Elementary particles in a curved space. II

C. Fronsdal*
International Centre for Theoretical Physics, Trieste, Italy
and Istituto di Fisica Teorica dell'Università di Trieste, Italy
(Received 24 September 1973)

This is an attempt to develop conventional, contemporary, elementary-particle physics in a Riemannian space of constant curvature. We study the global structure of the 3 + 2 de Sitter space, which we take to mean the covering space of the hyperboloid $y_0^2 - \vec{y}^2 + y_5^2 = \rho^{-1}$ in a five-dimensional Minkowski space. This space is *not* periodic in time. A causal structure is shown to exist and the commutation relations between free fields are shown to be causal. Elementary massive particles are associated with a class of irreducible representations of the universal covering group of SO(3, 2) for which the Hamiltonian has a discrete spectrum with a lower (positive) bound. A detailed study is made of the wave functions in "momentum space" and in configuration space. Free quantum fields are introduced with the help of a discrete set of creation and destruction operators and the commutator $[\phi_0(x), \phi_0(x')]$ is calculated. An appendix describes what we think is an interesting way to realize irreducible representations of the "discrete series."

I. INTRODUCTION

Dirac,[1] in 1935, was the first to consider wave equations invariant under the groups SO(3, 2) and SO(4, 1). Schrödinger[2] derived the same equations from the point of view of general covariance. Goto[3] derived invariant equations of the Duffin-Kemmer type. Gürsey[4,5] and Gürsey and Lee[6] studied Dirac's 1935 equation from the point of view of the irreducible representations of the de Sitter group. They discuss both positive and negative constant curvature and the corresponding groups O(3, 2) and O(4, 1). Although they seem to prefer the latter, they mention the serious difficulty that arises from the indefinite spectrum of the Hamiltonian, common to all irreducible representations of this group. Quantum fields in de Sitter space were discussed by Gutzwiller[7] and by Thirring and Nachtmann,[8] who also appear to prefer O(4, 1) to O(3, 2). Finally, several people have calculated Green's functions for the lowest spin values.[9]

This paper is a direct continuation of an earlier report[10] in which our motivation for undertaking an investigation of elementary-particle physics in a space of constant curvature was given. Here we merely add that the work is of considerable pedagogical value: One finds that it is necessary to acquire a deeper understanding of conventional flat-space physics.

Our program, barely initiated in our first report and only slightly advanced in this paper, is to make essential use of the group of motions of space-time, by virtue of which the foundations exist for applying the modern approach based on the irreducible representations of this group. The group of motions has ten parameters and the infinitesimal generators can be associated with the operators of energy-momentum and angular momentum in the usual way. Consequently, our work has no close relationship to papers that deal with curved spaces in general.

The point of departure of our 1964 paper[10] was to apply Wigner's method[11] to construct the most important irreducible representations of the group SO(3, 2). This group was chosen in preference to SO(4, 1) because it has representations for which the spectrum of the Hamiltonian has a minimum. Such representations could be associated with elementary particles. The "mass" m of the particle was identified with the lowest eigenvalue of the Hamiltonian. The subspace of states with energy $P_0 = m$ was found to carry an irreducible representation of the rotation group, and this allows a natural definition of the spin s of the particle. In fact, an irreducible representation is determined up to equivalence by m and s. A complete system of basis vectors, denoted $|p, s_z\rangle$, was defined as follows. Let $\tilde{p}$ be the four-vector $(m, \vec{0})$ and let $|\tilde{p}, s_z\rangle$, with $\tilde{p}$ fixed and $s_z = -s, -s+1, \ldots, s$, be the states with energy m. Let $|p, s_z\rangle$, where $p = (p_0, \vec{p})$ and $p_0 = +(m^2 + \vec{p}^2)^{1/2}$, be the result of applying a Lorentz boost to $|\tilde{p}, s_z\rangle$. This is precisely Wigner's method, and the transformation properties of $|p, s_z\rangle$ under (homogeneous) Lorentz transformations are exactly the same as in flat space. The effect of the translation operators on $|p, s_z\rangle$ was found by an extremely simple calculation (ρ is the curvature constant):

$$P_\mu \equiv \rho^{1/2} L_{\mu 5} = p_\mu - (i\rho^{1/2}/m) p^\nu L_{\nu\mu} \,.$$

These results hold for any spin, but the present paper is devoted entirely to spinless particles.

Section II of this paper is a detailed study of the states of a free particle in the "momentum" representation. The wave functions turn out to have an interesting dual structure of distributions and analytic functions.

Section III studies the local and global geometry of $3+2$ de Sitter space-time. Contrary to what has been said,[12] space-time is not periodic in time. We make use of the usual model given by a hyperboloid in five-dimensional Minkowski space (which is periodic), but identify space-time with the covering space. The group of motions is, therefore, the universal covering group of SO(3, 2). A causal structure, with invariant distinction between past and future, is shown to exist. A geodesic coordinate system (nonglobal), as well as a global coordinate system, is introduced, and the group generators are found. The invariant wave equation is calculated from the known expression for—and the known value of—the Casimir operator. It coincides with the covariant Klein-Gordon equation because the space-time wave functions are taken to transform like scalar functions under de Sitter transformations.

In Sec. IV a complete set of space-time wave functions are calculated. The transformation of wave functions between p space and x space (generalized Fourier transform) is found.

Section V deals with free fields; their canonical quantization is based on a discrete orthonormal system of basis states. The commutator $[\phi_0(x), \phi_0(x')]$ is calculated and shown to have correct causal properties and the correct flat-space limit.

II. STATES AND WAVE FUNCTIONS IN p SPACE

In our first report[10] we introduced a set of normalizable but not mutually orthogonal basis vectors $|p\rangle$, the main properties of which were recalled in the Introduction. The action of the SO(3, 2) generators on $|p\rangle$ is noted in Table I. Relatively to this basis we now define the wave function $\Psi(p)$ of a state Ψ by the expansion

$$\Psi = \int \Psi(p)|p\rangle(dp)_+ \tag{2.1}$$

with the Lorentz-invariant volume element

$$(dp)_+ \equiv d^4p\,\delta(p^2-m^2)\,\theta(p_0)\,.$$

Applying $L_{\mu\nu}$ and P_μ to (2.1), using the just-quoted rules for acting with the generators on $|p\rangle$, and transferring this action to $\Psi(p)$ by partial integration, we find the rules for applying SO(3, 2) generators to $\Psi(p)$. The result is entered in Table I.

