PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

ϕ_2^4 quantum field model in the single-phase region: Differentiability of the mass and bounds on critical exponents

James Glimm*† and Arthur Jaffe‡§
Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, Essonne, France
(Received 25 March 1974)

We prove that the physical mass m is a differentiable function of the bare mass m_0 down to the critical point (defined by the occurrence of $m=0$ or of the onset of symmetry breaking). We derive an upper bound on $m'(m_0)$. We show that the critical exponents η, ν, γ, ζ are bounded from below by their classical values.

The physical mass m in the ϕ_2^4 quantum field model depends on the bare mass $m_0 > 0$, as well as on the other parameters of the theory. The cluster expansion converges for large m_0, and shows that $m>0$ (Refs. 1 and 2) and that m is analytic in m_0.[3] As a consequence of correlation inequalities, m decreases as m_0 decreases for m_0 in an interval $(m_{0,c}, \infty)$.[4] According to the standard picture of symmetry breaking in the ground state (see Ref. 5), we expect that the lower end point $m_{0,c}$ of this interval is characterized by the condition $m=0$. Two other alternatives, however, have not been excluded: $m_{0,c}=0$ or $\langle \phi \rangle \neq 0$. In the Goldstone picture $m_0 > m_{0,c}$ corresponds to the single-phase region, and $m_{0,c}=0$ would mean the absence of phase transitions. We expect that in a pure phase $\langle \phi \rangle$ is a continuous function of m_0, so we expect $\langle \phi \rangle = 0$ for $m_0 = m_{0,c}$. The two-dimensional Ising model is known to behave in this manner, while the open questions for the three-dimensional Ising model are similar to those discussed above for ϕ_2^4.

One purpose of this note is to prove that m is differentiable in m_0, for $m_0 > m_{0,c}$. We define $m_{0,c}$ to be the supremum of the m_0 for which $m=0$ or $\langle \phi \rangle \neq 0$, and we define the ϕ_2^4 quantum field theory using Dirichlet boundary conditions. The convergence as $V\to\infty$ is based on monotonicity together with an upper bound on the Schwinger functions. For free boundary conditions, the upper bound follows from Ref. 6. The generalization of this upper bound to Dirichlet boundary conditions follows from the GKS inequality.[4,7] The monotonicity is established in Refs. 4 and 8. Our basic estimate is the correlation inequality that the four-point connected Euclidean Green's function is negative for $m_0 > m_{0,c}$.[5,9] We call this inequality the Lebowitz inequality.

To define m, choose $D>0$ and consider the Euclidean expectation

$$F(D) = \int_{|\vec{x}|,\, |\vec{y}| < D} \langle \phi(x)\phi(y)\rangle_T \, d\vec{x}\, d\vec{y}\, \big|_{x_0=0,\, y_0=D}, \qquad (1)$$

where $\langle\ \rangle_T$ denotes the truncated vacuum expectation value. For $m_0 > m_{0,c}$, $\langle\ \rangle_T = \langle\ \rangle$ in (1). Then

$$m = \lim_{D\to\infty} -\ln F(D)/D\,. \qquad (2)$$

The validity of (2) follows from the spectral representation of the two-point function and from standard properties of the two-point function.[6,10] We fix the Wick-ordering constants and vary $m_0 = m_0(\sigma)$ by the addition of a quadratic (mass) perturbation, $\frac{1}{2}\sigma \int :\phi(x)^2: dx$.[11] We define

$$m' \equiv \frac{dm}{d\sigma}\,. \qquad (3)$$

To bound the derivative m', we derive a bound on $-d[\ln F(D)/D]/d\sigma$ which is uniform in D and thus obtain the following:

Theorem 1. For each $\epsilon > 0$, $m(\sigma)$ is Lipschitz-continuous on the interval $\sigma \geq \sigma_c + \epsilon$. If $m(\sigma)\to 0$ as $\sigma \to \sigma_c$, then $m \leq (\sigma - \sigma_c)^{1/2}$.

