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Spontaneous symmetry breaking without scalar mesons. II *

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Recently, the formulation of spontaneously broken gauge theories (SBGT) without scalar mesons was carried out nonperturbatively for Abelian gauge models. These results are reformulated and extended to non-Abelian gauge theories using an effective Lagrangian, which allows symmetry-breaking coefficients to be calculated in perturbation theory. These coefficients appear in Callan-Symanzik equations for the SBGT, which differ only in what is conventionally called the right-hand side from the Callan-Symanzik equations for the symmetric theory. Spontaneous breakdown can only take place if these symmetry-breaking coefficients are positive, in which case the effective Lagrangian reduces, in a certain sense, to the Lagrangian of the symmetric theory. The question of positivity is studied in non-Abelian theories in lowest-order perturbation theory, and it is shown how to accommodate SBGT without scalars in the framework of asymptotic freedom. Some aspects of the utility of the light-cone gauge for calculations in non-Abelian theories are discussed in an appendix.

I. INTRODUCTION

There are many ways of inducing spontaneous symmetry breakdown in quantum mechanics, and surely the most widely studied recently is the Higgs-Kibble[1] mechanism for spontaneously broken gauge theories (SBGT).[2] Adding Higgs scalars to a gauge-theory Lagrangian is an excellent way to achieve a spectrum of massive vector mesons without sacrificing renormalizability, but there is no phenomenological evidence for particles associated with these scalar fields, and they are usually given very large masses and very small coupling constants by model builders. Another possibility proposed recently[3,4] is that the gauge symmetry is dynamically broken, that is, there exist nonperturbative solutions to the conventional gauge-theory field equations without scalars which contain massive vector mesons and are renormalizable. These works are very similar in spirit to the classic paper of Nambu[5] on superconductivity, in which the photon becomes massive (Meissner effect) and certain homogeneous, symmetry-breaking Dyson equations have nonzero solutions. A little while after Nambu, Englert and Brout[6] suggested that Nambu's work might have a counterpart in relativistic field theory.

Long ago Schwinger[7] pointed out the general kinematic basis for the appearance of massive vector mesons in gauge theories: The vector-meson self-energy acquires a pole at $q^2=0$. This pole appears in many Green's functions, but not in the S matrix.[3,4] In the absence of elementary scalar fields, the zero-mass pole has a dynamical origin and its residues obey homogeneous Bethe-Salpeter equations (as in superconductivity). The solutions to such equations, if they exist, are necessarily nonperturbative.

Whatever the mechanism for symmetry breakdown is, there are three major problems to be faced: (1) maintaining gauge invariance (i.e., the Ward-Takahashi identities), (2) controlling the asymptotic behavior of the theory so that no new divergences arise, and (3) calculating symmetry-breaking masses and coupling constants, which receive significant contributions both from the finite-momentum regime and from the asymptotic regime. We shall have nothing to say about this last, most difficult, problem. The purpose of the present paper is to give a formal prescription for solving problems (1) and (2) for the case of dynamical symmetry violation, without having to struggle with solving homogeneous integral equations (as was done in Refs. 3 and 4). The prescription—to be discussed in detail later—is to calculate the coefficients of the Callan-Symanzik (CS) equations[8,9] from a gauge-invariant, renormalizable effective Lagrangian which incorporates in its structure the dynamical Goldstone bosons which give the vector mesons mass. Only the CS coefficients which appear on what is by convention the right-hand side of the CS equations differ from those of the symmetric theory. This evolution of the problem from earlier efforts[3,4] is precisely analogous to the evolution of the Baker-Johnson[10] approach in quantum electrodynamics, which is now phrased in terms of the CS equation[11]; the main difference is that there are no Goldstone bosons in the Baker-Johnson program.

Problem (1), gauge invariance, was not discussed explicitly in Refs. 3 and 4. These works use the Landau gauge, which happens to give the correct answer, but it is not obvious (except perhaps to very experienced readers) why this should be so; no discussion of gauge invariance appears in the cited works. In fact, the Landau gauge is privileged, from the point of view of computational convenience, but the present development maintains manifest gauge invariance at all stages, and allows one to calculate in any gauge.

Problem (2)—controlling asymmetric divergences—assumes a new aspect in dynamically broken theories. We classify symmetry-breaking mechanisms as either soft, or potentially divergent. Soft mechanisms either have no divergences, as in nonrelativistic superconductivity,[5] or the divergences of the spontaneously broken theory can be incorporated in those of the symmetric theory order by order in perturbation theory (as in the σ model[12] and Higgs-scalar models[2]). In soft theories, addition of properly chosen symmetry-breaking terms to the Lagrangian does not affect the perturbative renormalizability of the theory, even though not every symmetry-breaking parameter is finite without renormalization. [We classify the work of Coleman and Weinberg[13] as a soft theory, because the radiative corrections to the potential can be calculated order by order in conventional perturbation theory. These authors have profitably used the renormalization-group equations (CS equations with no right-hand side) to study symmetry breakdown with scalar mesons.] But in potentially divergent theories, naive addition of symmetry-breaking mass terms may lead to nonrenormalizability, as in the Abelian models of Refs. 3 and 4. In a proper spontaneously broken theory, not only must nonrenormalizable divergences be absent, but also any divergence which requires a counterterm that is not symmetric. Crudely speaking, symmetry-breaking effects must disappear asymptotically, which ensures that there are well-behaved solutions to the symmetry-breaking homogeneous Dyson equations.[3,4] In the case of asymptotic power-law behavior $(p^2)^{-\epsilon}$, ϵ must be positive; if $\epsilon<0$, there is no spontaneous breakdown (corresponding to a superconductor at too high a temperature). Actually, ϵ refers to the rate of decrease of the symmetry-breaking propagators relative to the asymptotic behavior of the symmetric propagators; it is only this concept which is gauge-invariant.

There is no general requirement that symmetry-breaking effects vanish like a power, and there are a number of interesting circumstances where the decrease may be slower. In such a case the CS coefficient $\beta(g)$ (which we will show is the same for the symmetric theory and for the spontaneously broken theory) has a zero at the same place that $\epsilon(g)$ does. (Here ϵ appears as a power-law parameter if $\beta\equiv 0$; see Ref. 11.) A simultaneous zero always occurs at the origin $g=0$, and if $g=0$ is a stable fixed point (i.e., the theory is asymptotically free[14-16]) the asymptotic behavior of the symmetric as well as of the spontaneously broken theory can be calculated reliably in perturbation theory. For asymptotically free theories, the requirement that ϵ be positive in the neighborhood of the origin still holds, but now symmetry-breaking effects disappear only at a logarithmic rate. One may also consider the Gell-Mann–Low–Adler–Baker–Johnson program[10,17,18] in which $\beta(g)=0$ for $g\neq 0$. In general, this leads to power-law behavior, since there is no reason to expect $\epsilon(g)$ to vanish at the same point. This will be modified if $\epsilon(g)$ does vanish, however, but the exact circumstances depend on the nature of the zeros of β and ϵ. In principle, we cannot say anything reliable about this possibility without going beyond low-order perturbation theory for β and ϵ.

Dynamically broken non-Abelian gauge theories have one advantage over Higgs-scalar models, in the case of asymptotic freedom: It is easier to achieve a realistic vector mass spectrum. There often seems to be at least one massless vector meson, however, so that one does not have complete freedom in choosing a mass spectrum for dynamically broken theories either, and at the moment there is no outstanding candidate for a realistic dynamically broken theory, of either strong or weak interactions, which is asymptotically free.

The basic method of the present work is explained in Sec. II, using for clarity of explanation the spontaneously broken Abelian model of Ref. 3. To the usual symmetric Lagrangian we add *gauge-invariant* symmetry-breaking terms, which reduce to mass terms in the limit of zero coupling constant. These terms are nonpolynomial and nonlocal, which explains how we can achieve this seemingly contradictory result. The final Lagrangian bears a certain resemblance to the phenomenological Lagrangian of Jackiw and Johnson,[4] with some crucial differences: Our Lagrangian is renormalizable order by order, and it contains propagators for massless scalars where the Jackiw-Johnson Lagrangian would have inverse powers of the vector-meson mass. (Because of these inverse mass powers, the Jackiw-Johnson Lagrangian is unrenormalizable.) It turns out that we can use perturbation theory (in principle, at least) to calculate the asymptotic behavior of the theory, thus avoiding the difficulties of homogeneous integral equations for which conventional perturbation theory fails. It is then possible to write CS equations for the effective nonlocal Lagrangian, and recover all the results of the earlier works.[3,4] The coefficients usually termed β, γ are the same as in the symmetric theory, while the right-hand side of the CS equations has now terms which specifically refer to the symmetry-breaking parameters (called ϵ above). It is because we have zero-mass propagators present in the original effective Lagrangian that we can get away with using perturbation theory, just as Weinberg[19] showed that one can perform perturbation theory in the presence of a bound state if pole terms are included in the potential. It is worth noting that, with the aid of Lagrange-multiplier fields, it is possible to give a local, but still nonpolynomial description of the effective Lagrangian, which can thus be canonically quantized.