In order to discover the invariant normalization of basis states and wave functions, we consider the norm of (2.1)

$$\|\Psi\|^2 = \int \Psi^*(p)\langle p|p'\rangle\Psi(p')(dp)_+(dp')_+\,. \tag{2.2}$$

The kernel $\langle p|p'\rangle$, if it exists, is determined up to a constant factor by the invariance of (2.2) under the de Sitter group; that is, by the Hermiticity of the generators with respect to this norm. Lorentz invariance requires that $\langle p|p'\rangle$ be a function $K(p\cdot p')$ and Hermiticity of P_μ is equivalent to the differential equation

$$E_0K + (p\cdot p' + m^2)K' = 0\,. \tag{2.3}$$

We may choose the integration constant to suit our convenience, and take

$$\langle p|p'\rangle = (4\pi)^{-1}\left(\frac{2m^2}{p\cdot p'+m^2}\right)^{E_0}. \tag{2.4}$$

It will be seen that the norm (2.2) exists for every state in a Hilbert space $\mathcal{H}$ in which our unitary irreducible representation of the de Sitter group is realized. However, $\mathcal{H}$ is not the whole space defined by (2.2).

We next determine the wave functions $\Psi_{ELM}(p)$

TABLE I. A collection of results. Notation: $E_0=\rho^{-1/2}m$ is the lowest eigenvalue of $L_{05}=\rho^{-1/2}P_0$; it was earlier called $\overline{m}$.

$$\Psi = \int \Psi(p)|p\rangle(dp)_+ = \int \psi(p)|p)(dp)_+ = \int \psi(x)|x\rangle(dx)$$

$$\psi(p') = \int \langle p'|p\rangle\Psi(p)(dp)_+\,, \quad |p'\rangle = \int \langle p|p'\rangle|p)(dp)_+$$

$$\|\Psi\|^2 = \int \psi^*(p)\Psi(p)(dp)_+ = \int \Psi^*(p)\psi(p)(dp)_+ = \int |\psi(x)|^2(dx)$$

$$L_{\mu\nu} = -i(p_\mu\partial_\nu - p_\nu\partial_\mu) \quad \text{on } |p\rangle,\ |p),\ \Psi^*(p),\ \psi^*(p)$$

$$= +i(p_\mu\partial_\nu - p_\nu\partial_\mu) \quad \text{on } \langle p|,\ (p|,\ \Psi(p),\ \psi(p)$$

$$P_\mu = p_\mu - \frac{1}{E_0}p^\nu(p_\nu\partial_\mu - p_\mu\partial_\nu) \quad \text{on } |p\rangle,\ \langle p|,\ \psi(p),\ \psi^*(p)$$

$$= p_\mu + \frac{1}{E_0}(p_\nu\partial_\mu - p_\mu\partial_\nu)p^\nu \quad \text{on } |p),\ (p|,\ \Psi(p),\ \Psi^*(p)$$

$$L_{\alpha\beta} = i(y_\alpha\partial_\beta - y_\beta\partial_\alpha)\,, \quad P_\mu = \rho^{1/2}L_{\mu 5} \qquad \text{on } \psi(y) = \psi(x) = \psi(\vec{r}, t)$$

$$L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\,, \quad P_\mu : \text{Eq. (3.10)}, \quad y_\mu = f(x^2)x_\mu$$

$$Q = \rho E_0(E_0-3) = m^2(1-3/E_0) = -\Box = -(-g)^{-1/2}\partial_\mu g^{\mu\nu}(-g)^{1/2}\partial_\nu$$

$$= \rho\,\partial_\alpha y_\alpha y_\beta\partial_\beta - \partial_\alpha\partial_\alpha$$

$$(dy) = 2\rho^{-1/2}\delta(y^2-\rho^{-1})d^5y = (dx) = (-g)^{1/2}d^4x = d^3r\,dt$$

$$\langle x|p) = (p|x\rangle^* = e_0(\rho^{1/2}y_5 + ip\cdot y/E_0)^{E_0-3}\,, \quad e_0 = \frac{(E_0-1)(E_0-2)(E_0-\frac{3}{2})}{(2\pi E_0)^2B(E_0,\frac{3}{2})^{1/2}}$$

$$\langle x|p\rangle = \langle p|x\rangle^* = e_0'(\rho^{1/2}y_5 + ip\cdot y/E_0)^{-E_0}\,, \quad e_0' = \frac{\rho}{2\pi^2}B(E_0,\tfrac{3}{2})^{1/2}$$

$$\langle p|p'\rangle = \frac{1}{4\pi}\left(\frac{2m^2}{pp'+m^2}\right)^{E_0}, \quad \langle x|x'\rangle : \text{Eq. (4.5)}$$

$$g_{\mu\nu}(x) = f^2\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}\right) + \frac{(f+2x^2f')^2}{1-\rho x^2f^2}\,\frac{x_\mu x_\nu}{x^2}\,, \quad y_\mu = f(x^2)x_\mu$$

of the basis

$$|I\rangle = \int \Psi_I(p)|p\rangle(dp)_+, \quad I = ELM$$

in which L_{05}, $\vec{L}^2$, and L_{12} are diagonal:

$$(L_{05}-E)|ELM\rangle = (L_{12}-M)|ELM\rangle = 0\,,$$
$$[\vec{L}^2 - L(L+1)]|ELM\rangle = 0\,. \tag{2.5}$$

This turns out to be somewhat complicated, so we have relegated the details to the Appendix. The result is as follows.

(a) When $E_0 > 0$ and $E_0 \neq \frac{1}{2}$, the spectrum of ELM in an irreducible representation is given by

$$E = E_0 + L + 2K, \quad K = 0, 1, 2, \ldots,$$
$$L = 0, 1, 2, \ldots, \quad M = -L, -L+1, \ldots, L\,. \tag{2.6}$$

The basis vectors, normalized in the sense that

$$\langle I|I'\rangle = \int \Psi_I^*(p)\langle p|p'\rangle \Psi_{I'}(p')(dp)_+(dp')_+$$
$$= \delta_{II'} = \delta_{EE'}\delta_{LL'}\delta_{MM'}\,, \tag{2.7}$$

are

$$\Psi_{ELM}(p) = (E, L)\left(\frac{p_0+m}{2m}\right)^{E_0}\Delta^N(p)Y_{LM}(\hat{p})\,. \tag{2.8}$$

where $N \equiv E - E_0$, $\hat{p}$ is the direction of $\vec{p}$, $p_0 = +(\vec{p}^2+m^2)^{1/2}$, and $\Delta^N(p)$ is the distribution[13] defined by

$$\int A(p)\Delta^N(p)p^2dp/2p_0 = \frac{1}{N!}\left(\frac{p_0(p_0+m)}{m}\frac{\partial}{\partial p}\right)^N A(p)\bigg|_{p=0}, \tag{2.9}$$

where $p = |\vec{p}|$ and the normalizing coefficient is

$$(E, L) = \left[\frac{(E_0-1)!(E_0-\frac{3}{2})!(L+K+\frac{1}{2})!K!}{(\frac{1}{2})!(L+K+E_0-1)!(K+E_0-\frac{3}{2})!}\right]^{1/2}. \tag{2.10}$$