Proof. Assuming $\sigma > \sigma_c$, we apply the GKS[4] inequality and the Lebowitz inequality[5] to obtain

$$\begin{aligned} 0 &\leq \frac{d}{d\sigma}[-\ln F(D)/D] \\ &= \frac{1}{2DF(D)} \int [\langle \phi(x) :\phi(z)^2: \phi(y)\rangle \\ &\qquad - \langle \phi(x)\phi(y)\rangle\langle :\phi(z)^2:\rangle]\, d\vec{x}\, d\vec{y}\, dz \\ &\leq D^{-1}F(D)^{-1} \int \langle \phi(x)\phi(z)\rangle\langle\phi(z)\phi(y)\rangle\, d\vec{x}\, d\vec{y}\, dz\,. \end{aligned} \qquad (4)$$

To apply these inequalities we use a momentum cutoff as an intermediate step, in order to eliminate the Wick ordering.

The contribution of the region $z_0 \notin [0, D]$ or $|\vec{z}| \geq D$ to the upper bound in (4) is negligible, as $D\to\infty$ with σ fixed. As a consequence,

$$m' \leq \limsup_{D\to\infty} \sup_{0\leq z_0 \leq D} \frac{\int \langle \phi(x)\phi(z)\rangle\langle\phi(z)\phi(y)\rangle\, d\vec{x}\, d\vec{y}\, d\vec{z}}{\int \langle\phi(x)\phi(y)\rangle\, d\vec{x}\, d\vec{y}}\,, \qquad (5)$$

where the $\vec{z}$ integration is restricted to $|\vec{z}| \leq D$.

We now justify the exclusion of these regions. We first consider the region I defined by $0 \leq z_0 \leq D$, $|\vec{z}| \geq D + D^{3/4}$. The spectral representation for $\langle \phi(x)\phi(z)\rangle\langle \phi(z)\phi(y)\rangle$ is

$$\int da\, db\, \rho(a)\rho(b) C_a(x-z) C_b(z-y),$$

where $C_a(x-y)$ is the kernel of $(-\Delta + a)^{-1}$. We choose a point w on the line xy with $w_0 = z_0$. Then

$$|x-z| \geq |x-w| + O(1)\min\{|z-w|^2/D, |z-w|\}$$
$$\geq |x-w| + O(D^{1/2}) + O(|\vec{z}|^{1/2}),$$

as $|\vec{z}| \to \infty$. For a given mass $a \geq m^2$,

$$C_a(x) \sim |x|^{-1/2} e^{-\sqrt{a}\,|x|} \quad \text{as } |x| \to \infty .$$

Thus,

$$C_a(x-z)C_b(z-y) \leq C_a(x-w)C_b(w-y)e^{-O(\delta)},$$

where $\delta = D^{1/2} + |\vec{z}|^{1/2}$. Thus we see that the integral over region I is bounded by $e^{-O(\delta)}$ times the integral over a strip of unit width about the line xy. Hence the integral over region I is negligible.

Furthermore, we see from the translation invariance of the integrands that the numerator and denominator both have magnitude $O(D^2 e^{-mD})$. Thus the region II defined by $0 \leq z_0 \leq D$, $D \leq |\vec{z}| \leq D + D^{3/4}$ contributes to the numerator at most the magnitude $O(D^{7/4}e^{-mD})$, and is negligible in the limit $D \to \infty$. Similarly, the region III defined by $z_0 \geq D + D^{3/4}$ and the region IV defined by $D + D^{3/4} \geq z_0 \geq D$ are negligible in (4) in the limit $D \to \infty$.

Let χ_D be the characteristic function of the interval $|\vec{z}| \leq D$ and let

$$f_D(z) = \int \langle \phi(x)\phi(z)\rangle \chi_D(\vec{z}) d\vec{x},$$

$$g_D(z) = \int \langle \phi(z)\phi(y)\rangle \chi_D(\vec{z}) d\vec{y},$$

so that the numerator in (5) is the L_2 inner product $\langle f, g\rangle$. Again, as $D \to \infty$, the angle between f_D and χ_D or between g_D and χ_D tends to zero. Thus as $D \to \infty$,

$$|\langle f, g\rangle| \leq \frac{\langle f, \chi\rangle\langle \chi, g\rangle}{\|\chi\|^2} + o(1),$$

or

$$m' \leq \limsup_{D\to\infty} \sup_{z_0 = w_0} \frac{\int\langle \phi(x)\phi(z)\rangle d\vec{x}\, d\vec{z} \int \langle \phi(w)\phi(z)\rangle d\vec{w}\, d\vec{z}}{2D\int\langle \phi(x)\phi(y)\rangle d\vec{x}\, d\vec{y}}, \tag{6}$$