Section III sets up a similar scheme for non-Abelian gauge theories. The difficulties of maintaining gauge invariance are considerably exacerbated, but appear to be manageable; we indicate a prescription which seems to work, but which has only been studied in detail in lower orders. Some simple group theory is worked out to reveal the signs of the ϵ parameters in lowest order; the only striking thing is that in an asymptotically free theory, the fermions *cannot* be in the lowest-dimensional group representation. One model is exhibited—a parity-conserving chiral model—in which part of the symmetry breaking is soft (in the sense used above), ϵ vanishes identically, and the problem of symmetry breaking is entirely nonasymptotic.

After the conclusions in Sec. IV, there appears an appendix which gives certain results concerning Yang-Mills theories in the light-cone gauge.[20] This is a special case ($n^2=0$) of the ghost-free gauges $n\cdot B=0$, discussed by many authors.[21-24] Because this gauge is ghost-free, the Ward-Takahashi identities become no more complicated than those of Abelian theories, and various heuristic demonstrations are greatly simplified. The main point of the Appendix is to demonstrate that loop integrals can be done with no more effort in the light-cone gauge than in any covariant gauge, which is certainly not true for the axial gauge $n^2=-1$.[21,22] If that were all there is to it, then presumable everyone would rush to use the light-cone gauge. Unfortunately, this gauge *may* not even exist, because loop integrals acquire new divergences which appear in the Feynman parameter integrals. However, these can be regulated with the dimensional regularization techniques of 't Hooft and Veltman.[25] *Assuming* that a consistent renormalization procedure can be achieved in the light-cone gauge, we can give a simple heuristic demonstration of the analog of the Schwinger[7] mechanism in non-Abelian theories. The demonstration of the Schwinger mechanism in covariant gauges with ghosts is considerably more complicated.[26]

After the material in this appendix had been worked out, the author found out about related work of Chakrabarti and Darzens,[24] who give a special case of the general theorem in the Appendix.

II. ASYMPTOTIC SYMMETRY BREAKING: THE ABELIAN CASE

A. Review of previous work

We consider the model of Ref. 3; the model of Jackiw and Johnson[4] is equivalent. The symmetric Lagrangian is

$$\mathcal{L}_S=\bar{\Psi}\,(\tfrac{1}{2}i\,\gamma\cdot\overleftrightarrow{\partial}-M_0+g'\,\tau_2\gamma_\mu B^\mu+g\gamma_\mu A^\mu)\Psi -\tfrac{1}{4}A_{\mu\nu}{}^2-\tfrac{1}{4}B_{\mu\nu}{}^2\,, \quad (1)$$

where

$$A_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,\quad B_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu\,, \quad (2)$$

$\Psi=(\Psi_1,\Psi_2)$ is a two-component fermion field, and τ_2 is the usual Pauli matrix. The bare-mass matrix M_0 is a multiple of the two-dimensional identity matrix. This Lagrangian is symmetric under the local gauge transformations

$$\Psi\to e^{i\tau_2\theta}\Psi\,,\quad B_\mu\to B_\mu-\frac{1}{g'}\,\partial_\mu\theta\,, \quad (3)$$

and

$$\Psi\to e^{i\alpha}\Psi\,,\qquad A_\mu\to A_\mu-\frac{1}{g}\partial_\mu\,\alpha\,. \quad (4)$$

The gauge invariance (4) will remain unbroken, and we look for spontaneous breakdown of the gauge symmetry (3). This spontaneous breakdown must respect the Ward identity

$$(p-p')^\mu\,\Gamma^B_\mu(p',p)=\tau_2S^{-1}(p)-S^{-1}(p')\tau_2\,, \quad (5)$$

where Γ^B_μ is the proper $\bar{\Psi}\Psi\ B$ vertex, and S^{-1} is the two-dimensional inverse fermion propagator. The spontaneous breakdown reveals itself in a part of the proper fermion self-energy proportional to τ_3,

$$S^{-1}(p)=\not{p}-M_0-\Sigma(p)\,,$$
$$\Sigma(p)=\Sigma_S(p)+\tau_3\Sigma_V(p)\,. \quad (6)$$

As explained in Ref. 3, the Ward identity (5) can only be satisfied at $p'=p$ if there is a pole in Γ^B_μ:

$$\Gamma^B_\mu(p',p)=\frac{q_\mu}{q^2}\,i\tau_1\,\Gamma^B_V(p',p)+\cdots\quad(q=p-p')\,, \quad (7)$$

with

$$\Gamma_V^B(p,p)=2\Sigma_V(p). \tag{8}$$

The associated composite Goldstone boson ($q^2=0$ pole) does not appear in the S matrix. The symmetry-breaking Green's functions Γ_V^B, Σ_V obey homogeneous Dyson equations, and it is shown[3,4] that to lowest order the linearized equation for Σ_V has a solution (in the Landau gauge)

$$\Sigma_V(p)=\delta M(-p^2/M^2)^{-\epsilon(g,g')} \tag{9}$$

if the parameter ϵ has the value

$$\epsilon=\frac{3}{4\pi}(\alpha-\alpha'), \tag{10}$$

where $\alpha=g^2/4\pi$, $\alpha'=g'^2/4\pi$. This expression makes it clear why the A meson is needed; if $\alpha=0$, $\epsilon<0$. We take it that $\alpha>\alpha'$, so ϵ is positive. The pole in Γ_μ^B leads to a pole in the B-meson self-energy, and thus to a vector-meson mass via the Schwinger mechanism.[7]

It is shown that, again to lowest order,

$$\Pi_B(0)=\frac{2\alpha'}{\pi\epsilon}(\delta M)^2\simeq M_B{}^2, \tag{11}$$

where the B-meson self-energy is of the form

$$\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\Pi_B(q). \tag{12}$$

A curious feature about the mass formula (11) is that ϵ appears in the denominator, so that $\Pi_B(0)$ is actually of zeroth order in the coupling constants. Furthermore, ϵ must be positive so that $M_B{}^2$, the square of the B-meson mass, is positive. There is no limitation on the size of δM or of M_B even in the limit of zero coupling constant, as long as we require α'/ϵ to be finite and positive in the limit. The theory then describes free fermions and vectors, with a massive B and fermion mass splitting. However, if such mass terms are added to $\mathcal{L}_S$ in (1), the theory becomes unrenormalizable; according to the definition given in the Introduction, this is a potentially divergent theory. Only if $\epsilon>0$ are these potential divergences removed.

Two criticisms can be made against this development of the theory: (1) Gauge invariance is not manifest; (2) it is difficult to see how to proceed in higher orders, or indeed what the actual expansion parameter is, since expressions like (11) are of zero order, yet come from nontrivial loop graphs. We turn to another approach which avoids these defects.

B. The effective-Lagrangian method

It is not possible to add naive symmetry-breaking mass terms to the asymmetric Lagrangian $\mathcal{L}_S$ of (1), without destroying renormalizability. The essential reason is that terms violate gauge invariance [(i.e., the Ward identity (5)]. But it is possible to add certain nonlocal, nonpolynomial terms to (1) which are gauge-invariant in a restricted sense, and which are renormalizable. For example, the mass term

$$\tfrac{1}{2}M_{B0}{}^2\left(B_\mu-\frac{1}{\Box}\partial_\mu\partial\cdot B\right)^2 \tag{13}$$

is invariant under the gauge transformation (3), as long as $\Box\theta\neq 0$. The recipe for forming such a term is the following: (a) Write down a naive symmetry-breaking mass term. (b) Make a gauge transformation (3) on it. (c) Replace the gauge parameter θ by $g'\Box^{-1}\partial\cdot B$. In this way, we come to the full effective Lagrangian

$$\mathcal{L}_{\text{eff}}=\mathcal{L}_S-\overline{\Psi}e^{-ig'\tau_2\phi}\delta M_0\tau_3e^{ig'\tau_2\phi}\Psi+\tfrac{1}{2}M_{B0}{}^2(B_\mu-\partial_\mu\phi)^2\big|_{\phi=\Box^{-1}\partial\cdot B}. \tag{14}$$

This has the same form as the Jackiw-Johnson[4] effective Lagrangian, with one crucial difference: The scalar field ϕ has (naive) dimension zero. In the Jackiw and Johnson Lagrangian, the corresponding scalar field Φ has dimension 1, as an ordinary free scalar field would; we recover their Lagrangian by setting $\phi=M_{B0}{}^{-1}\Phi$ in (14), and removing the constraint $\phi=\Box^{-1}\partial\cdot B$. Thus Φ becomes an independent degree of freedom, with propagator behaving like k^{-2}. However, our ϕ field has a propagator behaving like k^{-4}. In consequence, some factors of k^{-2} needed for renormalizability are replaced by $M_{B0}{}^{-2}$ in the Jackiw-Johnson Lagrangian. As these authors point out, there may well be a hierarchy of effective Lagrangians to be used in different regimes of momenta. Their nonrenormalizable Lagrangian is useful at low energies, where it reveals clearly the pole structure (in particular, the cancellation of the Goldstone excitation in the S matrix, in tree approximation); our Lagrangian, on the other hand, is to be used (in this work, at least) only at asymptotic momenta. Actually, there is nothing in principle forbidding us from using (14) for all momenta, but that would get us into the incredibly difficult problem [(3) in the Introduction] of calculating both low- and high-energy symmetry-breaking effects.