A dual space of continuous functions $\psi_I(p)$ is defined through the transformation

$$\psi(p) \equiv \int \langle p|p'\rangle\Psi(p')(dp')_+\,. \tag{2.11}$$

In particular, the transform of (2.8) is

$$\psi_{ELM}(p) = (E, L)^{-1}\left(\frac{2m}{p_0+m}\right)^{E_0}\left(\frac{p}{p_0+m}\right)^N Y_{LM}(\hat{p})\,. \tag{2.12}$$

The norm (2.2) is then simply

$$\|\Psi\|^2 = \int \psi^*(p)\Psi(p)(dp)_+ \tag{2.13}$$

and

$$\int \psi_I^*(p)\Psi_{I'}(p)(dp)_+ = \delta_{II'} \tag{2.14}$$

However, it is impossible to invert (2.11), and one cannot express the norm in terms of an integral over $\psi(p)$ alone.[14]

(b) In the case $E_0 = \frac{1}{2}$ the above representation becomes reducible, since it turns out that the subspace $K = 0$ is invariant. (See the last paragraph of the Appendix.) In this subspace we find one of Dirac's "remarkable" representations.[15] The above formulas are valid if K is replaced by 0, and (E, L) by $(2L+1)^{1/2}$.

As we noted above, following Eq. (2.4), the Hilbert space $\mathcal{H}$ of the irreducible representation is not the whole L^2 space defined by (2.2). A rigorous and safe procedure is to define $\mathcal{H}$ as the space of distributions

$$\Psi(p) = \sum_I C_I\Psi_I(p)\,, \tag{2.15}$$

or as the space of functions[16]

$$\psi(p) = \sum_I C_I\psi_I(p)\,, \tag{2.16}$$

with square-summable coefficients

$$\sum_I |C_I|^2 < \infty\,. \tag{2.17}$$

or, better still, as the l^2 space of the ∞-tuples $\{C_I\}$. It is easy to show that the integrals (2.2) and (2.13) exist if (2.17) is satisfied and that, in this case,

$$\|\Psi\|^2 = \sum_I |C_I|^2\,. \tag{2.18}$$

We also note that the relation

$$\langle p|p'\rangle = \sum_I \psi_I(p)\psi_I^*(p')\,, \tag{2.19}$$

which is easily derived formally from (2.18), is meaningful beyond the distribution sense, since the sum is absolutely convergent when $\mathrm{Re}\,p_\mu$ and $\mathrm{Re}\,p'_\mu$ are both inside the forward cone.

Finally, one may attempt to introduce basis states $|p)$ that have the same relation to $\psi(p)$ as $|p\rangle$ has to $\Psi(p)$ by writing

$$\Psi = \int \psi(p)|p)(dp)_+\,. \tag{2.20}$$

Inserting (2.11) we find

$$|p'\rangle = \int \langle p|p'\rangle|p)(dp)_+\,. \tag{2.21}$$

If we take $\Psi = |I\rangle$ in (2.20), multiply by $\psi_I^*(p')$ and sum over I, and use (2.19) and (2.21), we obtain

$$|p) = \sum_I \psi_I^*(p)|I\rangle\,. \tag{2.22}$$

Notwithstanding the formal nature of the derivation, this result is easily justified in terms of the original definition of $|p\rangle$. In fact, (2.22) is covariant (see Table I) so it is sufficient to verify its validity for $\vec{p}=0$. But then (2.22) reduces to $|\tilde{p}\rangle = (4\pi)^{-1/2}|E_0, 0, 0\rangle$, which agrees with our definition of this state.

To complete the bra-ket formalism we note that

$$\psi_I(p) = \langle p|I\rangle, \tag{2.23}$$

$$\Psi_I(p) = (p|I\rangle. \tag{2.24}$$

III. GEOMETRY

In view of the interpretation of P_μ as translation generators, it seems natural to introduce configuration-space coordinates as parameters of translation. Let $\psi(x)$ be a scalar field and suppose that $P_\mu\psi(x)$ has been defined, then it may happen that[17]

$$\psi(x) = e^{-ix_\mu P_\mu}\psi(0), \quad -\infty < x_\mu < +\infty. \tag{3.1}$$

In this case we say that the x_μ are geodesic coordinates. Unfortunately we shall see that such coordinates are not global; nevertheless, they are of some use.

It is worth emphasizing the infinite range of x_0. In the case of a pure time translation,

$$\psi(x_0, \vec{0}) = \exp(-i\rho^{1/2}x_0 L_{05})\psi(0) \tag{3.2}$$

and, in particular,

$$\psi(2\pi\rho^{-1/2}, \vec{0}) = \exp(-2\pi i E_0)\psi(0), \tag{3.3}$$

since the eigenvalues of L_{05} differ from E_0 by integers. If E_0 is integer, then $\psi(2\pi\rho^{-1/2}, \vec{0}) = \psi(0)$, and if only integer values of E_0 were allowed, then the history of the universe would be periodic with period $2\pi\rho^{-1/2}$. But if different fields have different and incommensurable values of E_0, then the periodicity is lost. We wish to treat this more general case. The group of motions is therefore the universal covering group of SO(3, 2).

It would be convenient to interpret our Riemannian space of constant curvature as the hyperboloid

$$y_\alpha y_\alpha \equiv y_5^2 + y_0^2 - \vec{y}^2 = \rho^{-1} \tag{3.4}$$

imbedded in a (3 + 2)-dimensional Minkowski space. This is not possible globally because the time translation $\exp(-2\pi i L_{05})$ carries a point on the hyperboloid into itself. To remedy this we replace the hyperboloid by a covering space—an onion with countably infinitely many leaves—so that $\exp(-2\pi i L_{05})$ carries a point on leaf number n into a point with the same coordinates on leaf number $n+1$. A point in Riemannian space should therefore be denoted (y, n), $n = 0, \pm1, \pm2, \ldots$. A global set of coordinates $\vec{r}, t$ is given by

$$\begin{aligned} &\vec{r} = \vec{y}, \\ &\sin\rho^{1/2}t = -y_0/R, \\ &R \equiv (y_0^2 + y_5^2)^{1/2} = (\rho^{-1} + \vec{r}^2)^{1/2}, \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} &\rho^{1/2}t = \tau + 2\pi n, \\ &-\pi \le \tau < \pi, \quad n = 0, \pm1, \ldots. \end{aligned} \tag{3.6}$$

We shall show that a causal structure exists on the onion. The only continuous two-point invariant is[18]

$$z(x, x') \equiv \rho y_\alpha y'_\alpha = 1 - \tfrac{1}{2}\rho(y_\alpha - y'_\alpha)^2. \tag{3.7}$$

We shall say that the separation between two points is

spacelike if $|z| > 1$,

timelike if $|z| < 1$,

and lightlike if $|z| = 1$. Taking x' to be any fixed point, we find the regions that are spacelike or timelike with respect to it. The situation is illustrated in Fig. 1. As we see, there are two disconnected spacelike regions on the hyperboloid and infinitely many on the onion. The timelike regions are (barely) connected. We have named the various timelike regions by the leaf number n and the invariant sign function ϵ, defined for $|z| < 1$, by

$$\epsilon(x, x') = \mathrm{sgn}[\sin\rho^{1/2}(t - t')]. \tag{3.8}$$

Among the spacelike regions, we may distinguish two types:

spacelike-even: $z > 1$,

spacelike-odd: $z < -1$.

