In the case of three dimensions, $N=3$, this means that in the general partial-wave expansion

$$V(p,k)=\sum_{l=0}^{\infty}(2l+1)V_l(|p|,|k|)P_l(\cos\theta),$$

only the $l=0$ term is nonvanishing. A similar result holds for $N=1,2$. The interaction acts only in the s-wave subspace ("spherically symmetric" wave functions), which carries all the scattering and binding effects.

This is not surprising, as the underlying ϕ^4 interaction vertex in lowest order has the nature of a δ function in space (locality), giving rise only to s-wave scattering.

The eigenvalue equation for s-wave bound states is best written in terms of $\xi(\omega_p)=\Psi(p)$ as

$$E\xi(\omega)=2\omega\xi(\omega)-\tfrac{3}{4}\lambda m^{3-N}\int_{2m}^{\infty}D(\omega,\omega')(\omega\omega')^{-1}\rho_N(\omega')\times\xi(\omega')d\omega'. \tag{3.5}$$

The integral over momenta has been replaced by the single-particle density of states

$$\rho_N(\omega)=(2\pi)^{-N}\int\delta(\omega-\omega_k)d^Nk, \tag{3.6}$$

which has the form

$$\rho_1(\omega)=\pi^{-1}\omega(\omega^2-m^2)^{-1/2}, \tag{3.7a}$$

$$\rho_2(\omega)=(2\pi)^{-1}\omega, \tag{3.7b}$$

$$\rho_3(\omega)=(2\pi^2)^{-1}\omega(\omega^2-m^2)^{1/2}. \tag{3.7c}$$

For weak coupling ($|\lambda|\ll 1$) in the case of two space dimensions, a relation between the bound-state energy and the value of the (normalized) wave function at zero momentum is derived in Appendix A:

$$E_B=2m-E=4\pi m^{-1}|\Psi(0)|^{-2}. \tag{3.8}$$

Since the right-hand side is positive, the bound state always lies below the continuum threshold $2m$. It is possible to construct long-range local potentials with well-arranged bumps which possess bound states in the continuum.[9] Thus it is interesting that the nonlocal potential of Eq. (3.2) does not have such bound states, at least for the case of two dimensions. Note that Eq. (3.8) is valid for both signs of λ; however, a variational calculation shows the existence of a bound state below $2m$ only for $\lambda>0$.

If the (positive) coupling constant is turned off for $N=2$, $E_B\to 0$ and $|\Psi(0)|\to\infty$. Since Ψ is normalized and $\Psi(0)=\int\tilde{\Psi}(r)d^2r$, where $\tilde{\Psi}(r)$ is the (spatial) wave function in space of the physical particles, it follows that the wave function spreads out more and more as $\lambda\to 0$ from above. It is reasonable that stronger binding implies a more localized wave function.

IV. TWO-PARTICLE SCATTERING

The Hamiltonian of Eq. (3.4) is expected to have a pure continuous spectrum extending from $2m$, the two-particle threshold, to ∞. In the case of two space dimensions this is proved. In the other cases it is a plausible conjecture.

It is shown in Appendix B that for $N=1$, 2, and 3 the interaction potential of Eq. (3.4b) is relatively compact[10] with respect to the kinetic energy of Eq. (3.4a). The perturbation then leaves invariant the essential spectrum,[10] which is obtained by removing from the spectrum all isolated eigenvalues of finite multiplicity. Since the kinetic energy H_0 has the continuous (hence essential) spectrum $[2m,\infty]$, it follows that M has the essential spectrum $[2m,\infty]$. For two dimensions it is proved in Appendix A that M has no eigenvalues above $2m$ (at least in a small interval above $2m$), so that M has the purely continuous spectrum $[2m,\infty]$. The spectrum below $2m$ must consist of isolated eigenvalues of finite multiplicity in all cases.