Some readers may well recoil at the prospect of a Lagrangian which is both nonlocal and nonpolynomial. It is easy to remove the nonlocality by adding to (14) a Lagrange-multiplier term

$$-\chi(\Box\phi-\partial\cdot B) \tag{15}$$

(the first term can be integrated by parts, so that it depends only on first derivatives) and ignoring the constraint written in (14). The new constraint equation $\Box\phi = \partial\cdot B$ is not, in fact, different from that of (14) ($\phi = \Box^{-1}\partial\cdot B$), since there is no legitimate free field of dimension zero which could be added to ϕ. If χ is taken to be a gauge-invariant field, then the sum of (14) and (15) is gauge-invariant also (always assuming the gauge parameter θ is such that $\Box\theta \neq 0$). As with any gauge-invariant Lagrangian, the free propagators of the vector mesons are undetermined, and we must add special gauge-breaking terms, in a well-known way.[2] Then the final $\mathfrak{L}_{\text{eff}}$ is

$$\mathfrak{L}_{\text{eff}} = \mathfrak{L}_S - \bar{\Psi} e^{-ig'\tau_2\phi}\delta M_0 \tau_3 e^{ig'\tau_2\phi}\Psi + \tfrac{1}{2}M_{B0}{}^2(B_\mu - \partial_\mu\phi)^2 - \chi(\Box\phi - \partial\cdot B) - \frac{1}{2\lambda}\left[(\partial\cdot A)^2 + (\partial\cdot B)^2\right] . \tag{16}$$

(We could use different gauge parameters λ for A^μ, B^μ, but that only complicates the writing.) Without the last term, (16) can be canonically quantized in some noncovariant gauge, e.g., the Coulomb gauge. The A fields are treated conventionally. All four components of B_μ, and χ and ϕ as well have nonvanishing canonical momenta; however, these six momenta are constrained by one relation (the 0 component of the field equations for B_μ). The only point of carrying through the canonical quantization is to make sure that there are no covariant seagulls of the type discussed by Gerstein *et al.*[28] and Finkelstein *et al.*[29]; in fact, there are none. Therefore, the Feynman rules following from (16) including the covariant gauge-setting terms, are the naive ones based on the interaction-Lagrangian part of (14), with free vector propagators:

$$\Delta^A_{\mu\nu} = \frac{-g_{\mu\nu} + (1-\lambda)k_\mu k_\nu k^{-2}}{k^2 - i\epsilon} ,$$

$$\Delta^B_{\mu\nu} = \frac{-g_{\mu\nu} + k_\mu k_\nu k^{-2}}{k^2 - M_{B0}{}^2 + i\epsilon} - \frac{\lambda k_\mu k_\nu}{k^4} . \tag{17}$$

Of course, $\lambda = 0$ is the Landau gauge. These Feynman rules generate poles in the integrands of Green's functions (and in $\Pi^B_{\mu\nu}$ itself), which are clearly to be identified with a composite Goldstone excitation.

As we have stated several times, this Lagrangian is renormalizable. There are two aspects to this: First, the propagator of the ϕ field is just λk^{-4}, as is appropriate for a dimension-zero field. Second, the equations of motion derived from (16) show that the source of the B field is *conserved* (which is not the case for the Jackiw-Johnson Lagrangian). This key result, which is a consequence of the gauge invariance of $\mathfrak{L}_{\text{eff}}$, allows us to derive (at least formally) the Ward-Takahashi identities for proper B-field vertices; they can be verified order by order in perturbation theory. For example, (14) shows that, to $O(g')$, the proper vertex $g'\Gamma^B_\mu$ is

$$\Gamma^B_\mu = \tau_2\gamma_\mu - 2i\,\delta M_0 \tau \frac{q_\mu}{q^2} , \tag{18}$$

where q_μ is the outgoing B-field momentum. This vertex satisfies

$$q^\mu \Gamma^B_\mu(p'-p) = \tau_2 S^{-1}(p) - S^{-1}(p')\tau_2 \quad (q = p - p') , \tag{19}$$

with $S^{-1}(p) = \not{p} - M_0 - \tau_3\delta M_0$; (19) is the Ward-Takahashi identity (5). In higher orders (19) continues to be satisfied, as the exponential terms in (14) correct for the string of commutators which are created when the τ_2's arising from (19) as applied to an internal vertex are moved to the left or the right of the fermion self-energy part. As is well known, the identity (19) allows us to reduce the number of independent renormalization constants of the theory, as we discuss below.

Even assuming that $\mathfrak{L}_{\text{eff}}$ in (16) generates a perfectly well-behaved theory, what relation does it have to spontaneous symmetry breakdown? If there are in fact two new physical parameters δM_0, M_{B0} which do not appear in the original $\mathfrak{L}_S$, then we are simply discussing another Lagrangian, and not a spontaneously broken solution to $\mathfrak{L}_S$. The point is that the necessary conditions for spontaneous breakdown to be described in Sec. II C force δM_0 and M_{B0} to be zero, but do *not* force their renormalized counterparts to zero. (Of course, any *bare*-mass parameter is only well defined when a cutoff is introduced into the theory; we mean that when, e.g., δM_0 is expressed in terms of renormalized parameters and a cutoff Λ, the limit $\Lambda\to\infty$ of δM_0 is zero.) In effect, δM_0 and M_{B0} play the role of weak magnetic fields placed on an unaligned ferromagnet; the resulting spontaneous magnetization is independent of the weak external field.

We shall use $\mathfrak{L}_{\text{eff}}$ to construct Callan-Symanzik equations for spontaneously broken gauge theories, which will thus yield a perturbation-theory algorithm for the asymptotic behavior of these theories.

C. The Callan-Symanzik equations

There are two coupling constants and three mass terms in $\mathfrak{L}_{\text{eff}}$, and it is quite straightforward to write down the CS equations for a renormalized, one-particle irreducible, amputated vertex Γ[8,9]:

$$\left(D + \beta\frac{\partial}{\partial g} + \beta'\frac{\partial}{\partial g'} - \sum N_i\gamma_i\right)\Gamma = \Delta\Gamma . \tag{20}$$

Γ describes a process with N_i fields of type i, each with anomalous dimension γ_i ($i=1,2,A,B$). The operator D is given in terms of renormalized masses M, δM, M_B by

$$D=M\frac{\partial}{\partial M}+\delta M\frac{\partial}{\partial(\delta M)}+M_B\frac{\partial}{\partial M_B}\;; \tag{21}$$

the dimensionless functions β, β', γ_i are all finite and depend on renormalized coupling constants and mass ratios. The mass derivatives on the left-hand side of (20) or in (21) are applied holding the *renormalized* coupling constants fixed. However, on the right-hand side of (20) we shall need the same mass derivatives with *unrenormalized* coupling constants held fixed; we denote this operator by $\overline{D}$. The relation between them is simple: $\overline{D}=D+\beta(\partial/\partial g)+\beta'(\partial/\partial g')$. On the right-hand side of (20), $\Delta\Gamma$ is formed by (1) applying the operator $\overline{D}$ to the *unrenormalized* vertex Γ_u expressed in terms of bare masses and coupling constants; (2) multiplying the result by $\Pi(Z_{2i})^{1/2}$, that is, by the same factor which changes Γ_u to Γ. Another way to say it is the following: Expand Γ_u in powers of the unrenormalized mass operator

$$\begin{aligned}\mathfrak{M}&=M_0\overline{\Psi}\Psi+\overline{\Psi}e^{-ig'\tau_2\phi}\delta M_0\tau_3e^{ig'\tau_2\phi}\Psi\\&\quad-\tfrac{1}{2}M_{B0}{}^2(B_\mu-\partial_\mu\phi)^2,\end{aligned} \tag{22}$$

replace $\mathfrak{M}^N$ by $\overline{D}\mathfrak{M}^N$, and then renormalize. Only the first two powers of $\mathfrak{M}$ matter for the leading asymptotic behavior. $\mathfrak{M}$ is essentially the trace of the stress-energy tensor $T_{\mu\nu}$ and there are general arguments[30] that the matrix elements of $T_{\mu\nu}$ are made finite by multiplying by $\Pi(Z_{2i})^{1/2}$ (at least if there are no elementary scalar fields). The action of $\overline{D}$ on $\mathfrak{M}$ is expressed in terms of three functions δ_i, defined by

$$\begin{aligned}&\overline{D}\ln M_0=1+\delta_0,\\&\overline{D}\ln\delta M_0=1+\delta_1,\\&\overline{D}\ln M_{B0}=1+\delta_2.\end{aligned} \tag{23}$$