The former are identified on the hyperboloid with the set of spacelike geodesics through x'; for these the leaf number is defined and is equal to the leaf numbers of the timelike regions on either side. The spacelike-odd regions are identified on the hyperboloid with the set of spacelike geodesics through the point that is antipodal to x'. The leaf number is not defined for these regions. To identify a particular spacelike-odd region we may give the leaf numbers of the pair of timelike regions adjoining it (see Fig. 1).

We shall determine the relationship between the hyperbolic coordinates y_α and the geodesic coordinates defined by (3.1). Since x_μ is a Lorentz four-vector we must have

$$y_\mu = f(x^2)x, \quad y_5 = (\rho^{-1} - x^2 f^2)^{1/2}. \tag{3.9}$$

The generators P_μ are

$$P_\mu = i\rho^{1/2}(y_\mu \partial_5 - y_5 \partial_\mu)$$
$$= -iy_5\left[f^{-1}\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}\right) + \frac{1}{f + 2x^2 f'}\frac{x_\mu x_\nu}{x^2}\right]\frac{\partial}{\partial x^\nu}\,. \tag{3.10}$$

According to (3.1),

$$x_\mu \partial_\mu = -ix_\mu P_\mu\,,$$

or

$$-(1 - \rho x^2 f^2)^{1/2} = f + 2x^2 f'\,,$$

and this fixes $f(x^2)$ up to an irrelevant constant: $f = -\sin\lambda/\lambda$, where $\lambda = (\rho x^2)^{1/2}$. Thus

$$y_\mu = -x_\mu \sin\lambda/\lambda, \quad y_5 = \rho^{-1/2}\cos\lambda\,. \tag{3.11}$$

The geodesics through the origin are the orbits of the one-parameter groups of translations. According to (3.1), the parameters associated with a one-parameter translation group,

$$x_\mu = \sigma u_\mu\,, \quad -\infty < \sigma < \infty, \quad u_\mu \text{ fixed}\,, \tag{3.12}$$

are precisely the coordinates of the associated geodesic. According to whether $u^2 > 0/=0/<0$, we have a timelike/lightlike/spacelike geodesic.

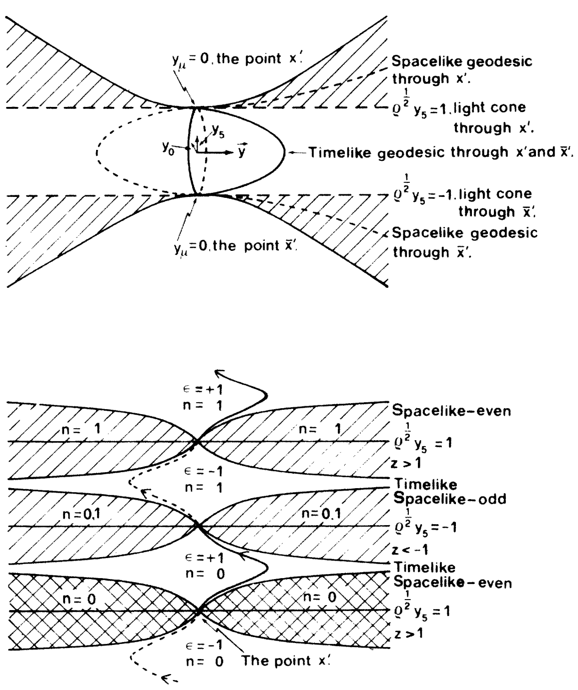

FIG. 1. View of the hyperboloid and the covering space. The space is homogeneous and the indicated structure refers to an arbitrary point x' selected as origin. Regions that are spacelike relative to x' are shaded. The timelike geodesic in the upper figure is the winding curve in the lower diagram. Geodesics through x' cover the unshaded (timelike) and the cross-hatched (spacelike) regions.

Equation (3.11) shows that (3.12) is the equation for a two-dimensional plane through $y_\alpha = 0$, hence the geodesics are just the intersections of the hyperboloid with planes passing through the point $y_\alpha = 0$. In the case of timelike geodesics, the intersection is a closed curve, so that a timelike geodesic is closed on the hyperboloid (but open on the onion). In the case of spacelike geodesics the intersection consists of two disconnected open lines, one passing through $y_\mu = 0$, $y_5 = 1$ and the other going through $y_\mu = 0$, $y_5 = -1$. The region defined by the set of all spacelike geodesics through the latter point is inaccessible by means of geodesics originating at the former; hence geodesic coordinates are not global on the hyperboloid,[19] let alone on the covering space.

Explicit expressions for the generators may easily be obtained in geodesic coordinates and in the coordinates (3.5) by means of the formula

$$L_{\alpha\beta} = i(y_\alpha \partial_\beta - y_\beta \partial_\alpha)\,. \tag{3.13}$$

The results are listed in Table I. The metric tensor is

$$g_{\mu\nu} = \frac{\partial y_\alpha}{\partial x^\mu}\frac{\partial y_\alpha}{\partial x^\nu}\,. \tag{3.14}$$

Explicit expressions are given in Table I. The invariant volume element is found by effecting the change of variables from y coordinates, in terms of which the form is self-evident, to x coordinates, with the result

$$(dy) \equiv 2\rho^{-1/2} d^5 y\, \delta(y^2 - \rho^{-1})$$
$$= (-g)^{1/2} d^4 x \equiv (dx)\,. \tag{3.15}$$

The invariant Laplace-Beltrami operator is the same as the Casimir operator of SO(3, 2),[20]

$$Q \equiv \tfrac{1}{2}\rho L_{\alpha\beta} L_{\alpha\beta} = P_\mu{}^2 + \tfrac{1}{2}\rho L_{\mu\nu} L_{\mu\nu} = -\Box\,, \tag{3.16}$$

$$\Box \equiv (-g)^{-1/2}\partial_\mu g^{\mu\nu}(-g)^{1/2}\partial_\nu\,. \tag{3.17}$$

The wave functions of our irreducible representation must therefore satisfy the free wave equation

$$\left[\Box + m^2\left(1 - \frac{3}{E_0}\right)\right]\psi(x) = 0\,. \tag{3.18}$$

IV. WAVE FUNCTIONS IN x SPACE AND FOURIER TRANSFORMS

Wave functions are determined most easily in terms of hyperbolic coordinates. Since it happens that the basis functions are square-integrable,[21] the calculation turns out to be quite standard and we give only the results.[22] In order to avoid ambiguities we shall use global coordinates.