The s-wave scattering amplitude can be calculated from the Lippmann-Schwinger equation,

$$T(z)=V+V(z-H_0)^{-1}T(z), \tag{4.1}$$

where $T(z)$ is the off-energy-shell T matrix. For the (relatively compact) interaction being studied, Eq. (4.1) has a unique solution[11] $T(z)$ for each z, provided $\mathrm{Im}z\neq 0$. In terms of plane-wave matrix elements ($\langle x|p\rangle=e^{ip\cdot x}$), Eq. (4.1) becomes

$$\langle p|T(z)|k\rangle=V(p,k)+(2\pi)^{-N}\times\int V(p,q)(z-2\omega_q)^{-1}\langle q|T(z)|k\rangle d^Nq. \tag{4.2}$$

Separating out the s-wave part, we obtain via the definition $t(\omega_p,\omega_q,z)=\langle p|T(z)|q\rangle$ the equation (for $\mathrm{Im}z\neq 0$)

$$t(\omega,\nu,z)=v(\omega,\nu)+\int_m^{\infty}v(\omega,\omega')(z-2\omega')^{-1}\times t(\omega',\nu,z)\rho_N(\omega')d\omega', \tag{4.3a}$$

$$v(\omega,\nu) = -\tfrac{3}{4}\lambda m^{3-N}\omega^{-1}\nu^{-1}D(\omega,\nu). \tag{4.3b}$$

For $|\lambda|$ small, $D(\omega,\omega')$ (which vanishes if $|\omega^2-\omega'^2|>|\lambda|m^2$) restricts the integration over ω' to a narrow interval centered at ω. An approximate solution of Eq. (4.3) is obtained by assuming that in this interval $t(\omega',\nu,z)$ can be replaced by the constant $t(\omega,\nu,z)$. It is assumed that the variation of t in the interval is small. Then we obtain the approximate solution

$$t(\omega,\nu,z) = \left[1 - \int_m^\infty v(\omega,\omega')\rho_N(\omega')(z-2\omega')^{-1}d\omega'\right]^{-1} \times v(\omega,\nu). \tag{4.4}$$

If ω is near m, $\omega - m = O(\lambda)$, the nonrelativistic approximation is accurate for ρ_N, v, and D, and the integral in Eq. (4.4) can be easily evaluated.

The approximate solution for t given by Eq. (4.4) is identical with the solution obtained from a separable-potential approximation in the nonrelativistic region of energies $m \leq \omega, \nu \leq m(1+|\lambda|)$. The separable potential approximation involves replacing v by $\bar{v}$:

$$\bar{v}(\omega,\nu) = -\tfrac{3}{4}\lambda m^{1-N}\chi(\omega)\chi(\nu), \tag{4.5}$$

where $\chi(\omega)=1$ if $m \leq \omega \leq m(1+|\lambda|)$ and is zero otherwise. Here only the low-energy part of the interaction is kept. This approximation was used to obtain upper and lower bounds on the bound-state energy[4] for the cases $N=1,2$. Moreover, it is expected that the approximate solution of Eq. (4.4) is quite accurate for weak coupling not only at low energies but also at high energies, where the small-variation assumption on t should be even better. The interval width shrinks as ω^{-1}, and the elastic threshold at $2m$ (which leads to rapid variation) is further away. For small $|\lambda|$ the approximation is uniformly accurate in energy.

The elastic s-wave scattering amplitude can be expressed in terms of the T matrix on the energy shell. Define

$$\tau(2\omega) = \lim_{\epsilon\to 0^+} t(\omega,\omega,2\omega+i\epsilon) \tag{4.6}$$

as the on-energy-shell T matrix, where the complex variable z approaches the physical (center-of-mass) energy 2ω from above the real axis. The nonrelativistic scattering amplitude for the various space dimensions is given by

$$f_1(2\omega) = m(2ik)^{-1}\tau_1(2\omega), \tag{4.7a}$$

$$f_2(2\omega) = m(8\pi k)^{-1/2}e^{i\pi/4}\tau_2(2\omega), \tag{4.7b}$$

$$f_3(2\omega) = -m(4\pi)^{-1}\tau_3(2\omega). \tag{4.7c}$$

Here $k = [2m(\omega-m)]^{1/2}$ is the magnitude of the wave vector of the relative motion in the center-of-mass coordinate system. In estimating the amount of (s-wave) scattering, it is useful to compare with the bounds set by unitarity. The unitarity bounds are $|f_1| \leq 1$, $|f_2| \leq (2/\pi k)^{1/2}$, and $|f_3| \leq k^{-1}$.