If one accepts the argument that the matrix elements of $\mathfrak{M}$, like those of $T_{\mu\nu}$, are made finite by $\Pi(Z_{2i})^{1/2}$ and makes the obvious remark that $\overline{D}\mathfrak{M}$ is renormalized with this factor [since the left-hand side of (20) is so renormalized], it follows that the δ_i are separately finite. With the aid of (23), we express the right-hand side of the CS equation in terms of

$$\begin{aligned}\overline{D}\mathfrak{M}&=+(1+\delta_0)M_0\overline{\Psi}\Psi+(1+\delta_1)\overline{\Psi}e^{-ig'\tau_2\phi}\delta M_0\tau_3e^{ig'\tau_2\phi}\Psi\\&\quad-(1+\delta_2)M_{B0}{}^2(B_\mu-\partial_\mu\phi)^2.\end{aligned} \tag{24}$$

Let us make these arguments concerning the renormalization of the mass operator $\mathfrak{M}$ more precise. There are three terms:

$$\begin{aligned}&\mathfrak{M}_0=M_0\overline{\Psi}\Psi,\\&\mathfrak{M}_1=\overline{\Psi}e^{-ig'\tau_2\phi}\delta M_0\tau_3e^{ig'\tau_2\phi}\Psi,\\&\mathfrak{M}_2=-\tfrac{1}{2}M_{B0}{}^2(B_\mu-\partial_\mu\phi)^2.\end{aligned} \tag{25}$$

The last term, $\mathfrak{M}_2$, has no divergent skeleton graphs, so it may be rendered finite without introducing any new renormalization constants, as we argued in the previous paragraph. Next, consider the effect of $\mathfrak{M}_0$ and $\mathfrak{M}_1$ on the two fermion propagators. The fermion proper self-energy has the Dirac-matrix decomposition

$$\Sigma_i(p)=A_i\not{p}+B_i\quad(i=1,2). \tag{26}$$

It is easy to see by counting γ matrices that A_i may contain only even powers of $\mathfrak{M}$, B_i only odd powers. But all skeleton graphs containing 2, 4, 6, ... $\mathfrak{M}$ vertices are superficially convergent, so the *divergent* parts of the A_i are independent of $\mathfrak{M}$. Therefore the cutoff dependence of A_1 and A_2 is the same, and is given by the cutoff dependence of the *symmetric* zero-mass theory. It follows that the two wave-function renormalization constants Z_{2i} ($i=1,2$) are each *finite* factors multiplying a common divergent factor (so their ratio is finite). It then follows that each of the two B_i are rendered finite by factoring out this same infinity. In particular, their difference $B_1-B_2=2\Sigma_V$ is rendered finite, but Σ_V is the two-fermion matrix element of $\mathfrak{M}_1$ [plus finite skeleton graphs of $O(\mathfrak{M}_1{}^3,\ldots)$]. So there are no new renormalization constants for two-fermion matrix elements of $\mathfrak{M}_1$, and it is known[11] that there are none for $\mathfrak{M}_0$. The Ward-Takahashi identities (5) or (19) then show that the vertex renormalization constants Z_{1A}, Z_{1B} have the same cutoff dependence as Z_{2i}. The asymmetric part of Γ_μ^B, which has odd powers of $\mathfrak{M}_1$ in it, therefore has the same cutoff dependence as the symmetric part of either vertex or either fermion propagator. The only remaining superficial divergence is in the B-meson self-energy, corresponding to graphs of $O(\mathfrak{M}_1{}^2)$, but the divergence is only logarithmic and can be removed by mass renormalization. Therefore the vector-meson wave-function renormalization constants Z_{2A}, Z_{2B} differ by finite factors from what they are in the symmetric theory.

These facts have important consequences for β, β' and the γ_i: They are the same as in the symmetric theory. This follows directly from the definitions

$$\begin{aligned}&\gamma_i=\tfrac{1}{2}\overline{D}\ln Z_{2i}\quad(i=1,2,A,B),\\&\beta=\tfrac{1}{2}g\overline{D}\ln Z_{2A},\\&\beta'=\tfrac{1}{2}g'\overline{D}\ln\left(\frac{Z_{2B}Z_{21}Z_{22}}{Z_{1B}{}^2}\right).\end{aligned} \tag{27}$$

The dimensionless functions Z depend on mass ratios, or on ratios of cutoff to mass. But $\bar{\mathcal{D}}$ gives zero when applied to a mass ratio, so the above formulas probe only the cutoff-dependent parts of the Z's; we have just shown that they are the same as in the symmetric theory. This result is somewhat analogous to the σ-model result[12] that all the divergences of the spontaneously broken σ model can be reduced to those of the normal model. In our case, we still have to deal with the cutoff dependence of δM_0 and M_{B0}. However, there are no infinities in the δ_i defined in (23).

D. Calculation of the CS coefficients in lowest order

All the coefficients $\beta, \beta', \gamma_i, \delta_0$ can be read off from well-known quantum-electrodynamic calculations; to lowest significant order, they are

$$\beta=\frac{g\alpha}{3\pi}, \quad \beta'=\frac{g'\alpha'}{3\pi}, \quad \gamma_A=\frac{\alpha}{3\pi}, \quad \gamma_B=\frac{\alpha}{3\pi},$$
$$\gamma_1=\gamma_2=\left(\frac{\alpha+\alpha'}{4\pi}\right)\lambda, \quad \delta_0=\frac{3}{2\pi}(\alpha+\alpha'). \tag{28}$$

[Recall that λ is the gauge parameter defined in (17); $\lambda=0$ is the Landau gauge.] It only remains to calculate δ_1, δ_2. To $O(g'^2)$, the B-meson contribution to the mass comes from inserting the effective vertex (18) in the usual self-energy graph *and* using the g'^2 part of $\mathfrak{M}_1$ [see (25)]. It is important to observe that the graph for M_1 has fermion 2 as an intermediate state. The formal expression for M_1 using only the $\gamma_\mu\tau_2$ part of the effective vertex (18) and the propagators (17) is

$$M_1=M_0+\delta M_0+\frac{ig'^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\left[\frac{2(\not p-\not k)-4(M_0-\delta M_0)}{(p-k)^2}\quad\frac{-(1-\lambda)(\not p-M_0+\delta M_0)}{k^2}\right]_{\not p=M}+O(\delta M_0{}^2)+A\text{-meson terms}, \tag{29}$$

where we have dropped all masses in denominators, since they do not contribute to the divergent part. However, we have carefully saved all mass dependence in the numerator. A similar expression holds for M_2, with the sign of δM_0 reversed. Subtract the two equations, and use the simple cutoff procedure

$$\int\frac{d^4k}{k^4}\to i\pi^2\ln\frac{\Lambda^2}{M^2}\equiv i\pi^2 L, \tag{30}$$

where Λ is the cutoff, and M^2 is *any* combination with dimension 2 of masses which occur in the theory. The result is (including the A terms)

$$\delta M^{(a)}\equiv\tfrac{1}{2}(M_1-M_2)$$
$$=\delta M_0\left\{1+L\left[\frac{3}{4\pi}(\alpha-\alpha')-\frac{\lambda\alpha'}{2\pi}\right]\right\}. \tag{31}$$

Note that this graph is gauge-dependent.

Next, calculate the graphs with $\lambda_\mu\tau_2$ at one vertex, $-2i\delta M_0\tau_1k^\mu k^{-2}$ at the other [see (18)]. There are two equal graphs, so we get the necessary result by multiplying one graph by two:

$$\delta M^{(b)}=\frac{2ig'^2}{(2\pi)^4}\int\frac{d^4k}{k^2(p-k)^2}(2\delta M_0)\gamma_\mu(\not p-\not k)\frac{k_\nu}{k^2}\left[-g^{\mu\nu}+(1-\lambda)\frac{k^\mu k^\nu}{k^2}\right]+\text{finite}\to+4\delta M_0\frac{\alpha'}{4\pi}\lambda L. \tag{32}$$

Finally, there is the contribution of the $O(g'^2)$ term in $\mathfrak{M}_1$, namely, $2g'^2\bar\Psi\delta M_0\tau_3\Psi(\Box^{-1}\partial\cdot B)^2$, which cancels off half of (32):

$$\delta M^{(C)}=-2\delta M_0\frac{\alpha'}{4\pi}\lambda L. \tag{33}$$

The sum of (31), (32), and (33) yields

$$\delta M=\delta M_0\left[1+\frac{3(\alpha-\alpha')}{4\pi}\right]L, \tag{34}$$

which is gauge-invariant; now using (23) we find

$$\delta_1=\frac{3}{2\pi}(\alpha-\alpha'), \tag{35}$$

also gauge-invariant. Note that $\delta_1=2\epsilon$, where ϵ [defined in (10)] is the asymptotic power of the symmetry-violating propagator.