The simultaneous eigenstates of L_{05}, $\vec{L}^2$, and L_{12} are, for $E_0 > \frac{3}{2}$,

$$\psi_{ELM}(\vec{r},t)=(E,L)^{-1}\rho[\pi B(\tfrac{3}{2},E_0-\tfrac{3}{2})]^{-1/2}Y_{LM}(\hat{r})\times e^{-i\rho^{1/2}Et}(\rho^{1/2}R)^{-E}(\rho^{1/2}r)^{L+2K}\times{}_2F_1(-K,-K-L-\tfrac{1}{2};E_0-\tfrac{1}{2};-1/\rho r^2)\,. \tag{4.1}$$

The range of K, L, M is given by (2.6), and R was defined in (3.5). The normalizing coefficient (E, L) was given by (2.10) and the over-all normalization is such that

$$\int d^3r\int dt\;\psi_I^*(\vec{r},t)\psi_{I'}(\vec{r},t)=\delta_{II'}\,. \tag{4.2}$$

(In these coordinates $-g=1$.) There is no dual structure such as was encountered in p space, and no distributions appear. Note that integrals like (4.2) are always understood to cover the hyperboloid only, not the covering space. The integral is independent of the leaf number n, so it cannot exist if extended to an infinite number of leaves.

When $0<E_0\leq\frac{3}{2}$, the functions (4.1) are not normalizable in the sense (4.2).[23] In the special case of the singleton representation, $E_0=\frac{1}{2}$, K takes the value zero only, and the basis functions are simply, up to an arbitrary common factor,

$$\psi_{ELM}(\vec{r},t)\sim(2L+1)^{-1/2}Y_{LM}(\hat{r})\times e^{-i\rho^{1/2}Et}(\rho^{1/2}R)^{-E}r^L\,, \tag{4.3}$$

with $E=L+\frac{1}{2}$.

We shall need to know the function defined by

$$\langle x|x'\rangle=\sum_I\psi_I(x)\psi_I^*(x')\,. \tag{4.4}$$

This function is an invariant and depends on the invariants z,ϵ,n only. We may evaluate the sum in the case $\vec{r}=\vec{r}\,'=0$ and obtain the general expression by expressing the result in terms of the invariants. The series, which in this way becomes hypergeometric, converges in the entire z plane, cut from -1 to $+1$, and the result is

$$\langle x|x'\rangle=(2\pi)^{-2}\rho^2B(\tfrac{3}{2},E_0-\tfrac{3}{2})[z+(z^2-1)^{1/2}]^{-E_0}\times{}_2F_1\left(E_0,\tfrac{3}{2};E_0-\tfrac{1}{2};\frac{z-(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right). \tag{4.5}$$

In addition to the cut from -1 to $+1$ introduced by the hypergeometric series, we need another cut from $-\infty$ to $+1$ due to the factor $[\;]^{-E_0}$. The square root is defined by

$$(z^2-1)^{1/2}=\begin{cases}\pm|z^2-1|^{1/2}, & z\gtrless{}^{1}_{-1}\\ +i|1-z^2|^{1/2}\epsilon(x,x'), & |z|<1\,.\end{cases} \tag{4.6}$$

When z is real and >1, the phase of the factor $[\;]^{-E_0}$ is

$$e^{-2\pi inE_0}, \tag{4.7}$$

where n is the leaf number (relative to x') of the spacelike-even region to which x belongs.

The first Riemann sheet—the cuts being drawn as in Fig. 2—corresponds to leaf number $n=0$. The details of the correspondence are the following:

$$\begin{array}{lll} z>1, & \text{spacelike-even}, & n=0\\ |z|<1, & \text{timelike}, & n=0\\ \quad \mathrm{Im}z>0, & \epsilon=+1 & \\ \quad \mathrm{Im}z<0, & \epsilon=-1 & \\ z<-1, & \text{spacelike-odd} & \\ \quad \mathrm{Im}z>0, & n=0,1 & \\ \quad \mathrm{Im}z<0, & n=-1,0\,. & \end{array} \tag{4.8}$$

The rest of the manifold is reached by analytic continuation across the line $z<-1$. The Riemann sheet that is reached by descending across $z<-1$ corresponds to the next higher leaf number.

Another formula for $\langle x|x'\rangle$ is

$$\langle x|x'\rangle=-(\rho^2/4\pi^3)(E_0-\tfrac{3}{2})(z^2-1)^{-1/2}\times Q^1_{E_0-2}(z)e^{-2\pi inE_0}\,. \tag{4.9}$$

To obtain $\langle x|x'\rangle$ in a spacelike-odd region with leaf number $(k, k+1)$, take either $k=n$, $\mathrm{Im}z>0$ or $k+1=n$, $\mathrm{Im}z<0$.

Next we shall determine the transforms that play the role of Fourier transforms in de Sitter space. We look for functions[18] $\langle x|p)$ and $\langle x|p\rangle$, such that

$$\psi_I(x)=\int\langle x|p)\psi_I(p)(dp)_+=\int\langle x|p\rangle\Psi_I(p)(dp)_+\,, \tag{4.10}$$

where $\Psi_I(p)$, $\psi_I(p)$, and $\psi_I(x)$ were given by (2.8), (2.12), and (4.1), respectively. If such functions exist, then they can be determined up to a constant factor by noting that (when $E_0>\frac{3}{2}$)

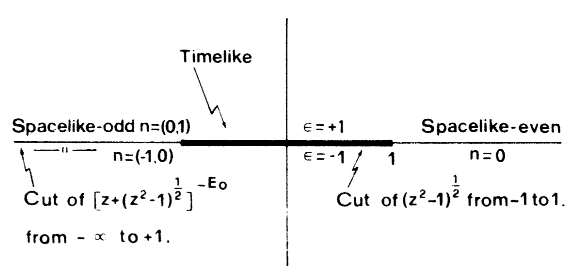

FIG. 2. The complex z plane, showing the sheet $n=0$, with the cuts of the function $K(x, x')$.

$$\|\psi\|^2 = \int |\psi(x)|^2(dx)$$
$$= \int \psi^*(x)\langle x|p\rangle\psi(p)(dp)_+$$
$$= \int \psi^*(x)\langle x|p\rangle\Psi(p)(dp)_+ \quad (4.11)$$

is invariant. By the same procedure that was used [following (2.2)] to calculate $\langle p|p'\rangle$ we easily discover that, for example,

$$\langle x|p\rangle \sim (\rho^{1/2}y_5 + ip\cdot y/E_0)^{E_0-3}. \quad (4.12)$$

The constant of proportionality is found by integrating (4.10). The final results are compiled in Table I. Applications follow easily; for example,

$$\psi(p) = \int \langle p|x\rangle\psi(x)(dx), \quad (4.13)$$

$$\langle x|p'\rangle = \int \langle x|p\rangle\langle p|p'\rangle(dp)_+, \quad (4.14)$$

$$\langle x|x'\rangle = \int \langle x|p\rangle\langle p|x'\rangle(dp)_+. \quad (4.15)$$

The integrals over x converge if $E_0 > \frac{3}{2}$; the p integrals converge almost everywhere. This is evident on inspection.