The amplitude τ_N is written in the form

$$\tau_N(2\omega) = -\tfrac{3}{4}\lambda m^{1-N}[1 - a_N(2\omega)]^{-1}. \tag{4.8}$$

Performing the nonrelativistic integrals in the case $N=1$, in the low-energy region $m \leq \omega \leq m+\Delta$ where $\Delta = \frac{1}{2}|\lambda|m$, leads to

$$a_1(2\omega) = -(3\lambda m/8\pi k)\{\ln[(\omega+\Delta-m)^{1/2}+(\omega-m)^{1/2}] - \ln[(\omega+\Delta-m)^{1/2}-(\omega-m)^{1/2}] - i\pi\}. \tag{4.9a}$$

In the next energy region, $m+\Delta \leq \omega \leq m+O(\lambda)$, the result is

$$a_1(2\omega) = -(3\lambda m/8\pi k)\{\ln[(\omega+\Delta-m)^{1/2}+(\omega-m)^{1/2}] - \ln[(\omega+\Delta-m)^{1/2}-(\omega-m)^{1/2}] + \ln[(\omega-m)^{1/2}-(\omega-\Delta-m)^{1/2}] - \ln[(\omega-m)^{1/2}+(\omega-\Delta-m)^{1/2}] - i\pi\}. \tag{4.9b}$$

For $N=2$ the formulas are much simpler because the nonrelativistic density of states is constant. If $m \leq \omega \leq m+\Delta$, then again in terms of Eq. (4.8)

$$a_2(2\omega) = -(3\lambda/16\pi)\{\ln[(\omega-m)\Delta^{-1}] - i\pi\}, \tag{4.10a}$$

and if $m+\Delta \leq \omega \leq m+O(\lambda)$,

$$a_2(2\omega) = 3\lambda i/16. \tag{4.10b}$$

For $N=3$, if $m \leq \omega \leq m+\Delta$, then in terms of Eq. (4.8)

$$a_3(2\omega) = -\tfrac{3}{4}\lambda\pi^{-2}(2m)^{-1/2}\left(\tfrac{1}{2}(\omega-m)^{1/2}\{\ln[(\omega+\Delta-m)^{1/2}+(\omega-m)^{1/2}] - \ln[(\omega+\Delta-m)^{1/2}-(\omega-m)^{1/2}] - i\pi\} - (\omega+\Delta-m)^{1/2}\right), \tag{4.11a}$$

and if $m+\Delta \leq \omega \leq m+O(\lambda)$, then

$$a_3(2\omega) = -\tfrac{3}{4}\lambda\pi^{-2}(2m)^{-1/2}\left(\tfrac{1}{2}(\omega-m)^{1/2}\{\ln[(\omega+\Delta-m)^{1/2}+(\omega-m)^{1/2}] - \ln[(\omega+\Delta-m)^{1/2}-(\omega-m)^{1/2}] + \ln[(\omega-m)^{1/2}-(\omega-\Delta-m)^{1/2}] - \ln[(\omega-m)^{1/2}+(\omega-\Delta-m)^{1/2}] - i\pi\} - (\omega+\Delta-m)^{1/2}\right). \quad (4.11b)$$

The properties of these scattering amplitudes are discussed in the next section.

V. DISCUSSION

It is interesting to compare the scattering amplitudes of Sec. IV with the results of the Bethe-Salpeter equation in the simplest approximation obtained by summing the "chain of bubbles" for the ϕ^4 interaction.[4] The result for the quantity $a_1(2\omega)$ defined by Eq. (4.8) is[12]

$$a_1 = 3\lambda m^2\pi^{-1}s^{-1/2}(4m^2-s)^{-1/2} \times\tan^{-1}[s(4m^2-s)^{-1}]^{1/2}, \quad (5.1)$$

where $s^{1/2}$ is the center-of-mass energy 2ω of the incoming particles, which is taken as real and less than $2m$ in Eq. (5.1). Passing to the nonrelativistic limit of Eq. (5.1), and analytically continuing to physical energies $2\omega > 2m$, we obtain

$$a_1(2\omega) = -(3\lambda m/8\pi k)\{\ln[m^{1/2}+(\omega-m)^{1/2}] - \ln[m^{1/2}-(\omega-m)^{1/2}] - i\pi\}. \quad (5.2)$$