If we had used the homogeneous-integral-equation approach[3,4] we would have included a symmetry-breaking vertex part with Goldstone pole at only *one* end of the fermion line. In this way, the contribution (32) would be cut in half, and there would be no contribution (33); the results for δM are unchanged. This reveals the computational significance of the Landau gauge in these earlier works: In this gauge, the Goldstone poles do not couple internally. (It is a useful coincidence that to lowest order the fermion anomalous dimensions are zero in this gauge.) In the present approach as well the Landau gauge greatly simplifies matters, since $\mathfrak{M}_1$ becomes a naive mass term and there is no nonpolynomial part to the Lagrangian.

Turn now to the computation of the symmetry-violating part of $\Pi^B_{\mu\nu}$. Here it is necessary to be less cavalier about the cutoff procedure. We can use conventional Pauli-Villars regularization, or much more conveniently the dimensional regularization of 't Hooft and Veltman.[25] A simple approach, valid for single-loop graphs, is merely to set

$$\frac{d^4k}{k^2} \equiv 0 \tag{36}$$

and then use the logarithmic regulator (30). It is *not* allowed to drop masses in the propagator denominators *ab initio*, because $\Pi^B_{\mu\nu}$ is quadratically divergent before regulation. However, after using (36), propagators may be expanded in powers of mass, and finite terms dropped.

At asymptotic momenta there is no purely kinematic way of isolating the symmetry-violating terms in $\Pi^B_{\mu\nu}$, as there was for the fermion propagators. We must *define* the symmetry-breaking self-energy $\Pi^{BV}_{\mu\nu}$ as that part which vanishes when δM_0 and M_{B0} are set to zero (keeping the cutoff finite). It is clear that odd powers of δM_0 cannot appear, as the leading corrections to the one-fermion-loop graph are $O(\delta M_0^2)$; there is no contribution from M_{B0}^2 to a one-loop graph. There are nine graphs: Three have conventional $\tau_2\gamma_\nu$ vertices and insertions of two $\mathfrak{M}_1$ vertices in all possible ways. Four graphs have one Goldstone vertex $-2i\delta M_0\tau_1 k_\mu k^{-2}$ and one $\mathfrak{M}_1$ correction; one graph has two Goldstone vertices; and the ninth graph comes from the $O(g'^2)$ part of $\mathfrak{M}_1$ [see above (33)]. This last graph is a seagull containing a closed fermion loop; its main role is to cancel a quadratic contribution to the two-Goldstone-vertex graph. If the rule (36) is applied, the ninth graph does not contribute to the divergent part.

The first three graphs (with conventional $\tau_2\gamma_\mu$ vertices) combine to give a result which can be rewritten as

$$\Pi^{BV}_{\mu\nu}(k) = \frac{-ig'^2}{(2\pi)^4}\int d^4p\tfrac{1}{2}\,\mathrm{Tr}\,\{\gamma_\mu[\tau_2, S(p)]\gamma_\nu[\tau_2, S(p)]\} \tag{37}$$

$$= \frac{-4\alpha'}{\pi}(\delta M_0)^2 L g_{\mu\nu} + \text{finite}, \tag{38}$$

using the regulator (30). All the rest of the graphs have one or more Goldstone-pole vertices, and they give a contribution which changes the $-g_{\mu\nu}$ in (38) to $-g_{\mu\nu} + k_\mu k_\nu k^{-2}$, thus ensuring gauge invariance. The integral (37) is just what occurs in the homogeneous-integral-equation approach,[3,4] only it converges because of the power-law decrease of $\Sigma_V(p)$ [see (9)]; for such a power-law decrease, replace L by $(2\epsilon)^{-1}$, as in (11). There is a profound difference in interpretation: Eq. (11) refers to $\Pi_{\mu\nu}$ at $k=0$, while (37) and (38) are to be used to find the asymptotic behavior of $\Pi_{\mu\nu}$.

The remaining CS parameter δ_2 is found by using (38) to write

$$M_B{}^2 = M_{B0}{}^2 + \frac{4\alpha'}{\pi}(\delta M_0)^2 L \tag{39}$$

and then using (23)

$$\delta_2 = \frac{4\alpha'}{\pi}\,\frac{(\delta M)^2}{M_B{}^2}\,; \tag{40}$$

just like δ_0 and δ_1, this coefficient is gauge-invariant and positive.

E. "Solution" of the Callan-Symanzik equations

Can we use the coefficients calculated in Sec. IID to rederive the homogeneous-integral-equation results (9), (10), and (11)? In a certain sense this is possible: We arbitrarily set β, $\beta'=0$, and use the rest of the CS coefficients as calculated. If we save β and β' to lowest order, the asymptotic solutions to the CS equations make no sense, because the theory is ultraviolet unstable (as all Abelian theories must be for sufficiently small coupling constants g, g'). To set β, $\beta'=0$ is to mimic a situation in which there is an ultraviolet-*stable* fixed point, where the coefficients δ_i are all positive, as we have calculated them. This is the same as the Baker-Johnson[10] approach to quantum electrodynamics.

The CS approach to the fermion propagator with β, $\beta'=0$ has already been discussed thoroughly by Adler and Bardeen.[11] For the symmetric part of the inverse propagator, we quote their results:

$$S_S{}^{-1}(p) \underset{p^2\to-\infty}{\sim} \not{p}A\left(\frac{-p^2}{M^2}\right)^{-\gamma} + MB\left(\frac{-p^2}{M^2}\right)^{\gamma-\delta_0/2}, \tag{41}$$

where $\gamma=\gamma_1=\gamma_2$ is the anomalous fermion dimension, M can be chosen without loss of generality as the renormalized symmetric mass $[M=\frac{1}{2}(M_1+M_2)]$, and A and B are dimensionless functions of mass ratios and coupling constants, independent of p. We briefly indicate how a similar result is found for the *symmetry-violating* inverse propagator, denoted by $-\Sigma_V(p)\,\tau_3$ as in (6). By construction, Σ_V is of first order in the mass operator $\mathfrak{M}$, and only the $\mathfrak{M}_1$ term is relevant (higher powers of $\mathfrak{M}_1$ are not asymptotically leading). With the help of (24), the right-hand side of the CS equation for $\Sigma_V(p)$ is simply $(1+\delta_1)\Sigma_V(p)$, so (20) becomes

$$(D-2\gamma)\Sigma_V(p) = (1+\delta_1)\Sigma_V(p). \tag{42}$$

The solution to (42) is

$$\Sigma_V(p) \underset{p\to\infty}{\sim} \delta M C\left(\frac{-p^2}{M^2}\right)^{-\gamma-\delta_1/2}, \tag{43}$$

where C is a dimensionless function of coupling constants and mass ratios. Note that δ_1, in (35), is twice ϵ given in (10), so that (43) agrees with (9) in the Landau gauge ($\gamma=0$). The significant point is that the gauge-dependent term $(-p^2/M^2)^{-\gamma}$ factors out of the full inverse propagator so that it is gauge-invariant and physically meaningful to compare the ratio of the symmetry-violating term with the symmetric term; if $\delta_1 > 0$, Σ_V vanishes with respect to the $\not p$ part of the symmetric self-energy.

The defining equation (23) for δ_1 is a differential equation which yields the cutoff dependence of the bare-mass splitting:

$$\delta M_0 = \delta M E\left(\frac{\Lambda^2}{M^2}\right)^{-\delta_1/2}, \tag{44}$$

where δM is the *renormalized* mass splitting, and E is a *cutoff-independent* function of renormalized mass ratios and coupling constants. If $\delta_1 > 0$, δM_0 vanishes as the cutoff approaches infinity. This is an essential requirement for spontaneous breakdown, for it means that $\mathcal{L}_{\text{eff}}$ in (14) is really the same as $\mathcal{L}_S$ (at least if $M_{B0}{}^2$ vanishes in the limit, which we show below).