The notation that we have used—$\langle x|p\rangle$, $\langle x|x'\rangle$, etc.—is justified by defining the basis states $|x\rangle$ by

$$|x\rangle = \sum_I \psi_I^*(x)|I\rangle. \quad (4.16)$$

Using the orthogonality condition (4.2) we find that

$$|I\rangle = \int \psi_I(x)|x\rangle(dx). \quad (4.17)$$

Like the basis states $|p\rangle$, the states $|x\rangle$ are normalizable and complete, but not orthogonal.

V. FREE FIELDS

Consider the normalized basis states

$$|ELM\rangle = \int \Psi_{ELM}(p)|p\rangle(dp)_+$$
$$= \int \psi_{ELM}(x)|x\rangle(dx) \quad (5.1)$$

and recall the inverse formulas

$$|p\rangle = \sum_I \psi_I^*(p)|I\rangle,$$
$$|x\rangle = \sum_I \psi_I^*(x)|I\rangle. \quad (5.2)$$

Let $|\Omega\rangle$ denote the vacuum state, and define creation and destruction operators by

$$|I\rangle = a_I^*|\Omega\rangle, \quad a_I|\Omega\rangle = 0 \quad (5.3)$$

$$[a_I, a_{I'}^*] = \delta_{II'},$$
$$[a_I, a_{I'}] = [a_I^*, a_{I'}^*] = 0. \quad (5.4)$$

A Hermitian free-field operator is defined by

$$\Phi_0(x)|\Omega\rangle = |x\rangle \quad (5.5)$$

or

$$\Phi_0(x) = \sum_I [\psi_I^*(x)a_I^* + \psi_I(x)a_I]. \quad (5.6)$$

The commutation relations follow from (5.3) and (5.4):

$$[\Phi_0(x), \Phi_0(x')] = \langle x|x'\rangle - \langle x'|x\rangle$$
$$\equiv -i\Delta_{m,\rho}(x, x'). \quad (5.7)$$

If the leaf number n of x, relative to x', is zero (see Fig. 1), then according to (4.9)

$$\Delta_{m,\rho}(x, x') = -i(\rho^2/4\pi^3)(E_0 - \tfrac{3}{2})$$
$$\times \mathrm{Disc}(z^2-1)^{-1/2}Q^1_{E_0-2}(z).$$

The function $(z^2-1)^{-1/2}Q^1_{E_0-2}(z)$ has simple poles at $z = \pm 1$ that give rise to δ-function contributions to the discontinuity. The discontinuity vanishes when $z > 1$. When $-1 < z \leqslant 1$ we get

$$\Delta_{m,\rho}(x, x') = -(\rho^2/4\pi^2)(E_0 - \tfrac{3}{2})$$
$$\times[(1-z^2)^{-1/2}P^1_{E_0-2}(z) + \delta(z-1)]$$
$$\times \epsilon(x, x'). \quad (5.8)$$

The δ functions appears on every Riemann sheet at $z = \pm 1$; this reflects the situation illustrated in Fig. 1. All geodesics through x' meet again twice for each circumnavigation of the hyperboloid.

As required by causality, $\Delta_{m,\rho}(x, x')$ vanishes when $n = 0$ and $z > 1$, that is, when x is in the region that is shown cross-hatched in Fig. 1. The commutator does not vanish when x is in any of the other "spacelike" regions. This result is physically reasonable. If x is anywhere outside the cross-hatched region, then a signal can reach x from x', or vice versa, by following a path that is piecewise timelike geodesic.

In the flat-space limit, $\rho \to 0$, $E_0 \to \infty$ with $m = \rho^{1/2}E_0$ fixed,

$$z \to 1 - \tfrac{1}{2}\rho\lambda^2, \quad (5.9)$$

$$\rho E_0 P^1_{E_0-2}(z) \to -m^2 J_1(m\lambda), \quad (5.10)$$

and

$$\Delta_{m,\rho}(x, x') \to \left(\frac{\rho^{1/2}}{2\pi}\right)(2m)\Delta_m(x - x'), \quad (5.11)$$

where

$$\Delta_m(x-x') = \left(\frac{m}{4\pi\lambda} J_1(m\lambda) - \frac{1}{2\pi}\delta(\lambda^2)\right)\epsilon(x-x') \tag{5.12}$$

is the familiar flat-space commutator function. The factors $(\rho^{1/2}/2\pi)$ and $(2m)$ are easily recognized as being due to the unconventional normalization adopted here [see Eq. (4.2)].

VI. PROBLEMS

Before we can define interpolating fields, it is necessary to understand what it means to go off the mass shell. A study of quantum mechanics in de Sitter space should be instructive.

"Massless" particles are of particular interest. Since $m=\rho^{1/2}E_0$, any particle is massless in the flat-space limit unless E_0 is of the order $\rho^{-1/2}$. Any "small" value of E_0, fixed in the process of letting ρ tend to zero, defines a massless particle. The values $E_0=1$ and $E_0=2$ are associated with conformal invariance, $E_0=\frac{1}{2}$ with the peculiarly degenerate Dirac representation.

Before we can study photons and gravitons we must introduce spin. As we explained in the Introduction, no difficulties arise. (It is curious that "mass" and "spin" are well defined even for "massless" particles.)

ACKNOWLEDGMENTS

I wish to thank Professor L. Castell, Professor M. Flato, Professor A. O. Barut, Professor R. Raczka, Professor I. Bialynicki-Birula, Professor J. Plebansky, and Dr. R. Haugen for helpful and stimulating discussions.

I should also like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

APPENDIX: BASIS FUNCTIONS IN p SPACE

Consider functions $f(b)$ defined on the real cone,

$$b_\alpha b_\alpha \equiv b_0^2 + b_5^2 - \vec{b}^2 = 0, \tag{A1}$$

with fixed degree of homogeneity

$$b_\alpha \partial_\alpha \ln f = n = \text{constant}. \tag{A2}$$

The de Sitter group is realized as the group of transformations

$$T(\Lambda):\ f(b) \to [D(\Lambda)f](b) = f(\Lambda^{-1}b). \tag{A3}$$

For infinitesimal transformations,

$$(\Lambda^{-1}b)_\alpha = b_\alpha + \theta_{\alpha\beta} b_\beta, \quad D(\Lambda) = 1 + \tfrac{1}{2} i\theta_{\alpha\beta} L_{\alpha\beta}, \tag{A4}$$

with $\partial_\alpha = \partial/\partial b^\alpha$ and

$$L_{\alpha\beta} = i(b_\alpha\partial_\beta - b_\beta\partial_\alpha). \tag{A5}$$