This is quite similar to the result of Eq. (4.9a) in the low-energy region; in fact if $\omega - m \ll \Delta$ then the only difference is the replacement of $\Delta = \frac{1}{2}|\lambda|m$ by m.

For the case of two space dimensions the corresponding approximate Bethe-Salpeter expression is[4]

$$a_2 = -3\lambda m(4\pi)^{-1}s^{-1/2}\ln[(2m+s^{1/2})(2m-s^{1/2})^{-1}], \quad (5.3)$$

again for $s^{1/2} < 2m$. The nonrelativistic limit analytically continued to physical energies is

$$a_2(2\omega) = -3\lambda(8\pi)^{-1}\{\ln[\tfrac{1}{2}m^{-1}(\omega-m)] - i\pi\}. \quad (5.4)$$

Now there is a replacement of Δ in Eq. (4.10a) by $2m$. Also there is an additional factor of 2 in the coefficient of Eq. (5.4) as compared with Eq. (4.10a).

For three space dimensions the Bethe-Salpeter scattering amplitude diverges, and a comparison cannot be made.[4] We note that no divergence occurs in the quasisecular scheme. Bound states, which appear as poles in the scattering amplitudes calculated from Eqs. (4.9a) and (4.10a) for $\omega < m$, occur for $\lambda > 0$. The analytic continuation of Eq. (4.9a) for $N = 1$ to $\omega < m$ is

$$a_1(2\omega) = 3\lambda m(8\pi)^{-1}[2m(m-\omega)]^{-1/2} \times[\pi - 2\tan^{-1}(m-\omega)^{1/2}\Delta^{-1/2}], \quad (5.5)$$

and $a_1(2\omega) = 1$ if $2m - 2\omega = E_B^{(1)}$, where the binding energy of the pair of particles takes the value[4] $E_B^{(1)} = (3\lambda/8)^2 m$. Similarly, for two space dimensions the pole of $f_2(2\omega)$ occurs when $E_B^{(2)} = \lambda m e^{-16\pi/3\lambda}$.

These are the only poles of the analytically continued scattering amplitude $f_N(2\omega)$, $N = 1, 2$, on the first sheet, and they occur only for $\lambda > 0$, which corresponds to an attractive potential in Eq. (3.1) with negative matrix elements between plane-wave states. As noted already, these bound states were discussed previously[4] via the separable-potential approximation of Eq. (4.5).

We turn to the behavior of the scattering amplitude of Eq. (4.7) as a function of energy. Near threshold for one dimension the logarithms in Eq. (4.9a) cancel and $a_1(2\omega) \approx 3\lambda m i/8k$ if $k \ll \Delta$. Then

$$f_1(2\omega) = 3\lambda i m(8k)^{-1}[1 - a_1(2\omega)]^{-1} = -1,$$

so that the scattering amplitude is at its unitarity bound, $|f_1| = 1$, in the low-energy region $\omega - m < \lambda^2 m$, irrespective of the sign of λ. The width of this region is of the same order as the binding energy $E_B^{(1)} = (3\lambda/8)^2 m$ for the case of $\lambda > 0$. As the energy increases, $|f_1|$ falls below the unitarity bound until when $\omega = m + \Delta$, $|f_1| \approx \frac{3}{8}|\lambda|^{1/2} \ll 1$ for weak coupling ($\lambda \leq 0.1$, for example), and $|f_1|$ continues to decrease with increasing energy. These results can be interpreted in terms of a resonance at the threshold energy $\omega = m$ with width of about $E_B^{(1)}$. The source of the resonance (and the bound state) is the large (singular) density of states at threshold for $N = 1$ as seen in Eq. (3.7a).