The CS equation for $\Pi^{BV}_{\mu\nu}$ presents some interesting developments. The full symmetry-violating inverse propagator consists of the bare-mass term $M_{B0}{}^2$, plus the contribution of the one-loop graphs [see (38)]. As mentioned before, terms of $O(\mathfrak{M}_1)$ or $O(\mathfrak{M}_1\mathfrak{M}_0)$ give identically zero, so we need only keep terms of $O(\mathfrak{M}_1{}^2)$ or $O(\mathfrak{M}_2)$ on the right-hand side. From (24), $\overline{D}\mathfrak{M}_1{}^2 = 2\mathfrak{M}_1 = 2(1+\delta_1)\mathfrak{M}_1{}^2$; $\overline{D}\mathfrak{M}_2 = 2(1+\delta_2)\mathfrak{M}_2$. Then the CS equation for the scalar symmetry-violating function Π^{BV} is (recall that $\beta'=0$ means $\gamma_B=0$)

$$D\Pi^{BV} = 2(1+\delta_1)\Pi^{BV}_1 + 2(1+\delta_2)\Pi^{BV}_2, \tag{45}$$

where Π^{BV}_1 is the sum of all insertions of $\mathfrak{M}_1{}^2$, and Π^{BV}_2 is the sum of all insertions of $\mathfrak{M}_2$. Of course, $\Pi^{BV} = \Pi^{BV}_1 + \Pi^{BV}_2$. There is no physical significance to the breaking of Π^{BV} into two separate parts, and we desire to use one differential equation, i.e., (45), for one quantity Π^{BV}, not for the separate parts. The only way that this is possible is if $\delta_1 = \delta_2$, which yields with the help of (35), (40), and the fact that $\delta_1 = 2\epsilon$,

$$M_B{}^2 = \frac{2\alpha'}{\pi\epsilon}(\delta M)^2 \tag{46}$$

—precisely the relation (11) derived from the homogeneous-integral-equation approach. If indeed $\delta_1=\delta_2$, (45) is easily solved, when we remember that Π^{BV}, the coefficient of $-g_{\mu\nu}+k_\mu k_\nu k^{-2}$ in $\Pi^{BV}_{\mu\nu}$, has dimension 2:

$$\Pi^{BV} \underset{k\to\infty}{\sim} F(\delta M)^2(-k^2/M^2)^{-\delta_1}. \tag{47}$$

As usual, F is a dimensionless function of mass ratios and coupling constants. Also (23) can be integrated to yield the cutoff dependence of M_{B0}:

$$M_{B0} = M_B G(\Lambda^2/M^2)^{-\delta_1/2}. \tag{48}$$

It vanishes as $\Lambda\to\infty$, but the ratio $M_{B0}/\delta M_0$ is finite.

The condition $\delta_1=\delta_2$ is not arbitrary or coincidental; it is essential to the soft asymptotic behavior (47). The point is that the Π^{BV}_1 and Π^{BV}_2 are not this soft; only their sum is. The *renormalized* Π^{BV}, calculated to second order, is (asymptotically)

$$\Pi^{BV} = M_B{}^2 - \frac{2\alpha'}{\pi}(\delta M)^2 \ln\left(\frac{-k^2}{M^2}\right). \tag{49}$$

If these are the first two terms in the expansion of a power-law $(-k^2/M^2)^{-2\epsilon}$, then $M_B{}^2$ and $(\delta M)^2$ must be related as in (46). Any other relation leaves an asymptotically constant piece, which we reject.

The fact that in a spontaneously broken theory not all the renormalized masses are independent is not surprising, but it forces us to reinterpret the meaning of the mass derivative D. In fact, D should be the sum over independent masses only of mass derivatives. It suffices to say that D is a distributive operator which counts mass dimensions, without specifying in detail which masses are dependent and which are independent.

II. EXTENSION TO THE NON-ABELIAN CASE

As usual, going from an Abelian gauge theory to a non-Abelian one is not exactly straightforward. There are the usual ghosts,[2] plus nonlocal couplings of Goldstone scalars like ϕ to the vectors, and a complete analysis of this problem has not been carried out. Fortunately, the only really new dynamical feature which differentiates the two cases—the presence of two-vector intermediate states in $\Pi^{\mu\nu}$—turns out to be trivial, in $O(g^2)$ so that there is a closer resemblance to the Abelian case than might have been expected.

In Sec. III A we develop the effective Lagrangian approach, and then we do some second-order model calculations of CS coefficients. This may actually be sensible for the asymptotically free models. We have not made a serious effort to find any realistic models.

A. The effective Lagrangian approach

First, let us develop the notation. A Yang-Mills theory is characterized by the real, totally antisymmetric structure constants ϵ_{abc} of a compact Lie group, and by a set of vector fields B_μ^a which transform as the adjoint representation of this group (under global gauge transformations). For simplicity we consider explicitly only semisimple groups; the extension to non-semi-simple groups is quite straightforward. By virtue of the Jacobi identity the real antisymmetric matrices $(T^a)_{bc} = -\epsilon_{abc}$ obey the group commutation laws

$$[T^a, T^b] = \epsilon_{abc} T^c . \tag{50}$$

When convenience and clarity permit, we drop the group indices and use conventional matrix notation. Also useful is the cross-product notation $A\times B$, where

$$(A\times B)^a = \epsilon_{abc} A^b B^c , \quad (A\times)^{ab} = \epsilon_{acb} A^c . \tag{51}$$

Consider first the pure Yang-Mills Lagrangian, with no fermions:

$$\mathfrak{L}_{\text{YM}} = -\tfrac{1}{4}(G_{\mu\nu}^a)^2 ,$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + g B_\mu \times B_\nu , \tag{52}$$

where g is the coupling constant. The equations of motion are

$$-D_\mu G^{\mu\nu} = 0 , \tag{53}$$

where the covariant derivative operator D_μ is

$$D_\mu = \partial_\mu + g B_\mu \times . \tag{54}$$

Independently of the equations of motion, there is the identity

$$D_\mu D_\nu G^{\mu\nu} \equiv 0 . \tag{55}$$

The Lagrangian is invariant under the gauge transformation

$$B_\mu' = \alpha(\overline{\theta}) B_\mu - \frac{1}{g} \beta(\overline{\theta}) \partial_\mu \theta , \tag{56}$$

where, in terms of the local gauge parameters θ^a,

$$\overline{\theta} = T^a \theta^a , \quad \alpha(\theta) = e^{\overline{\theta}} , \quad \beta(\overline{\theta}) = (e^{\overline{\theta}} - 1)/\overline{\theta} . \tag{57}$$

Just as in Sec. II, we desire to add a mass term to (52), which is gauge-invariant in the restructed sense (that is, for gauge functions obeying $\Box\theta^a \neq 0$). Of course, the simple construction (13) fails. The necessity of a gauge-invariant mass term is to be found in the identity (55): If a new term $\mathfrak{L}'$ is added to $\mathfrak{L}_{\text{YM}}$, so the equations of motion become

$$-D_\mu G^{\mu\nu} = S^\nu , \tag{58}$$

(55) requires the constraint

$$D_\nu S^\nu = 0 . \tag{59}$$

This is equivalent to the statement that the action integral $\int d^4x \mathfrak{L}'$ is gauge-invariant, as one may verify by using the infinitesimal form of (56):

$$B_\mu' = B_\mu - \frac{1}{g} D_\mu \theta \cdots \tag{60}$$

and integrating (59) by parts.

Suppose that we could find some scalar fields ϕ^a, whose gauge-transformation law was ($\overline{\phi} = T^a \phi^a$)

$$\alpha(\overline{\phi}') = \alpha(\overline{\phi}) \alpha^{-1}(\overline{\theta}) . \tag{61}$$

This is a nonlinear transformation on ϕ, whose form can be given explicitly for infinitesimal θ:

$$\phi' = \phi - \beta^{-1}(-\overline{\phi})\theta \tag{62}$$
$$= \phi - \theta - \tfrac{1}{2}(\phi\times\theta) \cdots .$$

It then follows from the usual group-composition laws that the field

$$Q_\mu^a[B,\phi] = \left[\alpha(\overline{\phi}) B_\mu - \frac{1}{g}\beta(\overline{\phi})\partial_\mu \phi\right]^a \tag{63}$$

is gauge-invariant:

$$Q_\mu^a[B',\phi'] = Q_\mu^a[B,\phi] . \tag{64}$$

Moreover, the field

$$R_\mu^a = [\alpha^{-1}(\overline{\phi}) Q_\mu]^a = \left[B_\mu - \frac{1}{g}\beta^T \partial_\mu \phi\right]^a \tag{65}$$

(T means the matrix transpose; this form follows from $\alpha^{-1}\beta = \beta^T$) transforms *homogeneously* under the gauge group, from (61) and (64):

$$R_\mu'[B,\phi] = R_\mu[B',\phi'] = \alpha(\overline{\theta}) R_\mu[B,\phi] . \tag{66}$$

Now consider the equations of motion following from the Lagrangian

$$\mathfrak{L}' = \tfrac{1}{2}\mu^2 R_\nu{}^2 , \tag{67}$$

where μ is an arbitrary (group-scalar) mass parameter, introduced only for the purpose of giving $\mathfrak{L}'$ dimension 4; it has nothing to do with the physical vector masses to be discussed later. To find the variation of $\mathfrak{L}'$ with respect to B_μ is quite simple, and the result [termed S^ν in (58)] is

$$S^\nu = \mu^2\left(B^\nu - \frac{1}{g}\beta^T \partial^\nu \phi\right) = \mu^2 R^\nu . \tag{68}$$

The variation of $\mathfrak{L}'$ with respect to the scalar variables is also straightforward, but lengthier; what is not quite so straightforward is to show that these scalar equations of motion can be factored in the form

$$\beta D_\nu S^\nu = 0 , \tag{69}$$

with S^ν as given in (68). The trick is to use the identity (71) below which follows from an identity well known to students of Yang-Mills theories:

$$\alpha^{-1}\frac{\partial\alpha}{\partial\phi_a}\equiv\beta_{ab}T^b\ . \tag{70}$$

The identity

$$\frac{\partial\beta_{ad}}{\partial\phi_c}-\frac{\partial\beta_{cd}}{\partial\phi_a}\equiv\beta_{ab}\beta_{ce}\epsilon_{bde} \tag{71}$$

follows from differentiating (70) and invoking the commutativity of differentiation with respect to ϕ, and observing that the T^b are linearly independent for semisimple groups.

