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Quantum corrections to the stress tensor in perturbation theory

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Quantum corrections to the stress tensor are studied within the framework of the perturbation theory. We use both the Feynman cutoff method and the Bogoliubov-Parasiuk-Hepp (BPH) cutoff method in the scalar theory with ϕ^4 coupling. In the case of the Feynman cutoff method, the quantum corrections to the stress tensor depend on the ratio of the regulator masses. To get the definite corrections, an additional renormalization condition is essential. To $O(\lambda^2)$, the finite stress tensor is $\theta_{\mu\nu} = T_{\mu\nu} + \frac{1}{6}[1+\lambda/(48\pi^2)](\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)\phi^2$, where $T_{\mu\nu}$ is the canonical stress tensor and λ is the coupling constant. The BPH cutoff method, on the other hand, gives us a fairly general basis to get the finite corrections. Explicit formulas are given to $O(\lambda^2)$. Finally the relation between the on-mass-shell renormalization and the off-mass-shell renormalization of the stress tensor is investigated in detail.

I. INTRODUCTION

The energy-momentum stress tensor in quantum field theory has been extensively studied in recent years[1] in connection with (1) the observed scaling law in particle physics, and (2) the construction of the finite stress tensor in perturbation theory. An important step in the latter is the proposal by Callan, Coleman, and Jackiw[2] (CCJ) of a modified stress tensor. This CCJ tensor has, in some cases, different consequences from the usual canonical stress tensor on both the classical and the quantum level. The main difference between various forms of the stress tensor will emerge when the quantum corrections are taken into account. There are several methods to obtain these corrections. When we use the Feynman cutoff method, we first regularize the Lagrangian and determine the cutoff-dependent coefficients of the necessary counterterms in the Lagrangian. Counterterms are determined by suitable renormalization conditions. In this method, a cutoff is the mass of a regulator field. Then we apply the Feynman-Dyson rule, using the entire Lagrangian supplied with the counterterms, to get the corrections to the scattering amplitude or the stress tensor. It is finite under the loop momentum integration. Finally we make the regulator masses arbitrarily large. The resultant scattering amplitude is the renormalized scattering amplitude, which is unique and finite as is well known. In CCJ it was suggested that the quantum corrections would also be finite for a suitably chosen stress tensor in the limit of the large regulator masses. This is a bold suggestion. So it will be absolutely necessary to know by explicit calculations whether or not the possible divergences, pointed out by Symanzik,[3] actually occur. In the ϕ^4 theory we will show that the Feynman cutoff method gives the divergent corrections to the stress tensor in the second order of the perturbation theory if $f=\frac{1}{6}$ in Eq. (2.15) below. This choice was done in CCJ. It is important, however, to note that there exists a unique choice of f which makes the second-order corrections finite. It is sufficient to add a term of order λ to f in CCJ, where λ is the coupling constant. The corrections are also unique, provided that the additional renormalization condition is imposed. This point will be clarified in Sec. II. When we use the Bogoliubov-Parasiuk-Hepp (BPH) cutoff method, on the other hand, explicit regulator fields are unnecessary. The possible divergences of the Feynman integral are transferred from the momentum space to the parameter space. The most general and the convenient way to perform the BPH subtractions systematically is contained in the method of the normal product due to Zimmermann.[4] The latter has been extensively applied and clarified by, among others, Lowenstein[5] and Schroer.[6,7] By using this method we find that the regulator contribution in the first-order quantum corrections to the stress tensor, as was found in CCJ, is identical to the contribution of the additional subtraction. The necessity of the additional subtraction comes in turn from the assumption[5,6] that the minimum canonical dimension of the stress tensor is four. The situation is in close analogy with the case of the axial-vector vertex in spinor electrodynamics. To obtain the expressions for the BPH-subtracted integrand and subsequent momentum integrations, we make extensive use of Appelquist's technique. Appelquist[8] has proved that Bogoliubov's R operation on the unrenormalized integrand I_G is equivalent to the product of the Taylor subtraction operators, $\prod(1-t_\gamma)$, where γ refers to the relevant

Feynman subgraph of G. We describe the BPH cutoff method and the related technical points in Sec. III. In this section the asymptotic forms of the quantum corrections are also given. In Sec. IV we consider the finite renormalization effect on the stress tensor. In the case of the scattering amplitude it is well known that the BPH subtraction itself does not give a unique finite result. The arbitrariness occurs when we expand the unrenormalized integrand in a Taylor series around some fixed values of the external momentum of the graph. This arbitrariness is completely eliminated by suitable renormalization conditions. In the case of the stress tensor, however, the renormalization conditions are necessary not only for the ordinary scattering amplitude but also for the Green's functions containing the stress-tensor vertex. We give such renormalization conditions for both the on-mass-shell renormalization (OR) and the intermediate renormalization (IR). We can show that the finite stress tensors obtained under these two renormalization conditions are related by a multiplication of a regular 6×6 matrix. This simply means that the finite renormalization from IR to OR of the stress tensor is done by multiplication of a finite factor. This fact has already been stated in CCJ in the case of the infinite renormalization. Section V is devoted to discussions.

II. FEYNMAN CUTOFF METHOD

We begin with a brief survey of the perturbation calculation due to Callan, Coleman, and Jackiw.[2] Then we will extend their method to the second order of the perturbation theory. The main purpose is to confirm the finiteness of the corrections to this order. To obtain the scattering amplitude, we need a set of relevant Feynman graphs. As the loop momentum integrations are generally divergent, a suitable regularization is necessary to have a well-defined value of a given graph. One way to do this is to use the Feynman cutoff. This cutoff can be derived from the Lagrangian by using the regulator fields. We shall restrict ourselves to the case of a single scalar field with quartic self-interaction. In this case, at least two regulator fields are necessary to regularize the scattering amplitude. A graph containing a stress tensor vertex is also regularized thereby. Let μ and λ be the physical mass of the scalar particle and the coupling constant normalized at the symmetric point on the mass shell, respectively. Following CCJ, the regularized Lagrangian is

$$L=-\tfrac{1}{2}(\partial_\mu\phi\partial_\mu\phi+\mu^2\phi^2)-\tfrac{1}{2}(\partial_\mu\phi_1\partial_\mu\phi_1+M_1^{\,2}\phi_1^{\,2})-\tfrac{1}{2}(\partial_\mu\phi_2\partial_\mu\phi_2+M_2^{\,2}\phi_2^{\,2})-(\lambda/4!)\Phi^4+L_c\,, \qquad (2.1)$$

where

$$1+c_1^{\,2}+c_2^{\,2}=0\,, \qquad (2.2)$$

$$\mu^2+c_1^{\,2}M_1^{\,2}+c_2^{\,2}M_2^{\,2}=0\,, \qquad (2.3)$$

and

$$\Phi=\phi+c_1\phi_1+c_2\phi_2\,. \qquad (2.4)$$

ϕ_1 and ϕ_2 are regulator fields. M_1 and M_2 are their respective masses. L_c is the renormalization counter term and is expressed by cutoff-dependent coefficients A, B, and C as

$$L_c=-\tfrac{1}{2}A\partial_\mu\Phi\partial_\mu\Phi-\tfrac{1}{2}B\Phi^2-(1/4!)C\Phi^4\,. \qquad (2.5)$$

A, B, and C can be expanded in power series of the coupling constant λ, and are determined by renormalization conditions on the 2- and 4-point Green's functions. The contribution of the regulator fields changes the free propagator to

$$\Delta(q)=-i\left[\frac{1}{q^2+\mu^2-i\epsilon}+c_1^{\,2}(\mu^2-M_1^{\,2})+c_2^{\,2}(\mu^2-M_2^{\,2})\right]. \qquad (2.6)$$

Convenient renormalization conditions will be

$$\Gamma^{(2)}(p,-p)|_{p^2=-\mu^2}=0\,, \qquad (2.7)$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}(p,-p)\Big|_{p^2=-\mu^2}=-i\,, \qquad (2.8)$$

and

$$\Gamma^{(4)}(p_1,\ldots,p_4)|_{\text{s.p.}}=-i\lambda\,, \qquad (2.9)$$

where s.p. indicates the symmetric point:

$$p_ip_j=\tfrac{1}{3}\mu^2(1-4\delta_{ij})\,. \qquad (2.10)$$

In (2.7)–(2.9), $\Gamma^{(N)}(p_1,\ldots,p_N)$ are amputated, one-particle-irreducible Green's functions.[9] We write

$$A=\sum_{n=1}^{\infty}A_n\lambda^n\,,\qquad B=\sum_{n=1}^{\infty}B_n\lambda^n\,, \qquad (2.11)$$

and

$$C=\sum_{n=1}^{\infty}C_n\lambda^n\,.$$

A_1, B_1, and C_1 were determined in CCJ from (2.7)–(2.9) $[(dq)\equiv d^4q/(2\pi)^4]$:

$$A_1=C_1=0\,,\qquad B_1=\tfrac{1}{2}\int(dq)\Delta(q)\,. \qquad (2.12)$$

We can get the second-order values by the same conditions. They are

$$A_2 = -\tfrac{1}{6}\frac{\partial}{\partial p^2} I(p^2)\bigg|_{p^2=-\mu^2},$$

$$B_2 = -\tfrac{1}{6}\left(1 - p^2\frac{\partial}{\partial p^2}\right) I(p^2)\bigg|_{p^2=-\mu^2} + \tfrac{3}{4} i \int (dq)(dr)\,\Delta(q)\Delta(r)\Delta(p_1 + p_2 - r)\bigg|_{\text{s.p.}}, \tag{2.13}$$

and

$$C_2 = -\tfrac{3}{2} i \int (dq)\Delta(q)\Delta(p_1 + p_2 - q)\bigg|_{\text{s.p.}}.$$