where the integrals extend over $|\vec{x}|$, $|\vec{y}|$, or $|\vec{z}| \leq D$. We now show that we may replace the sup in (6) by the value at $z_0 = 0$ or D. Let

$$|t\rangle = e^{-tH/2}\int_{|\vec{x}|\leq D} \phi(\vec{x})\Omega, \quad F(t) = \langle t|t\rangle,$$

so the numerator in (6) is $G(t) = F(t)F(D-t)$, with $0 \leq t = z_0 = w_0 \leq D$. Note

$$(-1)^n F^{(n)}(t) = \langle t|H^n|t\rangle > 0,$$

and

$$G''(t) = F''(t)F(D-t) + F(t)F''(D-t)$$
$$-2F'(t)F'(D-t). \tag{7}$$

By the Schwarz inequality, $-F'(t) \leq F(t)^{1/2}F''(t)^{1/2}$, so from (7) we have $G''(t) \geq 0$. Thus $G(t)$ has its maximum value at an end point $T = 0$ or D. It follows that (6) factors to give

$$m' \leq \int \langle \phi(\vec{x})\phi(\vec{0})\rangle d\vec{x} = \tfrac{1}{2}\int \rho(a)a^{-1/2}da \leq (2m)^{-1}. \tag{8}$$

Here $\rho(a)$ is the spectral measure for the two-point function. The existence of a Hilbert space ensures that $\rho(a)da$ is a positive measure, and the canonical commutation relations ensure $\int \rho da = 1$. Thus integration of (8) down to $\sigma = \sigma_c$ completes the proof of the theorem.

Furthermore, if the mass is defined for a limit of Dirichlet data in a strip of finite width L, the mass $m(\sigma)$ is upper semicontinuous. This follows since $m(\sigma, L)$ is continuous and decreasing to $m(\sigma)$ as $L \to \infty$. Thus

$$m(\sigma) = \lim_{\epsilon\to 0_+} m(\sigma_c + \epsilon).$$

We remark that our bound (8) also yields an inequality for the mass m as a function of the bare mass m_0.

Theorem 2. Assume a pure $\phi_2{}^4$ interaction which is Wick-ordered in the bare vacuum for a field of mass m_0, the bare mass.

(a) Let $\epsilon > 0$. The mass $m(m_0)$ is Lipshitz-continuous on the interval $m_0 > m_{0,c} + \epsilon$.

(b) Assume $m_{0,c} \neq 0$, and $m(m_0) \to 0$ as $m_0 \to m_{0,c}$. Then the mass m is bounded by its classical value, with a logarithmic correction,

$$m \leq \left(m_0{}^2 - m_{0,c}{}^2 + 3\lambda \ln \frac{m_0{}^2}{m_{0,c}{}^2}\right)^{1/2}. \tag{9}$$

This theorem concerns the change of variables $\sigma \to m_0{}^2$, since we now Wick-order in the bare mass m_0. In Ref. 11 we saw that the change in Wick ordering constants due to the change in bare mass yields

$$\frac{d\sigma}{dm_0^2}=1+3\lambda m_0^{-2}. \tag{10}$$

From this combined with theorem 1, we infer theorem 2a. From (8), we then obtain

$$\frac{dm^2}{dm_0^2}=2m\frac{dm}{d\sigma}\frac{d\sigma}{dm_0^2}\le 1+3\lambda m_0^{-2},$$

which on integration yields (9).

The inequality (9) yields an inequality on the critical exponent ν governing the mass. Assuming that $m(m_{0,c}+\epsilon)\sim\epsilon^\nu$ as $\epsilon\to 0_+$, we see that $\nu\ge\frac{1}{2}$. (We give another elementary proof of this bound below.) The value $\nu=\frac{1}{2}$ is the classical value predicted by mean field theory or the Goldstone picture, while the value $\nu=1$ occurs for the two-dimensional Ising model (see Ref. 12). We believe that $\nu=1$ is more likely for ϕ_2^4 than $\nu=\nu_{\rm cl}=\frac{1}{2}$, but we believe that $\nu\ge\frac{1}{2}$ is the best which can be obtained by methods which do not depend strongly on dimension, such as those used here. We note that Symanzik[13] has also used the Lebowitz inequality to bound an exponent.