The importance of the factorized form (69) is, of course, that the scalar equations of motion automatically respect the constraint (59), which is necessary from the vector equations of motion. [Note that β is not singular, and can be factored out of (69).] This self-consistency ensures that our supposition (61) concerning the gauge-transformation law of the field ϕ is indeed true. To verify this law, solve Eq. (69) for ϕ as a power series in g:

$$\phi=\frac{g}{\Box}\,\partial\cdot B-\frac{g^2}{\Box}\left[\tfrac{1}{2}(\partial\cdot B)\times\frac{1}{\Box}\,\partial\cdot B+B_\mu\times\partial_\mu\,\frac{1}{\Box}\,\partial\cdot B\right]+O(g^3)\,. \tag{72}$$

The infinitesimal gauge-transformation law for B_μ, Eq. (60), directly verifies that

$$\phi'=\phi-\theta-\tfrac{1}{2}\phi\times\theta+O(g^2) \tag{73}$$

as required by (62) (always assuming $\Box\theta\neq0$). If ϕ is solved correct to $O(g^N)$, it will gauge-transform correctly to $O(g^{N-1})$, because (60) contains g^{-1} in it.

With the help of ϕ, we can construct a gauge-invariant field corresponding to any gauge-covariant field. For B_μ, the gauge-invariant field is Q_μ, and for the fermion fields we wish to incorporate, the corresponding construction is

$$e^{it_a\phi_a}\Psi, \tag{74}$$

where the t_a are the Hermitian generators for the representation carried by Ψ. These fields allow for the building of symmetry-breaking mass terms for both fermions and vectors, analogous to (14).

The next step is to write down the complete Lagrangian, with fermions and mass-splitting terms, as well as a set of Lagrange multiplier fields χ^a to enforce the scalar-field equation (69), and finally some gauge-setting terms:

$$\mathcal{L}=\mathcal{L}_S-\overline{\Psi}e^{-it\cdot\phi}\delta M_0e^{it\cdot\phi}\Psi+\tfrac{1}{2}\mathrm{Tr}\,\{Q_\mu M_{B0}{}^2Q^\mu\}-\chi D^\nu\left(B_\nu-\frac{1}{g}\,\beta^T\partial_\nu\phi\right)-\frac{1}{2\lambda}\,(\partial\cdot B)^2\,, \tag{75}$$

where $\mathcal{L}_S$ is the symmetric Yang-Mills Lagrangian, including fermions, and the asymmetric bare-mass terms δM_0, $M_{B0}{}^2$ are matrices. Because of the complexity of the condition expressing ϕ in terms of B_μ [the first-order terms are given in (72)], it is very much more difficult to manipulate this Lagrangian than in the Abelian case. In addition, of course, the naive Feynman rules based on (75) must be supplemented with the usual ghost-loop terms[2] and possibly other determinantal terms.[28,29] The author has not yet carried out the relevant analysis, but hopes to do so in another publication. One thing is clear: The Landau gauge leads to a tremendous simplification of the Feynman rules, since any $\partial\cdot B$ term which occurs as one end of an internal B line gives zero. Thus in proper vertices, $\partial\cdot B$ can only appear at the external (off-shell) meson vertices. This fact allows us to draw certain ostensibly gauge-invariant conclusions, although these must be regarded as tentative until a manifestly gauge-invariant analysis is done.

Consider, for example, the contribution of the two-vector loop (plus associated ghosts and seagulls) to the vector self-energy. It is possible that the loop could contribute to the (matrix generalization of the) CS coefficient δ_2, in order g^2. If so, there is a term in $\Pi^{BV}_{\mu\nu ab}$ which is proportional to the square of meson masses, and has the kinematic structure $k_\mu k_\nu k^{-2}$. To find such a term, one writes out $\mathcal{L}$ to $O(g^2)$ [which means finding ϕ to $O(g^3)$; we spare the reader the unilluminating corrections to (72)] and seeks for vertices or seagulls of the appropriate form in the Landau gauge. Many candidates go to zero in this gauge, and only one vertex survives:

$$\mathcal{L}_1=g\,\mathrm{Tr}\left\{B^\mu M_{B0}{}^2\left(B_\mu\times\frac{1}{\Box}\,\partial\cdot B\right)\right\}+\cdots\ . \tag{76}$$

When coupled with an ordinary Yang-Mills vertex from $\mathcal{L}_S$, it yields the right sort of term, but with vanishing coefficient: The scalar self-energy Π_{ab} has as a factor

$$\epsilon_{acd}\epsilon_{bcd}(M_c{}^2-M_d{}^2)\equiv0\,, \tag{77}$$

where $M_c{}^2$ is an element of $M_{B0}{}^2$, assumed to be diagonal without loss of generality.

If this conclusion is verified by the full gauge-invariant analysis, it means that the $O(g^2)$ calculations of δ_1 and δ_2 can be carried out in rather obvious analogy to the Abelian calculations of Sec. II. Below, we list the results of such calculations, to show what bearing the choice of group representations has on these coefficients.

B. Simple group-theoretic considerations

Remember that two vector mesons were needed in the Abelian case to achieve $\delta_1>0$, as needed for spontaneous breakdown. The extra Abelian meson coupled to the fermion number current may or may not be necessary in the Yang-Mills case, as we shall see, but if it is necessary, the resulting theory is not asymptotically free. Since our $O(g^2)$ calculations only make sense if the theory is asymptotically free, we confine ourselves for the most part to simple groups.

Let us first calculate the components of $\delta_1 = \{\delta_1^i\}$ for a parity-conserving, pure vector group (no axial-vector mesons). There is one δ_1^i for each irreducible representation i occurring in the fermion mass matrix

$$\delta M_0 = \sum_i \delta M^i P^i \ , \qquad (78)$$

where P^i are matrix representatives of appropriately chosen symmetric traceless tensors. One easily finds, in analogy with the Abelian result (35), that

$$\delta_1^i = \frac{3\alpha}{2\pi} \Lambda_i \ , \qquad (79)$$

where Λ_i is defined by

$$\sum_a t^a P^i t^a = \Lambda^i P^i \qquad (80)$$

and $\alpha = g^2/4\pi$. The Λ^i are calculated by commuting t^a through P^i, thus producing the Casimir operator $\sum t_a{}^2$. Define: N is the dimension of the group $(=n^2-1)$ for SU(n); $C_2(A)$ is the Casimir operator $\sum T_a{}^2$ for the adjoint representation A; $C_2(R) = \sum t_a^2$ for the fermion representation R. Let P^i be the projector for a symmetric, traceless tensor of rank i (the tensor's indices are those of the adjoint representation, which has $i = 1$). Then we find

$$\Lambda^i = C_2(R) - \frac{i}{2}\left(\frac{N+i-2}{N-1}\right) C_2(A) \ . \qquad (81)$$

From this we can read off the main features which determine the sign of Λ^i. Large i tends to make Λ^i negative, which is bad; thus in SU(3) with $R = A = 8$, $i = 1$ corresponding to octet breaking yields $\Lambda^i > 0$, but $i = 2$ corresponding to 27 yields $\Lambda^i < 0$ ($i = 2$ gives $\Lambda^i < 0$ for all groups, for $R = A$). If $C_2(R)$ is too small, it is also bad. For SU(n) the Casimir operator for the spinor (fundamental) representations is conveniently chosen to be $C_2(S) = (n^2-1)/2n$, corresponding to $C_2(A) = n$. (81) then shows that $\Lambda^i < 0$ for all i; *in $O(g^2)$, there can be no spontaneous breakdown of SU(n) with the fermions in the fundamental representation.* Even with this proviso, there is plenty of room left for asymptotic freedom, which requires[16]

$$C_2(A) > \frac{4}{11N} \sum_R d(R) C_2(R) \ . \qquad (82)$$

There can be two sets of fermions in the adjoint representation, with (82) satisfied and $\Lambda^i > 0$.