In (2.13), $I(p^2)$ is the second-order self-energy integral:

$$I(p^2) = -i\int (dq)(dr)\Delta(q)\Delta(r)\Delta(p - q - r). \tag{2.14}$$

The modified stress tensor introduced in CCJ is of the form

$$\theta_{\mu\nu} = T_{\mu\nu} + f(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)\phi^2, \tag{2.15}$$

where $T_{\mu\nu}$ is the canonical stress tensor. In CCJ the choice $f = \frac{1}{6}$ was made. First let us see the quantum corrections to $\theta_{\mu\nu}$ in this case ($f = \frac{1}{6}$). When the regulator fields are added, it would be

$$\theta_{\mu\nu} = -\partial_\mu\phi\partial_\nu\phi - \partial_\mu\phi_1\partial_\nu\phi_1 - \partial_\mu\phi_2\partial_\nu\phi_2 - A\partial_\mu\Phi\partial_\nu\Phi - \delta_{\mu\nu}L + \tfrac{1}{6}(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)(\phi^2 + \phi_1{}^2 + \phi_2{}^2 + A\Phi^2), \tag{2.16}$$

where L is given by (2.1). As the coefficients of the counter terms are explicitly dependent on the coupling constant λ, L and $\theta_{\mu\nu}$ are also explicitly dependent on λ. We define the total interaction Lagrangian L_i by

$$L_i = -\frac{1}{4!}\lambda\Phi^4 + L_c. \tag{2.17}$$

The expansions in powers of λ are

$$L_i = \lambda L_i^{(1)} + \lambda^2 L_i^{(2)} + \cdots \tag{2.18}$$

and

$$\theta_{\mu\nu} = \theta_{\mu\nu}^{(0)} + \lambda\theta_{\mu\nu}^{(1)} + \lambda^2\theta_{\mu\nu}^{(2)} + \cdots. \tag{2.19}$$

The matrix elements between the two-particle state with the momenta p, p' and the vacuum can be written as ($k + p + p' = 0$):

$$\langle pp'|\theta_{\mu\nu}|0\rangle = M_{\mu\nu}^{(0)}(p,p') + \lambda M_{\mu\nu}^{(1)}(p,p') + \lambda^2 M_{\mu\nu}^{(2)}(p,p') + \cdots, \tag{2.20}$$

where

$$M_{\mu\nu}^{(0)}(p,p') = -\tfrac{2}{3}(k^2\delta_{\mu\nu} - k_\mu k_\nu) + \tfrac{1}{2}\delta_{\mu\nu}(k^2 + p^2 + p'^2 + 2\mu^2) - (p_\mu p_\nu + p'_\mu p'_\nu). \tag{2.21}$$

$M_{\mu\nu}^{(1)}$ was obtained in CCJ and is

$$M_{\mu\nu}^{(1)}(p,p') = \frac{\lambda\pi^2}{3(2\pi)^4}\mu^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \times \left\{\int_0^1 dx \ln\left[1 + x(1-x)\frac{k^2}{\mu^2}\right] - \frac{k^2}{6\mu^2}\right\}. \tag{2.22}$$

The relevant Feynman graphs are shown in Fig. 1. This is finite. The trace $M_{\mu\mu}^{(1)}(p,p')$ is given by ($\zeta^2 = 1 + 4\mu^2/k^2$)

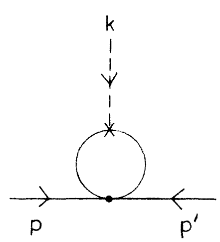

FIG. 1. Graph for $O(\lambda)$ correction to the stress tensor indicated by a cross.

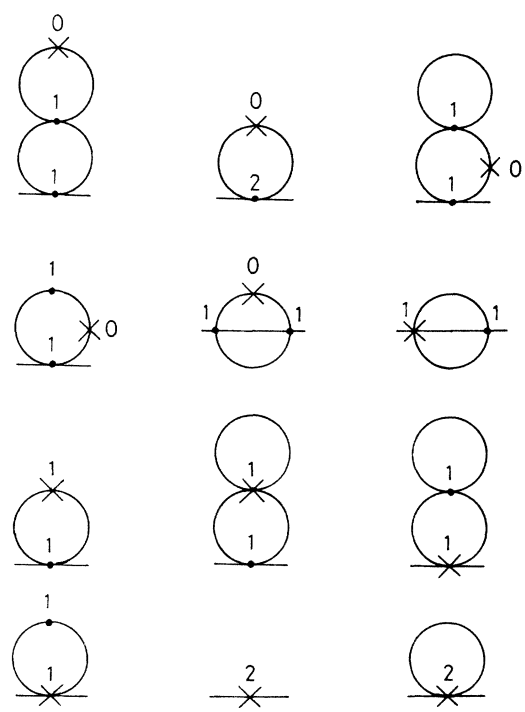

FIG. 2. Graphs for $O(\lambda^2)$ corrections to the stress tensor. Kinematics are the same as in Fig. 1. Dots indicate the vertices corresponding to L_i given by Eq. (2.17). The numbers attached to vertices indicate the order in λ introduced by Eqs. (2.18)–(2.19).

$$M_{\mu\mu}^{(1)}(p,p')=-\frac{\lambda\pi^2}{(2\pi)^4}\left[2\mu^2\left(1-\tfrac{1}{2}\zeta\ln\frac{1+\zeta}{1-\zeta}\right)+\tfrac{1}{6}k^2\right]. \tag{2.23}$$

The last term in the square bracket is the regulator contribution. It survives the limit $\mu^2\to 0$. The leading asymptotic form as $k^2\to\infty$ is given by this regulator contribution. We will see later that this is identical to that of the additional subtraction in the BPH cutoff method. Now let us turn our attention to the second-order corrections. The relevant graphs are shown in Fig. 2. We combine $L_i^{(m)}$ and $\theta_{\mu\nu}^{(n)}$ in such a way that $\sum(m+n)=2$. The application of the Feynman rules gives us immediately:

$$\begin{aligned}M_{\mu\nu}^{(2)}(p,p')=\tfrac{1}{4}\int(dq)(dr)&\left\{\frac{1}{(q^2+\mu^2)[(q+k)^2+\mu^2]}M_{\mu\nu}^{(0)}(q+k,-q)+c_1{}^2(\mu^2-M_1{}^2)+c_2{}^2(\mu^2-M_2{}^2)\right\}\\&\times\Delta(r)[\Delta(r+k)+\Delta(r+q-p)+\Delta(r+q-p')-3\Delta(r-p_1-p_2)|_{\text{s.p.}}]\\&+\tfrac{1}{12}\delta_{\mu\nu}\{[2I(p^2)-I(-\mu^2)-(p^2+\mu^2)I'(-\mu^2)]+(p\to p')\}\\&+\tfrac{1}{6}I'(-\mu^2)[p_\mu p_\nu+p'_\mu p'_\nu+\tfrac{1}{6}(k^2\delta_{\mu\nu}-4k_\mu k_\nu)].\end{aligned} \tag{2.24}$$

By using the identity

$$\int(dq_1)(dq_2)(dq_3)\delta\left(p-\sum_{i=1}^{3}q_i\right)\Delta(q_1)\Delta(q_2)\Delta(q_3)q_{1\nu}=\frac{1}{3(2\pi)^4}p_\nu I(p^2). \tag{2.25}$$

We find that

$$\begin{aligned}k_\mu M_{\mu\nu}^{(2)}(p,p')=&\tfrac{1}{12}k_\nu[I_R(p^2)+I_R(p'^2)]\\&+\tfrac{1}{12}(p-p')_\nu[I_R(p^2)-I_R(p'^2)],\end{aligned} \tag{2.26}$$

where

$$I_R(p^2)=I(p^2)-I(-\mu^2)-(p^2+\mu^2)I'(-\mu^2). \tag{2.27}$$

$I_R(p^2)$ is the second-order self-energy, renormalized on the mass shell. Thus $k_\mu M_{\mu\nu}^{(2)}(p,p')$ is finite. It vanishes on the mass shell $p^2=p'^2=-\mu^2$. In the following we will find that $M_{\mu\nu}^{(2)}(p,p')$ is divergent in the limit of the large regulator masses $(M_1,M_2\to\infty)$, and that the divergence in $M_{\mu\nu}^{(2)}(p,p')$ is exactly canceled by adding a finite $O(\lambda)$ term to f in Eq. (2.15) Then the stress tensor (2.15) with $f=\frac{1}{6}+O(\lambda)$ would be finite up to $O(\lambda^2)$. To be more precise, we may proceed in the following steps. We show that:

(i) $M_{\mu\mu}^{(2)}(p,p')$ is divergent as $M_1,M_2\to\infty$. This implies $M_{\mu\nu}^{(2)}(p,p')$ is divergent. $M_{\mu\nu}^{(2)}(p,p')$ is unambiguously separated into the divergent part and the finite part.

(ii) The divergent part of $M_{\mu\nu}^{(2)}(p,p')$ is a multiple of $(k^2\delta_{\mu\nu}-k_\mu k_\nu)$.

(iii) The divergence is canceled by the first-order corrections to the last term in Eq. (2.15). This implies that the quantum corrections to the stress tensor (2.15) can be made finite up to $O(\lambda^2)$ by choosing $f=\frac{1}{6}+O(\lambda)$.

(iv) The finite part of $M_{\mu\nu}^{(2)}(p,p')$ is uniquely determined by a renormalization condition on $M_{\mu\mu}^{(2)}(p,p')$.