We note that other exponents can also be bounded by their canonical values. The long-distance anomalous dimension η is greater than its classical value zero,

$$\eta\ge 0=\eta_{\rm cl},$$

as follows in two dimensions from the observation that the two-point function is a monotonic decreasing function of $|x-y|$ (see Ref. 10). In greater than two dimensions, the free field has an $m\to 0$ limit, so the proof that $\eta\ge 0$ is a straightforward consequence of the Källén-Lehmann representation.

If we assume the existence of an isolated single-particle state of mass m (this is proved for λ small[1]) then the field-strength renormalization constant Z is defined by

$$G^{(2)}(p)=\int\langle\Phi(x)\Phi(0)\rangle e^{ipx}dx$$

$$=\frac{Z}{p^2+m^2}+\int_{m^2+\delta}^{\infty}\frac{\rho(a)da}{p^2+a}.$$

Let ζ be the critical exponent for Z, $Z\sim\epsilon^\zeta$. Since $0\le Z\le 1$, we have $\zeta\ge 0$. As discussed in Ref. 1, $\zeta=0$ is the classical value and is compatible with the existence of zero-mass particles. Since ρ is positive with total integral 1,

$$Z/m^2\le G^{(2)}(0)\le 1/m^2. \tag{11}$$

Let γ be the critical exponent for the susceptibility $\chi=G^{(2)}(0)$,

$$G^{(2)}(0)=\int\langle\Phi(x)\Phi(0)\rangle dx\sim\epsilon^{-\gamma}.$$

Thus from (11) we obtain

$$2\nu-\zeta\le\gamma\le 2\nu. \tag{12}$$

It is a reasonable hypothesis (apparently not accessible at present to proof) that $m'\le O(1)Zm^{-1}$ and $\gamma=2\nu-\zeta$, improving the bounds (8) and (12), respectively.

From the Lebowitz inequality we obtain, as in (4), (9),

$$0\le-\frac{dG^{(2)}(0)}{dm_0^2}=-(1+3\lambda m_0^{-2})\frac{dG^{(2)}(0)}{d\sigma}$$

$$\le(1+3\lambda m_0^{-2})$$

$$\times\int\langle\Phi(x)\Phi(z)\rangle\langle\Phi(0)\Phi(z)\rangle dx\,dz$$

$$=(1+3\lambda m_0^{-2})G^{(2)}(0)^2. \tag{13}$$

Integrating (13) down to the critical point yields

$$G^{(2)}(0)^{-1}\le m_0^2-m_{0,c}^2+3\lambda\ln\frac{m_0^2}{m_{0,c}^2},$$

from which we obtain $\gamma\ge 1=\gamma_{\rm cl}$, and by the upper bound in (12), $\nu\ge\frac{1}{2}=\nu_{\rm cl}$. We summarize these bounds as follows:

Theorem 3. Suppose $m_{0,c}>0$ and $m\to 0$ as $m_0\to m_{0,c}$. Then the exponents η, ν, γ, and ζ are bounded from below by their classical values,

$$\eta\ge 0,\quad \nu\ge\tfrac{1}{2},\quad \gamma\ge 1,\quad \zeta\ge 0.$$

We raise the question whether other field-theory exponents are also bounded by their classical values.

Finally, we remark a simple inequality on α, the exponent for the specific heat C:

$$\epsilon^{-\alpha}\sim C\equiv\tfrac{1}{4}\int[\langle:\Phi(x)^2::\Phi(y)^2:\rangle$$

$$-\langle:\Phi(x)^2:\rangle\langle:\Phi(y)^2:\rangle]dy$$

$$\le\tfrac{1}{2}\int\langle\Phi(x)\Phi(y)\rangle^2dy=\tfrac{1}{2}\int G^{(2)}(p)^2dp,$$

where again we use the Lebowitz inequality. From

$$G^{(2)}(p)=\int\frac{\rho(a)da}{p^2+a},$$

we find that

$$\int G^{(2)}(p)^2dp=\pi\int\frac{\rho(a)\rho(b)\ln(b/a)}{b-a}da\,db$$

$$\le O(m^{-2}).$$

Here we employ an elementary estimate on $(b-a)^{-1}\ln(b/a)$, and hence we infer

$$\alpha\le 2\nu. \tag{14}$$

*Work supported in part by the National Science Foundation under Grant No. NSF-GP-24003.