For two space dimensions the behavior is quite different. Combining Eqs. (4.7b), (4.8), and Eq. (4.10a) we see that

$$f_2(2\omega) = -\tfrac{3}{8}\lambda e^{i\pi/4}(2\pi k)^{-1/2} \times(1 + 3\lambda(16\pi)^{-1}\{\ln[\Delta^{-1}(\omega-m)] - i\pi\})^{-1}. \quad (5.6)$$

The unitarity bound is $(2/\pi k)^{1/2} = F$, so that $|f_2|/F$ attains its maximum of unity when the real part of the denominator in Eq. (5.6) vanishes. Since

$\omega - m < \Delta$ in Eq. (5.6), this occurs only for $\lambda > 0$, the same sign which gives the bound state, and it occurs when $2\omega - 2m = E_B^{(2)}$, so that the total energy above the threshold at resonance equals the bound-state energy. The local behavior of the scattering cross section has the Breit-Wigner form

$$|f_2(2\omega)/F|^2 = \tfrac{1}{4}\Gamma^2[(2\omega - 2m - E_B^{(2)})^2 + \tfrac{1}{4}\Gamma^2]^{-1}, \tag{5.7}$$

where $\Gamma \approx E_B^{(2)}$. The width is thus about $E_B^{(2)}$, which is much less than Δ for weak coupling; hence the resonance is extremely sharp. The scattering amplitude decreases with increasing energy above the resonance until $|f_2|/F \approx 3|\lambda|/16\pi \ll 1$ at $\omega = m + \Delta$. Because of the divergence of the logarithm in Eq. (5.6), as $\omega \to m^+$, $|f_2|/F \to 0$. For $\lambda < 0$, $|f_2|/F$ increases smoothly with energy until it reaches its maximum of about $3|\lambda|/16\pi$ at $\omega = m + \Delta$. It is interesting that for $N = 2$ the resonant behavior occurs only for $\lambda > 0$.

Finally, for three space dimensions and weak coupling the scattering amplitude shows very little structure. For $m \leq \omega \leq m + \Delta$,

$$|f_3|/|f_3|_{\max} \leq (3/16\pi)|\lambda|^{3/2},$$

which is very small; the effect of $a_3(2\omega)$ from Eq. (4.11a) is negligible in the denominator. The same bound is valid in the entire nonrelativistic region from a detailed examination of Eq. (4.11b). Only for strong coupling does a bound state appear at $\lambda \approx 8.5$, and the scattering amplitude begins to show interesting structure. However, such strong coupling is probably well outside the limit of validity of the perturbative treatment.

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APPENDIX A

In the case of weak coupling the bound-state wave function $\Psi(p)$ is negligible for momenta p such that $\omega_p - m \geq c\lambda m$, where c is some constant of order one, because the interaction strength has a bound proportional to λ. As a consequence the nonrelativistic approximation to Eq. (3.5) is accurate for weak coupling. For two space dimensions the nonrelativistic bound-state equation is

$$E\xi(\omega) = 2\omega\xi(\omega) - 3\lambda(8\pi)^{-1}\int_m^\infty \tilde{D}(\omega, \omega')\xi(\omega')d\omega', \tag{A1}$$

where $\tilde{D}(\omega, \omega') = 1$ if $|\omega - \omega'| < \frac{1}{2}|\lambda|m$ and is zero otherwise. The norm of the bound-state wave function is chosen to be $(2\pi)^{-1}m\int_m^\infty |\xi(\omega)|^2 d\omega = 1$, so $\xi \in L^2$.