To show (i), we consider the trace $M_{\mu\mu}^{(2)}(p,p')$. From (2.24) it is given by

$$M_{\mu\mu}^{(2)}(p,p')=M_1^{(2)}(p,p')+M_2^{(2)}(p,k)+M_2^{(2)}(p',k)+\tfrac{1}{3}\left\{\left[I_R(p^2)-\tfrac{1}{2}p^2\int_{-\mu^2}^{p^2}dx\,I''(x)\right]+(p\to p')\right\}, \tag{2.28}$$

where

$$\begin{aligned}M_1^{(2)}(p,p')=\tfrac{1}{2}\int(dq)(dr)&\left\{\frac{\mu^2}{(q^2+\mu^2)[(q+k)^2+\mu^2]}-\frac{\mu^2}{(q^2+\mu^2)^2}+c_1{}^2(\mu^2-M_1{}^2)+c_2{}^2(\mu^2-M_2{}^2)\right\}\Delta(r)\\&\times[\Delta(r+k)-\Delta(r-p_1-p_2)|_{\text{s.p.}}]\\=&\frac{\pi^4}{(2\pi)^8}\left(3\sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}-\tfrac{1}{2}\zeta\ln\frac{1+\zeta}{1-\zeta}\right)\left[2\mu^2\left(1-\tfrac{1}{2}\zeta\ln\frac{1+\zeta}{1-\zeta}\right)+\tfrac{1}{6}k^2\right],\end{aligned} \tag{2.29}$$

and

$$M_2^{(2)}(p,k)=\tfrac{1}{2}\int(dq)(dr)\left\{\frac{\mu^2}{(q^2+\mu^2)[(q+k)^2+\mu^2]}-\frac{\mu^2}{(q^2+\mu^2)^2}+c_1{}^2(\mu^2-M_1{}^2)+c_2{}^2(\mu^2-M_2{}^2)\right\}$$
$$\times\Delta(r)[\Delta(r+q-p)-\Delta(r-p_1-p_2)|_{\text{s.p.}}]\,. \qquad (2.30)$$

In the right-hand side of (2.28), $I''(p^2)\equiv[\partial^2/\partial(p^2)^2]I(p^2)$ is a finite function depending only on μ^2, λ, and p^2 in the limit $M_1,M_2\to\infty$. $M_1^{(2)}(p,p')$ is also a finite function given by (2.29). So we have only to know the properties of $M_2^{(2)}(p,k)$. Apart from the masses, it depends only on three scalars k^2, p^2, and $(p_1+p_2)^2|_{\text{s.p.}}=-4\mu^2/3$. This will be verified below by explicit calculations. Note that $M_2^{(2)}(p,0)=0$. Hence

$$M_2^{(2)}(p,k)=-\tfrac{1}{2}\int_0^{k^2}dx\,J[x,p^2,(p_1+p_2)^2{}_{\text{s.p.}}]\,, \qquad (2.31)$$

where

$$J[x,p^2,(p_1+p_2)^2{}_{\text{s.p.}}]=\sum_{i,j,k}c_i{}^2c_j{}^2c_k{}^2J[x,p^2,(p_1+p_2)^2{}_{\text{s.p.}},M_i,M_j,M_k]\,, \qquad (2.32)$$

with

$$J[k^2,p^2,(p_1+p_2)^2{}_{\text{s.p.}},M_i,M_j,M_k]$$
$$=M_i{}^2\frac{\partial}{\partial k^2}\int(dq)(dr)\frac{1}{(q^2+M_i{}^2)}\frac{1}{[(q+k)^2+M_i{}^2](r^2+M_j{}^2)}\left[\frac{1}{(r+q-p)^2+M_k{}^2}-\frac{1}{(r-p_1-p_2)^2{}_{\text{s.p.}}+M_k{}^2}\right]. \qquad (2.33)$$

In (2.32), i, j, and k go through 0, 1, and 2 [$c_0^2\equiv1$ and $M_0^2\equiv\mu^2$]. The momentum integrations in (2.33) will most conveniently be performed by the parametric integration method explained in Sec. III. We obtain

$$J[k^2,\ldots,M_k]=\frac{\pi^4}{(2\pi)^8}M_i{}^2\int_0^\infty dx_1\cdots dx_4\,\delta\Big(1-\sum_{i=1}^4x_i\Big)\left[\frac{1}{U^3A}x_1x_2(x_3+x_4)-\frac{x_1x_2}{(x_1+x_2)^3(x_3+x_4)^2B}\right], \qquad (2.34)$$

where

$$A=\frac{1}{U}[x_1x_2(x_3+x_4)k^2+(x_1+x_2)x_3x_4p^2]+(x_1+x_2)M_i{}^2+x_3M_j{}^2+x_4M_k{}^2\,,$$
$$B=\frac{x_1x_2}{x_1+x_2}k^2+\frac{x_3x_4}{x_3+x_4}(p_1+p_2)^2{}_{\text{s.p.}}+(x_1+x_2)M_i{}^2+x_3M_j{}^2+x_4M_k{}^2\,, \qquad (2.35)$$

and

$$U=(x_1+x_2)(x_3+x_4)+x_3x_4\,.$$

We consider the following two cases separately:

(1) $i=j=k=0$ (i.e., $M_i=M_j=M_k=\mu$). By the change of variables

$$x_1=ux,\quad x_2=(1-u)x,\quad x_3=vy,\quad x_4=(1-v)y\,, \qquad (2.36)$$

Eq. (2.34) reduces, by introducing $\beta\equiv k^2/\mu^2$, to

$$J[k^2,p^2,(p_1+p_2)^2{}_{\text{s.p.}},\mu,\mu,\mu]=\frac{\pi^4}{(2\pi)^8}\int_0^1du\,dv\,dy\,u(1-u)(I_1+I_2)\,, \qquad (2.37)$$

where

$$I_1=\frac{(1-y)^3}{y[1-y+(1-v)vy]^2}[(1-u)u(1-y)^2\beta+(1-y)(1-v)vy(p^2/\mu^2)+1-y+(1-v)vy-i\epsilon]^{-1}\,, \qquad (2.38)$$

and

$$I_2=-\frac{1}{y}[(1-u)u(1-y)\beta+1-\tfrac{4}{3}(1-v)vy-i\epsilon]^{-1}\,. \qquad (2.39)$$

If $\beta=0$, and $p^2=-\mu^2$, then I_1 and I_2 satisfy the inequalities $(1>y>0)$:

$$\frac{1}{y}>I_1>\frac{(1-y)^3}{y(1-\frac{1}{3}y)}\,,$$

and

$$-\frac{1}{y}>I_2>-\frac{1}{y(1-\frac{1}{3}y)}\,. \qquad (2.40)$$

Hence, $0 > I_1 + I_2 > -3$ in the allowed ranges of the integration variables. Thus $J[0, -\mu^2, -4\mu^2/3, \mu, \mu, \mu]$ is finite. In general, the divergences can occur only at such values of y, u, and v where the denominator of the integrand vanishes. From (2.38) and (2.39) we see that the denominators of I_1 and I_2 never vanish if $\beta > -4$ and that the integrand is then finite. (We put $p^2 = -\mu^2$.) Note that $J[k^2, -\mu^2, -4\mu^2/3, \mu, \mu, \mu]$ is analytic throughout the entire complex k^2 plane with the exception of a cut running from $k^2 = -4\mu^2$ to $-\infty$ along the real k^2 axis.

(2) At least one of the M_i M_j, and M_k is the regulator mass. A and B reduce to

$$(x_1 + x_2)M_i^2 + x_3 M_j^2 + x_4 M_k^2 . \qquad (2.41)$$

The dependence on k^2, p^2, and $(p_1 + p_2)^2_{\text{s.p.}}$ of $J[k^2, \ldots, M_k]$ disappears. Equations (2.28) and (2.31) then imply that the contribution to the trace $M^{(2)}_{\mu\mu}$ of $J[k^2, \ldots, M_k]$ is a multiple of k^2. This feature is in common with the first-order result of (2.23) of CCJ. If $M_i = \mu$, then $J[k^2, \ldots, M_k]$ will vanish. So we can restrict ourselves to the case in which M_i is a regulator mass (i.e., $i = 1$ or 2). Using the abbreviation $J[M_i, M_j, M_k]$ for the left-hand side of (2.33), we rewrite it as

$$J[M_i, M_j, M_k] = -\frac{\pi^4}{6(2\pi)^8} \int_0^1 dv dy \{1 - (1-y)^3[1 - y + (1-v)vy]^{-3}\} \left[1 - y + vy\frac{M_j^2}{M_i^2} + (1-v)y\frac{M_k^2}{M_i^2}\right]^{-1} . \qquad (2.42)$$

From the inequality

$$1 \geq 1 - (1-y)^3[1 - y + (1-v)vy]^{-3} \geq 0 , \qquad (2.43)$$

which is valid for $1 \geq v \geq 0$, and $1 > y \geq 0$, we see that the divergence of the integral in the limit $M_1, M_2 \to \infty$, if any, comes from the second factor of the integral. The case (2) is further divided into three cases.

(2′) $j = k = 0$.

The integral has a logarithmic dependence on the regulator mass and is therefore divergent as $M_1, M_2 \to \infty$.

$$J[M_i, \mu, \mu] = -\frac{\pi^4}{6(2\pi)^8} \ln\left(\frac{M_i^2}{\mu^2}\right) . \qquad (2.44)$$

(2′′) $j = 0$, $k = 1, 2$ and vice versa.

$$0 \leq \text{integrand of (2.42)} \leq \frac{1}{1 - y + (1-v)y\theta} , \qquad (2.45)$$

where $\theta \equiv M_k^2/M_i^2$ (>0) is held fixed at some finite value. As the right-hand side of (2.45) is finite after integration, $J[M_i, \mu, M_{1,2}]$ is finite. However the precise value of it depends on θ.

(2′′′) $j \neq 0$, $k \neq 0$.

The integral (2.42) is finite. The value of it will be dependent on the relative ratio of the regulator masses. In particular, if $M_2^2/M_1^2 = 1$ in the limit $M_{1,2} \to \infty$, we would obtain $-\frac{1}{2}k^2 J[0, 0, 0, \mu, \mu, \mu]$ as the contribution to $M_2^{(2)}(p, k)$. But there is no apparent reason at this stage for choosing a particular value of M_2^2/M_1^2. The cases (1) and (2) exhaust all possibilities. The separation of the divergent part is unambiguous. So we conclude that $M_2^{(2)}(p, k)$ acquires the divergent contributions from the case (2′), which is not cancelled by other terms. The situation is exactly the same for $M_2^{(2)}(p', k)$ in (2.28). This establishes the validity of the step (i).