†Permanent address: Rockefeller University, New York, New York 10021.

‡Work supported in part by the National Science Foundation under Grant No. NSF-GP-40354X.

§Permanent address: Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138.

[1]J. Glimm, A. Jaffe, and T. Spencer, Ann. Math. (to be published).

[2]J. Glimm, A. Jaffe, and T. Spencer, in *Constructive Quantum Field Theory*, edited by G. Velo and A. Wightman (Springer, Berlin, 1973).

[3]T. Spencer, Commun. Math. Phys. (to be published).

[4]F. Guerra, L. Rosen, and B. Simon, Ann. Math. (to be published).

[5]J. Glimm, A. Jaffe, and T. Spencer, in *Constructive Quantum Field Theory*, edited by G. Velo and A. Wightman (Ref. 2).

[6]J. Glimm and A. Jaffe, J. Math. Phys. 13, 1568 (1972).

[7]J. Fröhlich, Ann. Inst. Henri Poincaré (to be published).

[8]E. Nelson, in *Constructive Quantum Field Theory*, edited by G. Velo and A. Wightman (Ref. 2).

[9]J. Lebowitz, Commun. Math. Phys. 35, 87 (1974); B. Simon and R. Griffiths, *ibid.* 33, 145 (1973).

[10]B. Simon, Commun. Math. Phys. 31, 127 (1973); Ann. Math. (to be published).

[11]J. Glimm and A. Jaffe, Ann. Inst. Henri Poincaré 21, (1974).

[12]H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford Univ. Press, Oxford, England, 1971).

[13]K. Symanzik, Cargèse lectures, 1973 (unpublished).

PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

Calculability and naturalness in gauge theories*

Howard Georgi†
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138

A. Pais
Rockefeller University, New York, New York 10021

(Received 7 February 1974)

Calculability conditions are discussed for local gauge theories with Higgs-type symmetry breaking. We focus on the naturalness of μe universality, the naturalness of the Cabibbo angle θ, the naturalness of CP-violating phases, and the naturalness of the nonleptonic $\Delta I = \frac{1}{2}$ rule. In this context we examine many published gauge models and construct others to illuminate the questions at hand. We note that naturalness of μe universality for charged currents does not necessarily imply universality for neutral currents (natural "restricted" universality), and we emphasize the need for ν_e-beam experiments. For SU(2)×U(1) and SU(2)×U(1)×U(1) we give first examples of how a nontrivial natural θ can appear. Models with CP violation are classified as to whether their CP-violating phases are natural or not. For O(4)×U(1) we give a first example in which all the above naturalness criteria can be implemented. Here the natural μe universality is necessarily restricted. The principal tool used in these investigations is the strict renormalizability relative to a gauge group enlarged by discrete symmetries, and the union of representations reducible under the gauge group to irreducible ones under the enlarged group. To implement this program, it is sometimes necessary to introduce Higgs couplings involving right-handed neutrinos; here the zero neutrino mass is associated with a discrete symmetry which remains unbroken upon spontaneous breakdown. We also find that strict renormalizability can lead to mass relations between fermions. In O(4)×U(1) models, such mass relations as well as right-handed neutrinos are necessary ingredients. Furthermore, for these models the spontaneity of CP violation acquired an operational significance, namely, as a discrete symmetry necessary (but not sufficient) to give a CP-violating phase a natural value (90°). While the models we discuss are rather cumbersome, particularly due to the complexity of the symmetry-breaking mechanism, we expect that the tools we have developed may well have wider applicability.

I. INTRODUCTION

Many gauge models of weak and electromagnetic interactions have been devised in the last few years. The basic strategy for their construction consists in a reconciliation of field-theoretical and phenomenological requirements. From the side of field theory one insists on the renormalizability of the scheme as the principal predictive theoretical tool. From the side of phenomenology one attempts to incorporate all the known regularities of the weak interactions. What is known here almost entirely concerns the rather low-energy and low-momentum-transfer domain. Indeed, it is