The Casimir operator is

$$Q = \tfrac{1}{2} L_{\alpha\beta} L_{\alpha\beta}. \tag{A6}$$

Using (A2) we find

$$Q = b_\alpha \partial_\alpha (b_\beta \partial_\beta + 3) = n(n+3). \tag{A7}$$

Applying Q to be ground state we get instead $Q = E_0(E_0-3)$; hence we have two possibilities:

$$\text{Case 1: } n=-E_0, \quad \text{Case 2: } n=E_0-3. \tag{A8}$$

Case 1. The simultaneous eigenfunctions of L_{05}, $\vec{L}^2$, and L_{12}, with eigenvalues E, $L(L+1)$, and M, are

$$f_{ELM}(b) = (E,L)^{-1} b^{-E_0} \left(\frac{b_0 - ib_5}{b_0+ib_5}\right)^{E/2} Y_{LM}(\hat{b}), \tag{A9}$$

where $\hat{b} = \vec{b}/b$ and $b^2 = \vec{b}^2 = b_0^2 + b_5^2$. The normalizing coefficients are fixed by requiring that these functions form a basis for a Hermitian matrix representation of the algebra. (The Hermitian matrices are given at the end of this appendix.) An absolutely straightforward calculation gives for (E,L) the expression given in the text by Eq. (2.10). The ranges of ELM are as follows. Let $K \equiv \frac{1}{2}(E - E_0 - L)$ and let $S_\pm$ be the index sets

S_+: $K=0,1,\ldots$; $L=0,1,\ldots$; $M=-L,\ldots,L$,

S_-: $K=-1,-2,\ldots$; $L=0,1,\ldots$; $M=-L,\ldots,L$.

Let $V_\pm$ be the linear vector spaces

$$f(b) \in V_\pm \Leftrightarrow f(b) = \sum_{S_\pm} C_{ELM} f_{ELM}(b), \tag{A10}$$

where a finite number of the complex coefficients are nonzero. If $E_0 \neq \frac{1}{2}$ we have (Case 1): The space V_+ is invariant and irreducible. We turn it into a Hilbert space $\mathcal{H}_1$ by completion with respect to the inner product

$$(f,f') = \sum_{S_+} C^*_{ELM} C'_{ELM}. \tag{A11}$$

(The space V_- is irrelevant, but compare Case 2.) In this case we may be allowed the usual abuse of language that confuses an element f with a function $f(b)$,

$$f = f(b) = \sum_{S_+} C_{ELM} f_{ELM}(b). \tag{A12}$$

Case 2. Instead of (A9),

$$F_{ELM}(b) = (E,L) b^{E_0-3} \left(\frac{b_0-ib_5}{b_0+ib_5}\right)^{E/2} Y_{LM}(\hat{b}). \tag{A13}$$

Let $S_\pm$ be as above and define $W_\pm$ by

$$F(b)\in W_\pm \Leftrightarrow F(b)=\sum_{S_\pm} C_{ELM}F_{ELM}(b)\,. \tag{A14}$$

In this case W_-, rather than W_+, is invariant; this complicates matters. If $E_0 \neq \frac{1}{2}$, we have (Case 2): The space $(W_+\oplus W_-)/W_-$ is invariant and irreducible. An element F of this space is an equivalence class of functions belonging to $W_+\oplus W_-$. Each equivalence class F contains one and only one function $F(b)$ that belongs to W_+; hence we have a natural bijection $F\Leftrightarrow F(b)$. A Hilbert space $\mathcal{K}_2$ of equivalence classes—and at the same time a Hilbert space $\mathcal{K}_2'$ of functions $F(b)$—is obtained by completion with respect to the inner product

$$(F, F')=\sum_{S_+} C^*_{ELM}C'_{ELM}\,. \tag{A15}$$

Here C_{ELM} are the expansion coefficients of the function $F(b)$ that is associated with the equivalence class F,

$$F\Leftrightarrow F(b)=\sum_{S_+} C_{ELM}\,F_{ELM}(b)\,. \tag{A16}$$

(The situation can be described in simpler terms as follows: Some of the generators, when applied to a basis function with $K=0$, lead out of W_+. The unwanted terms belong to W_- but, since W_- is invariant, we obtain an irreducible representation by simply ignoring such terms.)

Of course the representations obtained in $\mathcal{K}_1$ and in $\mathcal{K}_2$ are equivalent.

We wish to obtain a characterization of the two Hilbert spaces in terms of functions that are, in some sense, square-integrable. Let the angle u, $0\leq u<2\pi$, be defined by

$$(b_0+ib_5)/b=e^{iu}\,, \tag{A17}$$

and introduce functions h and H by

$$f(b)=(2\pi)^{1/2}(b_0+ib_5)^{-E_0}h(u,\hat{b})\,, \tag{A18}$$

$$F(b)=(2\pi)^{1/2}b^{-3}(b_0-ib_5)^{E_0}H(u,\hat{b})\,. \tag{A19}$$

In particular,

$$h_{ELM}(u,\hat{b})=(2\pi)^{-1/2}(E,L)^{-1}e^{-iu(E-E_0)}Y_{LM}(\hat{b})\,, \tag{A20}$$

$$H_{ELM}(u,\hat{b})=(2\pi)^{-1/2}(E,L)e^{-iu(E-E_0)}Y_{LM}(\hat{b})\,. \tag{A21}$$

Note that

$$\int h_I^*(u,\hat{b})H_{I'}(u,\hat{b})du\,d\Omega=\delta_{II'}\,, \tag{A22}$$

so that the inner product is

$$(f,f')=(F,F')=(h,h')=(H,H')=\sum_I C_I^*C_I'=\int h^*(u,\hat{b})H(u,\hat{b})du\,d\Omega\,. \tag{A23}$$

The question is whether we can find kernels k and K such that

$$(h,h')=\int h^*(u,\hat{b})k(u,\hat{b};u',\hat{b}')h'(u',\hat{b}')du\,du'd\Omega\,d\Omega'\,, \tag{A24}$$

$$(H,H')=\int H^*(u,\hat{b})K(u,\hat{b};u',\hat{b}')H'(u',\hat{b}')du\,du'd\Omega\,d\Omega'\,. \tag{A25}$$

The answer is no as far as k is concerned, and yes in the case of K.

The function k, if it exists, is determined by the Hermiticity of the generators to be a constant multiple of ($z\equiv e^{+iu}$, $z'=e^{-iu'}$):

$$(zz')^{-3}[(zz')^{-2}+1-2\hat{b}\cdot\hat{b}'/zz']^{E_0-3}\,. \tag{A26}$$

This function must be defined for real u,u' as the boundary value from the domain $|zz'|>1$ in which it is one-valued. Unfortunately (A24) is equal to zero by Cauchy's theorem.

For K we find

$$K=(8\pi^2)^{-1}[(zz')^2+1-2zz'\hat{b}\cdot\hat{b}']^{-E_0} \tag{A27}$$

defined for real u,u' as the boundary value from $|zz'|<1$, since this function is one-valued there. One easily verifies that the integral (A25) agrees with (A23) when both exist.