We next turn to the step (ii). From the Lorentz invariance we can write the symmetric tensor $M^{(2)}_{\mu\nu}(p, p')$ as

$$M^{(2)}_{\mu\nu}(p, p') = \delta_{\mu\nu}F_1 + (p_\mu p_\nu + p'_\mu p'_\nu)F_2 + k_\mu k_\nu F_3 , \qquad (2.46)$$

where the functions F_i ($i = 1$, 2, and 3) can depend only on three scalars p^2, p'^2, and k^2. Hence,

$$k_\mu M^{(2)}_{\mu\nu}(p, p') = k_\nu[F_1 + k^2(\tfrac{1}{2}F_2 + F_3)] - \tfrac{1}{2}(p^2 - p'^2)(p - p')_\nu F_2 . \qquad (2.47)$$

Comparing this with (2.26), we obtain

$$F_2(p^2, p'^2, k^2) = -\frac{1}{6} \frac{I_R(p^2) - I_R(p'^2)}{p^2 - p'^2} , \qquad (2.48)$$

and

$$F_1(p^2, p'^2, k^2) = \tfrac{1}{12}[I_R(p^2) + I_R(p'^2)] + \tfrac{1}{12}\left(\frac{k^2}{p^2 - p'^2}\right)[I_R(p^2) - I_R(p'^2)] - k^2 F_3 . \qquad (2.49)$$

The function F_2 is finite and has a definite value $-\frac{1}{6}I_R'(p^2)$ at $p^2 = p'^2$, which vanishes on the mass shell. Substituting (2.48) and (2.49) into (2.46), we get

$$M^{(2)}_{\mu\nu}(p, p') = -(k^2\delta_{\mu\nu} - k_\mu k_\nu)F_3 + \text{finite functions} . \qquad (2.50)$$

Thus the divergent part of $M^{(2)}_{\mu\nu}(p, p')$ is a multiple of $(k^2\delta_{\mu\nu} - k_\mu k_\nu)$. This proves (ii). Next we show (iii). As we have seen, the divergence of the trace $M^{(2)}_{\mu\mu}(p, p')$ comes from (2.44). Hence

$$M^{(2)}_{\mu\mu}(p,p')_{\rm div}=\frac{\pi^4}{6(2\pi)^8}k^2 \times\left[c_1^2\ln\left(\frac{M_1^2}{\mu^2}\right)+c_2^2\ln\left(\frac{M_2^2}{\mu^2}\right)\right]. \tag{2.51}$$

Equation (2.50) then implies that F_3 is logarithmically divergent as $M_1, M_2\to\infty$. From the $O(\lambda)$ result in CCJ, we know that the $(\partial_\mu\partial_\nu-\delta_{\mu\nu}\partial^2)\phi^2$ term in (2.15) has a logarithmically divergent correction in $O(\lambda)$ and that its tensor structure is of the form of a multiple of $(k^2\delta_{\mu\nu}-k_\mu k_\nu)$. This suggests a possibility that the $O(\lambda)$ part of f in (2.15) cancels the divergence in (2.50) through the first-order quantum corrections to the stress tensor. To see more precisely we put

$$f=\tfrac{1}{6}+a\lambda\,. \tag{2.52}$$

The $a\lambda$ term in this equation gives, through the zeroth-order perturbation theory, a finite $O(\lambda)$ contribution to $M^{(1)}_{\mu\nu}(p,p')$. The contribution to $M^{(2)}_{\mu\nu}(p,p')$, through the first-order perturbation theory, is

$$M^{(2)}_{\mu\nu}(p,p')|_{a\lambda\ \rm term}=\frac{a\pi^2}{(2\pi)^4}(k^2\delta_{\mu\nu}-k_\mu k_\nu) \times\left[c_1^2\ln\left(\frac{M_1^2}{\mu^2}\right)+c_2^2\ln\left(\frac{M_2^2}{\mu^2}\right)\right]. \tag{2.53}$$

This is divergent in the limit $M_1, M_2\to\infty$. We find that the net correction in $O(\lambda^2)$ is finite if and only if

$$a=\frac{\pi^2}{18(2\pi)^4}\,. \tag{2.54}$$

This establishes (iii). Note that the above value differs from the one given by Coleman and Jackiw,[10] who added an additional $O(\lambda)$ term to $f=\frac{1}{6}$ to eliminate the regulator contribution to the trace in the first-order correction. Their choice $a=[6(2\pi)^2]^{-1}$ would result in the divergence of the second-order corrections in the present method. We must remember that the finite parts of the regulator contribution to $M^{(2)}_{\mu\mu}(p,p')$ depend on the value of M_1^2/M_2^2 in the limit $M_1, M_2\to\infty$. This is a very unsatisfactory point in the Feynman cutoff method.[11] However, physicists are familiar with such ambiguities in quantum electrodynamics.[12] When ambiguities arise, one invokes some general principle, such as the gauge invariance, to eliminate them. In our case, a renormalization condition must be imposed on $(\partial/\partial k^2)M^{(2)}_{\mu\mu}(p,p')$. If this is done, $M^{(2)}_{\mu\nu}(p,p')$ is also completely determined. This proves (iv). In general, if a local operator product contains many derivatives, many regulators will be necessary to regularize the Feynman integral which contains a corresponding vertex. It follows that the number of independent ratios of regulator masses is correspondingly large. To get the unambiguous corrections, many renormalization conditions will be required. The situation is the same for the BPH cutoff method used in Sec. III. We further justify our procedure leading to (iv) in this manner. In concluding Sec. II, it should be noted that the contribution of the regulator fields makes it very difficult in practice to get the higher-order corrections in this method.

III. BPH CUTOFF METHOD

In this section we consider another cutoff method, which makes each Feynman integral finite without introducing the regulator fields. Consider a renormalizable Lagrangian field theory. Every meaningful cutoff method will give us the same scattering amplitude after renormalizations. This is not the case for the matrix elements of the general local operators such as the stress tensor. For example, the N-point Green's function $\langle T\phi(x_1)\cdots\phi(x_N)\rangle_0$ can be made unique and finite in each order of the perturbation theory after the renormalization. When some of the arguments x_i coincide, it is divergent however. Graphically we must make a special subtraction in the subgraph containing this vertex. It is at this point that the dependence on the method of the cutoff comes in. In the following we employ the Bogoliubov-Parasiuk-Hepp (BPH) cutoff method. This will also give us a simple interpretation of the regulator contribution in the Feynman cutoff method.

A. Stress tensor and normal product

The most convenient way to utilize the BPH cutoff method systematically will be to use the normal products. Let us summarize briefly the definition of the normal product given by Zimmermann.[13] Suppose a Gell-Mann–Low formula for the N-point Green's function in the presence of a single scalar field with the interaction Lagrangian L_I

$$\left\langle T\prod_{i=1}^N\phi(x_i)\right\rangle_0=\text{finite part of }\left\langle T\prod_{i=1}^N\phi_0(x_i)\exp\left\{i\int d^4z\,L_I[\phi_0(z)]\right\}\right\rangle_0, \tag{3.1}$$

where $\phi(x_i)$ is a renormalized Heisenberg field operator and $\phi_0(x_i)$ is a free-field operator. The finite part is obtained by BPH subtraction operations applied to the every unrenormalized amplitude. For a particular connected graph G, which contributes to the right-hand side of (3.1), there exists a corresponding unrenormalized integrand I_G. The renormalized integrand R_G is given by Zimmermann's formula:

$$R_G=\sum_{U\in F}\prod_{\gamma\in U}(-t^{\gamma}_{d(\gamma)})I_G\,. \tag{3.2}$$

In this formula, γ stands for a proper subgraph of G with a non-negative superficial degree of divergence and is called a renormalization part. The renormalization parts $\gamma_1, \gamma_2, \ldots, \gamma_N$ of G satisfying $\gamma_1 \quad \gamma_2 \cdots \supset \gamma_N$ constitute a set called a forest U, which may also be empty. All possible forests of a proper graph G constitute a set denoted by F. The t^{γ} stands for a Taylor expansion operator. Note that

$$(-t^{\gamma}_{d(\gamma)})I_G=I_{G/\gamma}(-t^{\gamma}_{d(\gamma)})I_\gamma\,,$$

where $I_{G/\gamma}$ corresponds to a reduced graph G/γ. $t^{\gamma}_{d(\gamma)}I_\gamma$ is obtained from I_γ by keeping only the first $d(\gamma)+1$ terms of the Taylor expansion of I_γ with respect to its external momenta p_i around $p_i=0$, where $d(\gamma)$ is the degree of the superficial divergence of γ. When integrated over internal loop momenta, the resultant amplitude is finite. It is the renormalized amplitude. The normal product of the field operators is defined by a slight generalization of the above rule. Let $M[\phi]$ be a monomial of the field operator $\phi(x)$ and its derivatives. The normal product of the degree δ, $N_\delta\{M[\phi]\}$, is defined by

$$\langle TN_\delta\{M[\phi]\}(x)\phi(y_1)\cdots\phi(y_N)\rangle_0=\text{finite part of }\left\langle TM[\phi_0](x)\phi_0(y_1)\cdots\phi_0(y_N)\exp\left\{i\int d^4z\, L_I[\phi_0](z)\right\}\right\rangle_0. \tag{3.3}$$