The interaction in Eq. (A1) is real and ξ can be chosen real. Multiplying Eq. (A1) by $d\xi/d\omega$ and integrating both sides from $2m$ to ∞ gives

$$\tfrac{1}{2}E\xi(\omega)^2|_m^\infty = \omega\xi(\omega)^2|_m^\infty - \int_m^\infty \xi(\omega)^2 d\omega - 3\lambda(8\pi)^{-1}\int_m^\infty \frac{d\xi}{d\omega}d\omega \times \int_m^\infty \tilde{D}(\omega, \omega')\xi(\omega')d\omega'. \tag{A2}$$

Since $\xi \in L^2$, we have $\lim_{\omega\to\infty}\xi(\omega) = 0$; also ξ falls off faster than $\omega^{-1/2}$ and $\lim_{\omega\to\infty}\omega\xi(\omega)^2 = 0$ as well. With an exchange in the order of integration, the last integral in Eq. (A2) becomes

$$3\lambda(8\pi)^{-1}\xi(m)\int_m^{(1+|\lambda|/2)m}\xi(\omega')d\omega' = (2m - E)\xi(m)^2. \tag{A3}$$

Substituting in Eq. (A2) and using the normalization condition, we obtain

$$2m - E = E_B = 4\pi m^{-1}\xi(m)^{-2}, \tag{A4}$$

where E_B is the binding energy of the two-particle state.

It follows from Eq. (A4) that the eigenvalues are nondegenerate, since if one were degenerate both eigenfunctions ξ_1, ξ_2 would obey the equation and the difference $\xi = \xi_1 - \xi_2$, which would be an eigenfunction (not identically zero), could not obey it, since $\xi(m) = \xi_1(m) - \xi_2(m) = 0$.

APPENDIX B

This appendix demonstrates that the nonlocal interaction of Eq. (3.4b) is a relatively compact perturbation with respect to the kinetic energy[10] of Eq. (3.4a) for $N = 1$, 2, and 3.

In the cases of one and two space dimensions the perturbation is actually compact, since it is a Hilbert-Schmidt kernel:

$$\int |V(p,k)|^2 d^N p\, d^N k = B\int D(p,k)\omega_p^{-2}\omega_k^{-2} d^N p\, d^N k = B\int_m^\infty\int_m^\infty D(\omega,\omega')\omega^{-2}\omega'^{-2}\rho_N(\omega)\rho_N(\omega')d\omega d\omega' < \infty. \tag{B1}$$

Here B is a positive constant. The function $D(\omega, \omega')$ picks out a diagonal strip (shrinking

in width like ω^{-1} for large ω) in the (ω, ω') plane. In the case of one dimension, the behavior of the integral for large ω, ω' takes the form $\int \omega^{-5} d\omega$, which is convergent. In the case of two dimensions it is $\int \omega^{-3} d\omega$, again convergent. For three dimensions $\int \omega^{-1} d\omega$ just fails to converge; however, the interaction V is relatively compact with respect to the kinetic energy H_0.

The domain of H_0, $\mathfrak{D}(H_0)$, equipped with the scalar product $\{\Psi, \phi\} = (\Psi, \phi) + (H_0\Psi, H_0\phi)$, is itself a Hilbert space since H_0 is a closed operator. Here (Ψ, ϕ) denotes the usual L^2 inner product in momentum space. Then V is said to be relatively compact with respect to H_0 if V is a compact operator from $\mathfrak{D}(H_0)$ (with the inner product $\{\ \}$) into the original Hilbert space L^2. This condition is equivalent to the condition that $V(H_0 - z)^{-1}$ is compact in L^2 for all z in the resolvent set of H_0.[13]

For three dimensions $V(H_0 - z)^{-1}$ is again of the Hilbert-Schmidt type and thus compact:

$$\int |V(H_0 - z)^{-1}(p,k)|^2 d^3p\, d^3k$$

$$= \int |V(p,k)|^2 |2\omega_k - z|^{-2} d^3p\, d^3k$$

$$= B \int_m^\infty \int_m^\infty D(\omega, \omega') \omega^{-2} \omega'^{-2} |2\omega' - z|^{-2} \times \rho_3(\omega)\rho_3(\omega') d\omega\, d\omega' . \quad \text{(B2)}$$

The behavior of the integral for large ω, ω' is given by $\int \omega^{-3} d\omega$, which converges.

Kato[10] has extended a theorem of Weyl to prove that the essential spectrum of a closed operator is unchanged under a relatively compact perturbation. The essential spectrum of a self-adjoint operator is obtained by removing from its spectrum all isolated eigenvalues of finite multiplicity.

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