The negative result regarding (A24) is a familiar feature of representations of this type.[24] If the existence of (A25) is surprising at first glance, it should be recalled that the Hilbert space $\mathcal{K}_2$, rather than the function space $\mathcal{K}_2'$, carries the representation. Of course the representation can just as well be realized in $\mathcal{K}_2'$, but not in terms of differential operators. How it can be done was shown in a paper by Barut and Fronsdal.[25]

The rephrasing of these results in terms of the p-space functions $\psi(p)$ and $\Psi(p)$ was carried out as follows. The action of $L_{\alpha\beta}$ on $F(b)$ is given by (A5). If we define

$$p_\mu=imb_\mu/b_5\,,$$
$$\Psi(p)=b_5{}^{3-E_0}F(b)\,,$$
$$\psi(p)=b_5{}^{E_0}f(b)\,,$$

then the action of $L_{\alpha\beta}$ on $\Psi(p)$ and on $\psi(p)$ is presisely the same as was found in the text by other means. But this is as yet only formal, since an analytic continuation from real b_α to real p_μ must be carried out. At first we deal only with the basis vectors and try to express the integral (A22) in terms of real p_μ. Since $H_{ELM}(u, b)$ has a pole at at $z=0$, while $h^*_{ELM}(u, \hat{b})$ has a zero there, we are naturally led to the structure described in the text. In fact, let $A(z)$ be analytic in the region $|z|<1$, and let n be a non-negative integer. Then

$$\int_0^{2\pi} A(e^{iu})e^{-inu}du = \int_0^1 A(z)2\pi\delta^n(z)dz\,,$$

where, by definition,

$$\int_0^1 A(z)\delta^n(z)dz = \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n A(z)\bigg|_{z=0}.$$

The domain of the z integration corresponds to real p_μ, and the derivation of the results for $\psi(p)$ and $\Psi(p)$ given in the test is now straightforward.

For completeness we give the explicit form of the Hermitian matrix representation, to the extent that it was needed:

$$(L_{35} \pm iL_{30})|E, L, 0\rangle = (L+1)\left[\frac{(E_0 \pm E + L)(3 - E_0 \pm E + L)}{(2L+1)(2L+3)}\right]^{1/2}|E\pm 1, L+1, 0\rangle + L\left[\frac{(E_0 - 1 \pm E - L)(2 - E_0 \pm E - L)}{(2L-1)(2L+1)}\right]^{1/2}|E\pm 1, L-1, 0\rangle\,.$$

This formula makes the invariance and irreducibility of the space spanned by $E - E_0 - L = 0, 2, \ldots$ explicit, unless $E_0 = \frac{1}{2}$, in which case the space spanned by $E - E_0 - L = 0$ is invariant and irreducible.

---

*On leave of absence from University of California, Los Angeles, California.

[1]P. A. M. Dirac, Ann. Math. 36, 657 (1935); Proc. R. Soc. A155, 447 (1936); Max-Planck Festschrift, Berlin, 1958 (unpublished).

[2]E. Schrödinger, Proc. R. Irish Acad. A46, 25 (1940).

[3]K. Goto, Prog. Theor. Phys. 12, 311 (1954).

[4]F. Gursey, in *Istanbul Summer School 1962* (Gordon and Breach, New York, 1963).

[5]F. Gürsey, in *Relativity, Groups and Topology,* Les Houches Summer School 1963 (Gordon and Breach, New York, 1963).

[6]F. Gürsey and T. D. Lee, Proc. Natl. Acad. Sci. USA 49, 179 (1963).

[7]M. Gutzwiller, Helv. Phys. Acta 29, 313 (1956).

[8]W. Thirring, Acta. Phys. Austr. Suppl. 4; O. Nachtmann, Commun. Math. Phys. 6, 1 (1967).

[9]G. Börner and H. P. Dürr, Nuovo Cimento 64, 669 (1969) and references cited therein.

[10]C. Fronsdal, Rev. Mod. Phys. 37, 221 (1965).

[11]E. P. Wigner, Ann. Math. 40, 149 (1939).

[12]See, for example, Ref. 8.

[13]It is sufficient to define this distribution on the space of functions $A(p)$ analytic in $\mathrm{Re}\,p_0 > 0$. See footnote 16.

[14]One may proceed, as following Eq. (2.2), to determine a kernel $(p|p')$ so that $\int\psi^*(p)(p|p')\psi(p')(dp)_+$ is invariant. One finds the same result (2.4) with E_0 replaced by $3-E_0$. However, on checking *a posteriori* the convergence of the integrals, one finds that certain surface integrals, which were tentatively ignored in the process of partial integration, fail to vanish. This invalidates the conclusion that the integral is invariant, and one easily verifies that it is not.

[15]P. A. M. Dirac, J. Math. Phys. 4, 901 (1963).

[16]These functions, if (2.17) holds, are analytic in $\mathrm{Re}\,p_0 > 0$ if $\hat{p}$ is real. This is the justification for footnote 13. The full, invariant domain of analyticity of $\psi(p)$ is $\mathrm{Re}\,p_\mu \in$ forward cone.

[17]The quantities x_μ, $L_{\mu\nu}$ are tensors with respect to Lorentz transformations only. Their indices are raised and lowered by means of the Minkowski metric which we denote $\delta_{\mu\nu}$. We use the summation convention in the form $x_\mu x_\mu = \delta^{\mu\nu}x_\mu x_\nu = x_0^2 - \vec{x}^2 = x^2$. The quantities dx^μ are contravariant vectors in the sense of general coordinate transformations.

[18]The symbol x will be used as a general reference to a point in space, without any implication about the choice of coordinate system.

[19]In other words, the hyperboloid is not geodesically convex. Compare J. A. Wolf, *Spaces of Constant Curvature* (McGraw Hill, New York, 1967). See lemma 11.2.1 on p. 338, taking $n \to 4$, $s \to 3$, $K \to \rho$, $Kb(x,y) \to z(x,x')$, and exchanging "timelike" ⟷ "spacelike."

[20]The connection between the Casimir operator Q and the Laplace-Beltrami operator $\Box$ ($Q = -\Box$) can be checked by direct calculation. It is an immediate consequence of the scalar nature of the wave function—that is, of the absence of non-derivative terms in (3.10)—that $\Box$ is an invariant with respect to SO(3,2) and hence a function of Q.

[21]Actually, they are square-integrable only if $E_0 > \frac{3}{2}$.

[22]Compare R. Raczka, N. Limić, and J. Niederle, J. Math. Phys. 7, 1861 (1966).

[23]The problem of expressing the norm as an integral over space-time for the case $0 < E_0 \leq \frac{3}{2}$ will be taken up in the course of a general study of massless particles.

[24]These are "analytic representations"; see I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. 5, pp. 400–413.

[25]A. O. Barut and C. Fronsdal, Proc. R. Soc. A287, 532 (1965). See Eq. (5.7).