The finite-part prescription is the same as in (3.2), except for a rule that $-t^{\gamma}_{d(\gamma)}$ should be replaced by $-t^{\gamma}_{\delta(\gamma)}$ if the renormalization part γ contains a special vertex corresponding to $M[\phi]$. We denote the canonical operator dimension of M by d. Then $\delta(\gamma)\equiv d(\gamma)+\delta-d(\delta\geq d)$. With these preliminaries, we follow Lowenstein[5] and Schroer[6] to introduce a finite stress tensor $\theta_{\mu\nu}$:

$$\theta_{\mu\nu}=N_4[T_{\mu\nu}+f(\partial_\mu\partial_\nu-\delta_{\mu\nu}\partial^2)\phi^2]\,. \tag{3.4}$$

This is the finite form of (2.15). By Lowenstein's lemma:

$$\partial_\mu N_\delta\{M[\phi]\}=N_{\delta+1}\{\partial_\mu M[\phi]\}\,, \tag{3.5}$$

(3.4) takes the form:

$$\begin{aligned}\theta_{\mu\nu}=&[-\tfrac{1}{2}\partial_\mu\partial_\nu+\tfrac{1}{4}\delta_{\mu\nu}\partial^2+f(\partial_\mu\partial_\nu-\delta_{\mu\nu}\partial^2)]N_2[\phi^2]\\&+\tfrac{1}{2}\overline{\mu}^2\delta_{\mu\nu}N_4[\phi^2]+N_4[\phi\partial_\mu\partial_\nu\phi]-\tfrac{1}{2}\delta_{\mu\nu}N_4[\phi\partial^2\phi]\\&+(1/4!)\,\overline{\lambda}\delta_{\mu\nu}N_4[\phi^4]\,.\end{aligned} \tag{3.6}$$

In (3.6) we have written the mass and the coupling constant as $\overline{\mu}$ and $\overline{\lambda}$, respectively. This is because the normal product is defined through the subtraction at p_i (external momenta) = 0. The precise relations with μ and λ will be given later. In Sec. III, we always use the $(\overline{\mu}, \overline{\lambda})$ Lagrangian. Vertex functions are defined by keeping only the contribution of the proper graphs to (3.3). Denoting the Fourier transform of $\phi(x)$ by $\tilde{\phi}(p)$, the 2-point vertex functions are expanded into the power series of $\overline{\lambda}$:

$$\begin{aligned}&\langle TN_2[\phi^2](k)\tilde{\phi}(p)\tilde{\phi}(p')\rangle_0^{\text{prop}}\equiv(2\pi)^4\delta(p+p'+k)\sum_{i=0}^{\infty}R_1^i(\overline{\lambda})^i\,,\\&\langle TN_4[\phi^2](k)\tilde{\phi}(p)\tilde{\phi}(p')\rangle_0^{\text{prop}}\equiv(2\pi)^4\delta(p+p'+k)\sum_{i=0}^{\infty}R_2^i(\overline{\lambda})^i\,,\\&\langle TN_4[\phi\partial_\mu\partial_\nu\phi](k)\tilde{\phi}(p)\tilde{\phi}(p')\rangle_0^{\text{prop}}\equiv(2\pi)^4\delta(p+p'+k)\sum_{i=0}^{\infty}R_{\mu\nu}^i(\overline{\lambda})^i\,,\\&\langle TN_4[\phi^4](k)\tilde{\phi}(p)\tilde{\phi}(p')\rangle_0^{\text{prop}}\equiv(2\pi)^4\delta(p+p'+k)\sum_{i=0}^{\infty}R_3^i(\overline{\lambda})^i\,,\end{aligned} \tag{3.7}$$

where we used the same symbol for both N_δ and its Fourier transform, and the superscript "prop" indicates that only one-particle irreducible diagrams are included.

B. Parametric integrations

The unrenormalized integrand I_G corresponding to a proper graph G containing a stress tensor vertex is of the form

$$I_G = M_{\mu\nu}(q,p) \prod_{2\le j\le N} \delta\Big(\sum q - p_j\Big) \prod_{i=1} \Delta(q_i), \tag{3.8}$$

where

$$\Delta(q) = \frac{-i}{q^2 + \bar{\mu}^2 - i\epsilon}. \tag{3.9}$$

$M_{\mu\nu}(q,p)$ is a tensor made from the internal and the external momenta. The momentum dependence of I_G is, except for δ functions, absorbed into the exponential factor by using

$$\frac{-iZ(q_\mu)}{q^2 + \bar{\mu}^2 - i\epsilon} = \int_0^\infty dx\, Z\Big(\frac{1}{-ix}\frac{\partial}{\partial l_\mu}\Big) \exp[-ix(q^2 + q\cdot l + \bar{\mu}^2 - i\epsilon)]\Big|_{l=0}, \tag{3.10}$$

where $Z(q_\mu)$ represents a polynomial of q_μ. To obtain the renormalized integrand R_G we may use (3.2) directly. It is more convenient, however, to perform the loop momentum integrations first. To do this, following Appelquist,[8] we multiply the external momenta of the each renormalization part γ_i by ξ_i. Then we perform the loop momentum integrations to get the unrenormalized amplitude $F_G(\xi_1, \ldots, \xi_n, \{p_j\})$, where $\{p_j\}$ represents a set of the external momenta. The Taylor subtraction operator is defined by

$$[1 - t^{\gamma_i}_{\delta(\gamma_i)}] F_G(\xi_1, \ldots, \xi_n, \{p_j\}) = F_G(\xi_1, \ldots, \xi_i = 1, \ldots \xi_n, \{p_j\}) - F_G(\xi_1, \ldots, \xi_i = 0, \ldots, \xi_n, \{p_j\}) - \cdots$$
$$- \frac{1}{\delta(\gamma_i)!}\Big[\Big(\frac{\partial}{\partial \xi_i}\Big)^{\delta(\gamma_i)} F_G(\xi_1, \ldots, \xi_n, \{p_j\})\Big]_{\xi_i = 0}. \tag{3.11}$$

In the absence of the overlapping divergence, (3.2) reduces to

$$R_G = \prod_i (1 - t^{\gamma_i}_{\delta(\gamma_i)}) I_G. \tag{3.12}$$

After the loop momentum integrations it takes the form:

$$J_G(\{p_j\}) = \prod (1 - t^{\gamma_i}_{\delta(\gamma_i)}) F_G(\xi_1, \ldots, \xi_n, \{p_j\}), \tag{3.13}$$

where $J_G(\{p_j\})$ is the renormalized amplitude. Appelquist[8] has proved that (3.13) holds also in the presence of the overlapping divergences. It can easily be checked in the examples described below. Let us first consider the $O(\lambda)$ corrections to the stress tensor. By (3.6) and (3.7),

$$M^{(1)}_{\mu\nu}(p,p') = [(\tfrac{1}{2} + f)k_\mu k_\nu - (\tfrac{1}{4} + f)k^2 \delta_{\mu\nu}] R^1_1 + \tfrac{1}{2}\bar{\mu}^2 \delta_{\mu\nu} R^1_2 + R^1_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu} R^1_{\lambda\lambda}. \tag{3.14}$$

$M^{(1)}_{\mu\nu}(p,p')$ is defined by (2.20). The relevant graph γ is identical to the one shown in Fig. 1. The unrenormalized integrands for R^1_1 and $R^1_{\mu\nu}$ are

$$\begin{aligned} I_\gamma &= \Delta(q)\Delta(q+\xi k) \text{ for } R^1_1 \\ &= -[q_\mu q_\nu + (q+\xi k)_\mu (q+\xi k)_\nu] \\ &\quad \times \Delta(q)\Delta(q+\xi k) \text{ for } R^1_{\mu\nu}. \end{aligned} \tag{3.15}$$

The set of forests is, for both R^1_1 and $R^1_{\mu\nu}$,

$$F = \{U_0, U_1\}, \tag{3.16}$$

where U_0 is the empty forest and $U_1 = \gamma$. Substituting (3.10) for the propagator and the momentum factor in I_γ, we perform the indicated integration by using the identity

$$1 = \int_0^\infty \frac{d\lambda}{\lambda}\, \delta\Big(1 - \frac{1}{\lambda}\sum x_i\Big), \tag{3.17}$$

and (2.36) to get

$$\begin{aligned} R^1_1 &= -i(1 - t^\gamma_0) \int (dq) I_\gamma \\ &= \frac{\pi^2}{(2\pi)^4} \int_0^1 du \ln[1 + u(1-u)\bar{\beta}], \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} R^1_{\mu\nu} &= -\tfrac{1}{2} i (1 - t^\gamma_0) \int (dq) I_\gamma \\ &= \frac{\pi^2}{2(2\pi)^4} \int_0^1 du \{\delta_{\mu\nu}[1 + u(1-u)\bar{\beta}] \ln[1 + u(1-u)\bar{\beta}] - \delta_{\mu\nu} u(1-u)\bar{\beta} - \bar{\beta}_{\mu\nu}(1 - 2u + 2u^2) \ln[1 + u(1-u)\bar{\beta}]\}, \end{aligned} \tag{3.19}$$

where $\overline{\beta}_{\mu\nu}\equiv k_\mu k_\nu/\overline{\mu}^2$ and $\overline{\beta}^2\equiv k^2/\overline{\mu}^2$. R_2^1 is obtained by replacing t_0^γ in (3.18) by t_2^γ. Thus

$$R_2^1=-i(1-t_2^\gamma)\int(dq)I_\gamma$$
$$=R_1^1-i(t_0^\gamma)-t_2^\gamma)\int(dq)I_\gamma$$
$$=\frac{\pi^2}{(2\pi)^4}\int_0^1 du\{\ln[1+u(1-u)\overline{\beta}]-\tfrac{1}{6}\overline{\beta}\}. \quad (3.20)$$

From (3.14) we get

$$k_\mu M_{\mu\nu}^{(1)}(p,p')=0 \quad (3.21)$$

$M_{\mu\nu}^{(1)}(p,p')$ depends on the external momenta p and p' only through k. So (3.21) tells us that $M_{\mu\nu}^{(1)}(p,p')$ is a multiple of $(k^2\delta_{\mu\nu}-k_\mu k_\nu)$. It is given by

$$M_{\mu\nu}^{(1)}(p,p')=\frac{\pi^2\overline{\mu}^2}{3(2\pi)^4}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\times\left\{[1+3\overline{\beta}(f-\tfrac{1}{6})]\times\int_0^1 du\ln[1+u(1-u)\overline{\beta}]-\tfrac{1}{6}\overline{\beta}\right\}. \quad (3.22)$$

Apart from the finite renormalization, this is identical to (2.22) if $f=\frac{1}{6}$. In particular we find that the regulator contribution in the first-order corrections (2.22) is identical to the additional subtraction term in the present method. The reason for the necessity of the additional subtractions is that the degree four is assigned to the stress tensor as a whole. A constituent of the stress tensor whose operator dimension is less than four acquires additional subtractions. However, its interpretation is very simple and transparent as compared with the previous method. Now let us turn our attention to the $O(\lambda^2)$ corrections. For the vertex functions containing (i) $N_2[\phi^2]$, (ii) $N_4[\phi^2]$, (iii) $N_4[\phi\partial_\mu\partial_\nu\phi]$, and (iv) $N_4[\phi^4]$, the relevant graphs for $N=2$ are shown in Fig. 3. The renormalized amplitudes are considered below separately. Only in special cases they are expressed by elementary functions.

(i) The unrenormalized integrands I_a and I_b for the graphs in Fig. 3(a) and Fig. 3(b), respectively, are, apart from the numerical constants,

$$I_a=\Delta(q)\Delta(q+\xi_2\xi_3 k)\Delta(r)\Delta(r+\xi_1\xi_3 k) \quad (3.23a)$$

and

$$I_b=\Delta(q)\Delta(q+\xi_3 k)\Delta(r)\Delta(\xi_1\xi_3 p-\xi_1 q-r)+(p\to p'). \quad (3.23b)$$

Here we introduced the parameters ξ_i in the way stated before. The renormalization parts are γ_1, γ_2, and γ_3 for I_a, and γ_1 and γ_3 for I_b. Then the $O(\lambda^2)$ corrections arising from Fig. 3(a) and Fig. 3(b), respectively, are given by

$$(R_1^2)_a=-\frac{1}{2}\prod_{i=1}^{3}(1-t_0^{\gamma_i})\int(dq)(dr)I_a \quad (3.24a)$$

and

$$(R_1^2)_b=-\frac{1}{2}\prod_{i=1,3}(1-t_0^{\gamma_i})\int(dq)(dr)I_b. \quad (3.24b)$$

These are convergent integrals by construction.

(ii) $(R_2^2)_a$ and $(R_2^2)_b$ are simply obtained by making the additional subtraction to the above results. We get

$$(R_2^2)_a=-\tfrac{1}{2}(1-t_2^{\gamma_3})(1-t_2^{\gamma_2})(1-t_0^{\gamma_1})\int(dq)(dr)I_a$$
$$=(R_1^2)_a-\frac{\pi^4}{12(2\pi)^8}\overline{\beta}\int du\ln[1+u(1-u)\overline{\beta}], \quad (3.25a)$$

and

$$(R_2^2)_b=-\tfrac{1}{2}(1-t_2^{\gamma_3})(1-t_0^{\gamma_1})\int(dq)(dr)I_b$$
$$=(R_1^2)_b-\tfrac{1}{2}(t_0^{\gamma_3}-t_2^{\gamma_3})(1-t_0^{\gamma_1})\int(dq)(dr)I_b. \quad (3.25b)$$

The additional subtraction terms in these expressions are finite by themselves. $(R_2^2)_a$ and $(R_2^2)_b$ are thus finite.

(iii) The unrenormalized integrands $I_{\mu\nu a}$ and $I_{\mu\nu b}$ are

$$I_{\mu\nu a}=-\Delta(q)\Delta(q+\xi_2\xi_3 k)\Delta(r)\Delta(r+\xi_1\xi_3 k)\times[q_\mu q_\nu+(q+\xi_2\xi_3 k)_\mu(q+\xi_2\xi_3 k)_\nu], \quad (3.26a)$$

and

$$I_{\mu\nu b}=-\Delta(q)\Delta(q+\xi_2\xi_3 k)\Delta(r)\Delta(\xi_1\xi_2\xi_3 p-\xi_1 q-\xi_2 r)\times[q_\mu q_\nu+(q+\xi_2\xi_3 k)_\mu(q+\xi_2\xi_3 k)_\nu], \quad (3.26b)$$

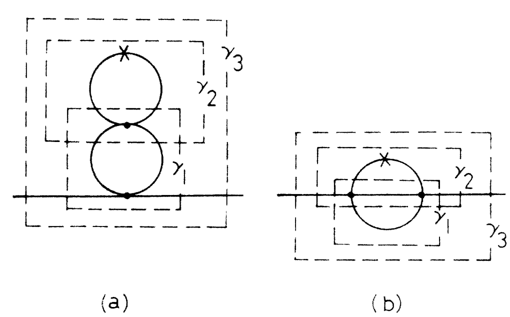

FIG. 3. Graphs for $O(\lambda^2)$ corrections to the stress tensor. Subgraphs γ_i ($i=1$, 2, and 3) denote the renormalization parts. Kinematics are the same as in Fig. 1.

respectively. The renormalization parts are γ_1, γ_2, and γ_3. Further $d(\gamma_1)=0$ and $d(\gamma_2)=d(\gamma_3)=2$. These hold for both $I_{\mu\nu a}$ and $I_{\mu\nu b}$. Thus we get

$$(R^2_{\mu\nu})_a = -\tfrac{1}{4}(1-t_2^{\gamma_3})(1-t_2^{\gamma_2})(1-t_0^{\gamma_1}) \times \int (dq)(dr) I_{\mu\nu a}\,, \tag{3.27a}$$

$$(R^2_{\mu\nu})_b = -\tfrac{1}{4}(1-t_2^{\gamma_3})(1-t_2^{\gamma_2})(1-t_0^{\gamma_1}) \times \int (dq)(dr) I_{\mu\nu b}\,, \tag{3.27b}$$

(iv) The unrenormalized integrand is given by

$$I_c = \Delta(q)\Delta(r)\Delta(\xi_1\xi_2 p - \xi_1 r - q) + (p \leftrightarrow p')\,. \tag{3.28}$$

From this we get

$$R^2_3 = -4i(1-t_2^{\gamma_2})(1-t_0^{\gamma_1}) \int (dq)(dr) I_c\,. \tag{3.29}$$

The finite amplitudes obtained above will also be necessary in Sec. IV to determine the finite renormalization matrix.

C. Asymptotic forms

In order to get more explicit forms of the corrections we may consider the asymptotic forms of them as $\bar\beta \to \infty$ with $\bar\alpha$ $(\equiv p^2/\bar\mu^2)$ and $\bar\alpha'$ $(\equiv p'^2/\bar\mu^2)$ fixed. In some cases we can factorize $\ln\bar\beta$ from the integrands with finite coefficients. In the other cases the coefficients are logarithmically divergent, if we factorize $\ln\bar\beta$ formally. The latter cases can be treated just as the former after differentiating the amplitudes with respect to $\bar\beta$. We put $\bar\alpha = \bar\alpha' = -1$ (i.e., $p^2 = p'^2 = -\bar\mu^2$). For the $O(\bar\lambda)$ corrections we have simply

$$R^1_1 = \frac{\pi^2}{(2\pi)^4}(\ln\bar\beta - 2 + \cdots)\,,$$

$$R^1_2 = \frac{\pi^2}{(2\pi)^4}(-\tfrac{1}{6}\bar\beta + \ln\bar\beta - 2 + \cdots)\,, \tag{3.30}$$

$$R^1_{\mu\nu} = \frac{\pi^2}{2(2\pi)^4}\bar\mu^2\Big[\tfrac{1}{6}(\bar\beta\delta_{\mu\nu} - 4\bar\beta_{\mu\nu})\ln\bar\beta + \tfrac{1}{9}(-4\bar\beta\delta_{\mu\nu} + 13\bar\beta_{\mu\nu}) + \frac{1}{\bar\beta}(\bar\beta\delta_{\mu\nu} - 2\bar\beta_{\mu\nu})\ln\bar\beta - \delta_{\mu\nu} + \cdots\Big]\,,$$

and

$$R^1_3 = 0\,.$$

For the $O(\bar\lambda^2)$ corrections we have $[c \equiv \pi^4/(2\pi)^8]$

$$(R^2_1)_a = \tfrac{1}{2}c[(\ln\bar\beta)^2 - 4\ln\bar\beta + \cdots]\,,$$

$$(R^2_1)_b = \tfrac{1}{2}c[(\ln\bar\beta)^2 - 6\ln\bar\beta + \cdots]\,,$$

$$(R^2_2)_a = (R^2_1)_a - \tfrac{1}{12}c(\bar\beta\ln\bar\beta - 2\bar\beta + 2\ln\bar\beta + \cdots)\,,$$

$$(R^2_2)_b = (R^2_1)_b + \tfrac{1}{6}cc'\bar\beta\,,$$

$$(R^2_{\mu\nu})_a = \tfrac{1}{24}c\bar\mu^2[(\bar\beta\delta_{\mu\nu} - 4\bar\beta_{\mu\nu})(\ln\bar\beta)^2 + (-\tfrac{14}{3}\bar\beta\delta_{\mu\nu} + \tfrac{50}{3}\bar\beta_{\mu\nu})\ln\bar\beta + \tfrac{16}{3}\bar\beta\delta_{\mu\nu} - \tfrac{52}{3}\bar\beta_{\mu\nu} + \cdots]\,, \tag{3.31}$$

$$(R^2_{\mu\nu})_b = \tfrac{1}{24}c\bar\mu^2[(\bar\beta\delta_{\mu\nu} - 4\bar\beta_{\mu\nu})(\ln\bar\beta)^2 - \tfrac{17}{3}(\bar\beta\delta_{\mu\nu} - 4\bar\beta_{\mu\nu})\ln\bar\beta + 2(\bar\alpha_{\mu\nu} + \bar\alpha'_{\mu\nu})\ln\bar\beta + \bar\beta\delta_{\mu\nu}\times O(1) + \bar\beta_{\mu\nu}\times O(1) + (\bar\alpha_{\mu\nu} + \bar\alpha'_{\mu\nu})\times O(1) + \cdots]\,,$$

and

$$R^2_3 = O(1)\,,$$

where c' is a finite constant determined from (3.25b). We introduced $\bar\alpha_{\mu\nu} \equiv p_\mu p_\nu/\bar\mu^2$ and $\bar\alpha'_{\mu\nu} \equiv p'_\mu p'_\nu/\bar\mu^2$. The expressions for the corrections to the stress tensor are given by

$$M^{(1)}_{\mu\nu}(p,p') = \frac{\pi^2}{(2\pi)^4}\bar\mu^2(\bar\beta\delta_{\mu\nu} - \bar\beta_{\mu\nu})[(f - \tfrac{1}{6})\ln\bar\beta - (2f - \tfrac{5}{18}) + \cdots]\,, \tag{3.32}$$

and

$$M^{(2)}_{\mu\nu}(p,p') = \frac{\pi^4}{(2\pi)^8}\bar\mu^2\{[(f - \tfrac{1}{6})(\ln\bar\beta)^2 - (5f - \tfrac{59}{72})\ln\bar\beta + \cdots]\bar\beta\delta_{\mu\nu} - [(f - \tfrac{1}{6})(\ln\bar\beta)^2 - (5f - \tfrac{31}{36})\ln\bar\beta + \cdots]\bar\beta_{\mu\nu} + \tfrac{1}{12}(\bar\alpha_{\mu\nu} + \bar\alpha'_{\mu\nu} + \delta_{\mu\nu})\ln\bar\beta + \cdots\}\,. \tag{3.33}$$

Note that the leading terms in $M^{(1)}_{\mu\nu}$ and $M^{(2)}_{\mu\nu}$ drop out with the choice $f = \frac{1}{6}$. $M^{(1)}_{\mu\nu}$ is explicitly of the conserved form, i.e., satisfies $k_\mu M^{(1)}_{\mu\nu}(p,p') = 0$. As was derived by Lowenstein,[5] the Ward identity for the $\theta_{\mu\nu}$ vertex can easily be obtained in the normal-product method. In particular it implies $\langle pp'|\partial_\mu\theta_{\mu\nu}|0\rangle^{\text{prop}} = 0$ for $p^2 = p'^2 = -\bar\mu^2$. This equation, together with the equation of motion, will fix the ratios F_2/F_1 and F_3/F_1 in (2.47) on the mass shell. Except for an over-all factor, this will also give us $M^{(2)}_{\mu\nu}$ $(p^2 = p'^2 = -\bar\mu^2)$ in terms of R^2_2 and R^2_3. Equations (3.32) and (3.33) were obtained, however, by taking all the constituents of $\theta_{\mu\nu}$ directly. Note also that, in $k_\mu M^{(2)}_{\mu\nu}$, not only the $(\ln\bar\beta)^2$ terms but also the $\ln\bar\beta$ terms drop out for $p^2 = p'^2 = -\bar\mu^2$. This is in accordance with

$k_\mu M^{(2)}_{\mu\nu}(p,p')=0$ on the mass shell and supports the correctness of (3.33). The fact that the coefficient of $(\ln\bar\beta)^2$ term in $R^2_{\mu\nu}$ vanishes by taking the trace follows from the equation of motion. From (3.31) we see that this is satisfied for the two graphs separately. In concluding this section, we stress that there are some ambiguities in the choice of $\theta_{\mu\nu}$. The $O(1)$ term in f has physical effects on the classical level and will be determined by gravitational interactions. However, the higher-order terms in f and the choice of the degree δ affect the quantum behavior. They cannot be fixed *a priori*.

IV. FINITE RENORMALIZATIONS

The BPH renormalization scheme as well as Zimmermann's normal product was originally formulated through the intermediate renormalization (IR). The on-mass-shell renormalization (OR) is also possible and is sometimes more convenient in practical applications. In this section we study the transformation law of the stress tensor when we go from IR to OR. Let us first consider the case of the ordinary scattering amplitudes, which also appear in the corrections to the stress tensor.

A. Scattering amplitude

The existence and the properties of the finite renormalization between IR and OR are well known for the scattering amplitude, or equivalently, for the Green's functions containing no generalized vertex. So we will restrict our attention only to an important step proved by Brandt and then will make a working rule to perform the finite renormalization. Consider Zimmermann's formula (3.2). If we change the subtraction point from $p_i=0$ to some nonzero fixed point, the resultant integrand is also finite after loop-momentum integrations. The difference can be expressed by finite counterterms in Lagrangian. This ambiguity of the amplitude associated with the subtraction point is completely eliminated by renormalization conditions. In particular, Hepp[14] has proved that there exists a choice of the finite counterterms consistent with the given renormalization conditions. However, an elaborate BPH technique is necessary in order to prove that the effect of the finite counterterms on the Green's functions is equivalent to finite changes of the mass and the coupling constant.

Let $\Sigma_r(p^2{:}\mu,\lambda)$ and $V_r(p_1,\ldots,p_4{:}\mu,\lambda)$ be 2- and 4-point vertex functions, respectively, with the lowest-order contributions subtracted, and normalized at $p_i=0$. In terms of them, the intermediate renormalized vertex functions are given by

$$\Gamma^{(2)}_r(p,-p{:}\mu,\lambda)=-i[p^2+\mu^2+\Sigma_r(p^2{:}\mu,\lambda)]\,, \tag{4.1a}$$

and

$$\Gamma^{(4)}_r(p_1,\ldots,p_4{:}\mu,\lambda)=-i[\lambda+V_r(p_1,\ldots,p_4{:}\mu,\lambda)]\,. \tag{4.1b}$$

The values of $\Gamma^{(2)}_r$, $(\partial/\partial p^2)\Gamma^{(2)}_r$, and $\Gamma^{(4)}_r$ at $p_i=0$ are the same as the corresponding values in (2.7)–(2.9). The result of Brandt[15] implies that there is a choice of $\bar\mu$, $\bar\lambda$, z_1, and z_2, so that

$$\Gamma^{(2)}_r(p,-p{:}\bar\mu,\bar\lambda)=z_1\Gamma^{(2)}(p,-p{:}\mu,\lambda)\,, \tag{4.2a}$$

and

$$\Gamma^{(4)}_r(p_1,\ldots,p_4{:}\bar\mu,\bar\lambda)=z_2\Gamma^{(4)}(p_1,\ldots,p_4{:}\mu,\lambda)\,, \tag{4.2b}$$

where $\Gamma^{(N)}$ $(N=2,4)$ were defined in Sec. II. The renormalization conditions imply

$$-\mu^2+\bar\mu^2+\Sigma_r(p^2{:}\mu,\lambda)|_{p^2=-\mu^2}=0\,, \tag{4.3}$$

$$z_1=1+\frac{\partial}{\partial p^2}\Sigma_r(p^2{:}\mu,\lambda)\Big|_{p^2=-\mu^2}\,, \tag{4.4}$$

and

$$z_2\lambda=\bar\lambda+V_r(p_1,\ldots,p_4{:}\bar\mu,\bar\lambda)|_{p_ip_j=(\mu^2/3)(1-4\delta_{ij})}\,. \tag{4.5}$$

From the dimensional reason we can write as

$$\bar\mu^2=f(\mu^2,\lambda)\,,\quad \bar\lambda=g(\lambda)\,. \tag{4.6}$$

In order to define λ as a function of $\bar\lambda$, we may choose $z_2=1$, which implies $\partial\lambda(\bar\lambda)/\partial\bar\lambda=1$ at $\lambda=0$. Then $f(\mu^2,\lambda)$, $g(\lambda)$, and z_1 are uniquely determined from Eqs. (4.3)–(4.5). The desired rule is: Make substitutions $\mu^2\to f(\mu^2,\lambda)$ and $\lambda\to g(\lambda)$ in $\Gamma^{(N)}_r(p_1,\ldots,p_N{:}\mu,\lambda)$ and divide by z_1 for $N=2$. Then we obtain $\Gamma^{(N)}(p_1,\ldots,p_N{:}\mu,\lambda)$ $(N=2,4)$.

B. Stress tensor

As in the case of the scattering amplitude, the problem is to make a rule to obtain the OR-value of the stress tensor when the IR-value of it is known. We will find that the relative ratios of the mixing of the normal products are rather complex in $\theta_{\mu\nu}$ if the latter is expressed by N_δ. $\theta_{\mu\nu}(x)$ is a linear combination of six tensors A_i $(i=1,\ldots,6)$. They are $A_1=\mu^2\phi^2\delta_{\mu\nu}$, $A_2=\partial^2\phi^2\delta_{\mu\nu}$, $A_3=\partial_\mu\partial_\nu\phi^2$, $A_4=\phi^4\delta_{\mu\nu}$, $A_5=\phi\partial^2\phi\delta_{\mu\nu}$, and $A_6=\phi\partial_\mu\partial_\nu\phi$ (with tensor indices of A_i suppressed). Suppose a proper graph G which contains a A_i vertex. For a renormalization part of G, which contains no A_i vertex, the subtraction is the same as for the scattering amplitude. So the OR-value of the amplitude is obtained by the previous method. For a renormalization part which contains a A_i vertex, we make necessary subtractions at $p_i=0$. However, even if we start with the $(\bar{\mu},\bar{\lambda})$ Lagrangian, the resulting finite amplitude containing a A_i vertex, obtained in this way, will not in general be simply proportional to the OR-value of the amplitude in the (μ,λ) theory. This is because the operator A_i (i fixed) alone is not necessarily closed under renormalization.[2] To be more precise, we define the vertex functions (i.e., amputated, one-particle irreducible Green's functions):

$$F^{ir}_{\mu\nu}(p,p':\mu,\lambda)=\langle TN_4[A_i(\mu,\lambda)]\,\tilde{\phi}(p)\tilde{\phi}(p')\rangle_0^{\text{prop}} \tag{4.7}$$

and

$$G^{ir}_{\mu\nu}(p_1,p_2,p_3,p_4:\mu,\lambda)=\langle TN_4[A_i(\mu,\lambda)]\,\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)\rangle_0^{\text{prop}}\,. \tag{4.8}$$

The (μ,λ) in the right-hand side implies that it must be calculated using the (μ,λ) Lagrangian. We need also the new normal product $N_\delta'[A_i]$, which is defined by the on-mass-shell subtraction[4]:

$$F^{i}_{\mu\nu}(p,p':\mu,\lambda)=\langle TN_4'[A_i(\mu,\lambda)]\,\tilde{\phi}(p)\tilde{\phi}(p')\rangle_0^{\text{prop}} \tag{4.9}$$

and

$$G^{i}_{\mu\nu}(p_1,p_2,p_3,p_4:\mu,\lambda)=\langle TN_4'[A_i(\mu,\lambda)]\,\tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4)\rangle_0^{\text{prop}}\,. \tag{4.10}$$

Then there exists a unique choice of a 6×6 matrix (α_{ij}) so that for $N=2$ and $N=4$

$$\langle TN_4'[A_i(\mu,\lambda)]\,\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_N)\rangle_0^{\text{prop}}=\alpha_{ij}\langle TN_4[A_j(\bar{\mu},\bar{\lambda})]\,\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_N)\rangle_0^{\text{prop}}\,, \tag{4.11}$$

where μ and λ are related to $\bar{\mu}$ and $\bar{\lambda}$ in the way described in Sec. IV A. To show (4.11), we first note that the normal product method of renormalization is equivalent to add to the Lagrangian the counterterms of special types corresponding to the local operator products. As in the case of the scattering amplitude, we can choose the finite counterterms such that the renormalization conditions are satisfied for each A_i amplitude $F^i_{\mu\nu}$ and $G^i_{\mu\nu}$. This is guaranteed by the generalized BPH theorem. It tells us that for Green's function containing a generalized vertex, there exists a choice of the finite counter terms consistent with the renormalization conditions, and that any such choice leads to the same Green's function. The proof of this theorem is straightforward by using the original Hepp's method[14] and is omitted here.[16] The next step is to prove the equivalence of the effect of the finite counter terms and the finite renormalization of the A_i amplitude. This step is trivial because the A_i vertex appears once and only once in a graph contributing to (4.11). Thus we arrive at (4.11). To determine (α_{ij}), we write

$$\begin{aligned}F^i_{\mu\nu}(p,p':\mu,\lambda)=\delta_{\mu\nu}\mu^2F^i_1(p,p':\mu,\lambda)\\+(p_\mu p_\nu+p'_\mu p'_\nu)F^i_2(p,p':\mu,\lambda)\\+k_\mu k_\nu F^i_3(p,p':\mu,\lambda)\,.\end{aligned} \tag{4.12}$$

F^i_m ($m=1$, 2, and 3) are functions of three scalars p^2, p'^2, and k^2. The renormalization conditions are necessary for F^i_1, $\partial F^i_1/\partial p^2=\partial F^i_1/\partial p'^2$, $\partial F^i_1/\partial k^2$, F^i_2, and F^i_3 at $p^2+\mu^2=p'^2+\mu^2=k=0$, and also for G^i ($i=4$, 5, and 6) at $k_\mu=0$, $p_i\,p_j=\frac{1}{3}\mu^2(1-4\delta_{ij})$. For a particular i, Eq. (4.11) expresses six (five for $i=1$, 2, and 3) equations corresponding to these renormalization conditions. The superficial degree of the divergence is negative for $G^i_{\mu\nu}$ ($i=1$, 2, and 3). The renormalization conditions are unnecessary for them. This occurs partly because the operators A_i ($i=1$, 2, and 3) are closed by themselves under renormalization. The decrease in the number of the independent renormalization conditions does not cause any trouble in the determination of α_{ij}, since the corresponding α_{ij} vanish identically. The (α_{ij}) is thus determined. At the subtraction point, the renormalization conditions are:

$$\begin{aligned}&F^1_1=-2\,,\quad F^2_1=0\,,\quad F^4_1=0\,,\\&F^5_1=2\,,\quad F^6_1=0\,,\\&\frac{\partial F^1_1}{\partial p^2}=0\,,\quad\frac{\partial F^2_1}{\partial p^2}=0\,,\quad\frac{\partial F^4_1}{\partial p^2}=0\,,\\&\frac{\partial F^5_1}{\partial p^2}=-1\,,\quad\frac{\partial F^6_1}{\partial p^2}=0\,,\\&\frac{\partial F^1_1}{\partial k^2}=0\,,\quad\frac{\partial F^2_1}{\partial k^2}=-2\,,\quad\frac{\partial F^4_1}{\partial k^2}=0\,,\\&\frac{\partial F^5_1}{\partial k^2}=\frac{\partial F^6_1}{\partial k^2}=0\,,\\&F^6_2=-1\,,\quad F^3_3=-2\,,\quad F^6_3=0\,,\end{aligned} \tag{4.13}$$

and

$$G^4_{\mu\nu}=4!\,\delta_{\mu\nu}\,,\quad G^5_{\mu\nu}=0\,,\quad G^6_{\mu\nu}=0\,. \tag{4.14}$$

In (4.13) other amplitudes or their derivatives not described here vanish identically. From (4.11) we get at the subtraction points (for a particular i):

$$F_1^i = \alpha_{ij} F_1^{jr} ,$$
$$F_2^i = \alpha_{ij} F_2^{jr} ,$$
$$F_3^i = \alpha_{ij} F_3^{jr} ,$$
$$\frac{\partial}{\partial p^2} F_1^i = \alpha_{ij} \frac{\partial}{\partial p^2} F_1^{jr} , \qquad (4.15)$$
$$\frac{\partial}{\partial k^2} F_1^i = \alpha_{ij} \frac{\partial}{\partial k^2} F_1^{jr} ,$$
$$G_{\mu\nu}^i = \alpha_{ij} G_{\mu\nu}^{jr} .$$

In order to be able to solve uniquely α_{ij}, it is necessary and sufficient that

$$F_1^{1r} F_2^{6r} F_3^{3r} \frac{\partial}{\partial p^2} F_1^{5r} \frac{\partial}{\partial k^2} F_1^{2r} G_{\mu\nu}^{4r} \neq 0 . \qquad (4.16)$$

The left-hand side of (4.16) is $2^3 4! \mu^2 \delta_{\mu\nu} + O(\lambda) + \cdots$ and is nonvanishing. (α_{ij}) is thus uniquely determined by (4.15). This justifies (4.11). Then it will be natural to define the stress tensor in terms of the new normal product:

$$\theta_{\mu\nu} = h_i N_4'[A_i(\mu, \lambda)] , \qquad (4.17)$$

where h_i $(i = 1, 2, \ldots, 6)$ are constants. Equation (4.11) then implies

$$\theta_{\mu\nu} = h_i \alpha_{ij} N_4[A_j(\bar{\mu}, \bar{\lambda})] . \qquad (4.18)$$

We see that the coefficients in (4.18) cannot be absorbed into the finite renormalizations of μ and λ. They are characteristic to the $\theta_{\mu\nu}$ vertex.[17]

If $F_{\mu\nu}^{ir}$ and $G_{\mu\nu}^{ir}$ are known, (4.13)–(4.15) give us (α_{ij}). Then the on-mass-shell renormalized stress tensor (4.17) is obtained from (4.18). Our main problem is thus solved. Note that (α_{ij}) is of the form:

$$(\alpha_{ij}) = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 & 0 & 0 \\ \alpha_{41} & \alpha_{42} & 0 & \alpha_{44} & \alpha_{45} & 0 \\ \alpha_{51} & \alpha_{52} & 0 & \alpha_{54} & \alpha_{55} & 0 \\ \alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} & \alpha_{65} & \alpha_{66} \end{bmatrix} , \qquad (4.19)$$

and that α_{ij} $(\lambda = 0) = \delta_{ij}$. From the general consideration we know that only eleven elements of (α_{ij}) are independent. Note also that a product of α's and an inverse of α are of the same form as (4.19).

V. DISCUSSION

We have learned the two possible ways to get the finite quantum corrections to the stress tensor. The first is the Feynman cutoff method, in which the Lagrangian counterterms are of the same type as those contained in the initial Lagrangian. We confirmed the finiteness of the corrections up to $O(\lambda^2)$. The second is the BPH cutoff method, in which the counter terms of special types are added to the Lagrangian. The essential difference between these two methods lies, not in the way of the regularization of the Feynman integral, but in the way of the subtraction. In the Feynman cutoff method, the cancellation of the divergences occurs only for a special value of the parameter f. In the BPH cutoff method we have made over-all subtractions for the graph which contains a stress tensor vertex. The number of the subtractions in the latter case is so chosen that the corrections become finite. This is always possible. Thus the corrections are finite by construction. At first sight it may seem that there is less arbitrariness in the Feynman cutoff method. This is not necessarily so. Remember that the divergence has appeared only in the quantum corrections. The analysis in Sec. II shows only that the method applies to a particular form of $\theta_{\mu\nu}$ only. The value of f will be determined, for example, by Schroer's condition[6] on the Callan-Symanzik function $\beta(\lambda)$. But such a choice is not compelling. So it would be preferable to have the finite corrections for the stress tensor without specifying the value of f. The BPH cutoff method is particularly suitable for this purpose.

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[14]K. Hepp, Commun. Math. Phys. 2, 301 (1966).
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