Now consider the vector-meson masses. The analog of the Abelian mass formula (39) is [see (37) also]

$$(M_B{}^2)_{ab} = (M_{B0}{}^2)_{ab} - \frac{\alpha}{2\pi} \mathrm{Tr}\{[t_a, \delta M_0][t_b, \delta M_0] L\} \ . \qquad (83)$$

The trace in (83) is non-positive-definite, which corresponds to a non-negative-definite set of δ_2^i. Just as for the fermions, one may analyze $M_B{}^2$ in terms of irreducible representations; more than one always contributes because $M_B{}^2$ is not traceless. In the simple case where δM_0 is a number times a fixed generator t_0 of the group, one finds that $(M_B{}^2)_{ab} \sim (T_0{}^2)_{ab}$. Then there is at least one massless vector ($M_{00}{}^2 = 0$) corresponding to the nonbroken Abelian symmetry generated by t_0. The appearance of massless vectors can be avoided if the fermion representation is sufficiently large, but then the constraint (82) of asymptotic freedom is in danger of being violated. As an artificial example for SU(2), the fermion representation with isospin $I = \frac{3}{2}$ is asymptotically free (barely), allows positive Λ's for $I = 1$ and 2, and can give mass to all three mesons. When some masses vanish, there are corresponding δ_1's which vanish; this is irrelevant since they refer to the asymptotic behavior of symmetry-breaking self-energy parts which vanish identically, because of the conserved subgroup which gives rise to the massless particles.

Turn now to parity-conserving chiral groups; here each vector meson has an axial-vector partner, and the left- and right-handed fermion representations R_L, R_R occur symmetrically: $(R_L, R_R) \oplus (R_R, R_L)$. There is an interesting cancellation in the fermion mass operator, which makes this object finite in lowest order. But the mass operator will not vanish identically if the vector mass M_V is different from the axial-vector mass M_A. This mass difference is, in turn, nonzero if the fermion mass operator does not vanish identically, but again it is not divergent. These theories are therefore not potentially divergent but instead they are soft, and the considerations of the present work do not apply. Whether spontaneous breakdown actually takes place or not can only be decided by investigating the nonasymptotic part of the theory.

Finally we consider parity-violating groups, in which the left- and right-handed fermion representations occur asymmetrically. For each simple subgroup, there is a lowest-order symmetric Yukawa vertex

$$\gamma^\mu \tfrac{1}{2}(1+\gamma_5)t_R^a + \gamma^\mu \tfrac{1}{2}(1-\gamma_5)t_L^a , \qquad (84)$$

where $\frac{1}{2}(1\pm\gamma_5)$ are the projection operators for right- and left-handed fermions. The symmetry-violation fermion mass matrix takes the form

$$M = M_R \tfrac{1}{2}(1+\gamma_5) + M_L \tfrac{1}{2}(1-\gamma_5) \qquad (85)$$

and TCP invariance means that $M_L^\dagger = M_R$. The physical masses are the positive eigenvalues of $(M_L M_R)^{1/2}$. M_L (or M_R) can be decomposed into irreducible tensors as before; calculations analogous to those of the parity-conserving groups yield formula (79) but with Λ^i defined as

$$\sum_a t_L^a P_R^i t_R^a = \Lambda^i P_R^i \qquad (86)$$

or by (86), with L and R interchanged. If this formula gives $\Lambda^i = 0$, it means that the symmetry breaking (if any is possible) is soft, not potentially divergent, and the asymptotic considerations of the present work are inapplicable. This happens for any fermion field (such as a two-component neutrino field) with only one chirality, for then either $t_L^a = 0$ or $t_R^a = 0$. Thus in the Weinberg model[31] only the photon couples both chiralities and it gives a positive δ_1 for the electron, as in Baker-Johnson[10] electrodynamics. Of course, the Weinberg model is not asymptotically free.

For the mesons, the mass formula from which the δ_2's can be derived is

$$(M_B{}^2)_{ab} = (M_{B0}{}^2)_{ab} - \frac{\alpha}{2\pi}\,\mathrm{Tr}\,\{t_L^a M_R M_L t_L^b + t_R^a M_L M_R t_R^b - 2 t_L^a M_R t_R^b M_L\}\,L \qquad (87)$$

(where the L outside the brackets stands for the cutoff-dependent logarithm). Again, the trace is non-positive-definite, yielding non-negative-definite δ_2's.

There is at least one weak-interaction model which is asymptotically free and which spontaneously breaks down without Higgs scalars: the Georgi-Glashow[32] model with two or fewer fermion triplets.

Armed with all the CS coefficients calculated in ths section, plus those referring to the symmetric sector[14-16] one may carry out an analysis of asymptotic behavior such as that given in Sec. II. However, for asymptotically free theories it is consistent (and necessary) to keep β in the CS equations, and this changes the asymptotic behavior from a power of momentum to a power of the logarithm of the momentum, as is well known. However, the condition that δ_1^i and δ_2^i are positive leads to symmetry-breaking Green's functions which vanish asymptotically compared with the symmetric Green's functions. Let us define coefficients a, b, c by power-series expansions of β, γ, δ_i near the origin:

$$\begin{aligned}\beta(g) &= -ag^3 + \cdots ,\\ \gamma(g) &= bg^2 + \cdots ,\\ \delta_i(g) &= c_i g^2 + \cdots .\end{aligned} \qquad (88)$$

For an asymptotically free theory, $a > 0$, and we define a spontaneously broken theory as one for which $\delta_i > 0$, that is $c_i > 0$. We now read off from Eq. (4.16) of Gross and Wilczek[16] that, for example, the $A\not p$ part of the fermion self-energy has the asymptotic behavior [cf. (41)]

$$A \underset{p\to\infty}{\sim} \left[\ln\left(-\frac{p^2}{M^2}\right)\right]^{-b/a}, \qquad (89)$$

while the symmetry-violating self-energy behaves like [cf. (43)]

$$\Sigma_V^i(p) \underset{p\to\infty}{\sim} \delta M_i C_i \left[\ln\left(-\frac{p^2}{M^2}\right)\right]^{-(2b+c_i)/2a}. \qquad (90)$$

With $c_i/a > 0$, Σ_V/A vanishes asymptotically. As before, the bare symmetry-breaking masses vanish, but only logarithmically, in the limit of infinite cutoff:

$$\delta M_0^i = \delta M^i E^i \left(\ln\frac{\Lambda^2}{M^2}\right)^{-c^i/2a} . \qquad (91)$$

Similar conclusions hold for the vector self-energy and bare mass.

IV. CONCLUSIONS

We have given a prescription for calculating, in principle (and even in practice, for asymptotically free theories), the asymptotic behavior of the symmetry-breaking Green's functions in gauge theories without scalar mesons. A necessary criterion for spontaneous breakdown is that the CS coefficients δ_1, δ_2 be positive at the UV stable fixed point, but of course this is not a sufficient condition, any more than the necessary condition that a bound-state Schrödinger wave function behave like $e^{-|\kappa|r}$ at infinite r ensures the existence of the bound state. The rate at which the symmetry-breaking Green's functions decrease compared with the symmetric ones depends on the nature of the zeros (if any) in δ_1 and δ_2 at the fixed point; for asymptotically free theories the decrease is only logarithmic, which may well mean in practice that symmetry-breaking Green's functions will not be observed to decrease relative to symmetric ones.

No realistic models which are asymptotically free and which allow spontaneous breakdown without scalars have yet been found either for the strong or the weak interactions (unless the Georgi-

Glashow model with no more than six fermions is realistic). In the strong-interaction case, removing all massless vectors requires a number of fermions so large that the conditions of asymptotic freedom may be violated, yet the flexibility in models without scalars seems to be greater than in models with scalars.[16] (Of course, if the broken strong symmetry is a color group it may not matter that all vector mesons be massive.)

These results were based on an effective Lagrangian which is nonlocal and nonpolynomial but renormalizable, and which is gauge-invariant only for gauge functions θ obeying $\Box\theta \neq 0$. $\mathcal{L}_{\text{eff}}$ is *not* gauge-invariant for constant θ, and the author believes that there is no renormalizable $\mathcal{L}_{\text{eff}}$ which is invariant for constant θ. The reason why only restricted gauge invariance is possible is that the operators which implement global gauge transformations—the charges—do not exist for spontaneously broken theories.[33] However, the current densities do, and it is these which generate local gauge transformations if $\theta(x)$ is sufficiently well behaved at spatial infinity, through the operator $Q[\theta] = \int d^3x\, J^0(x)\theta(x)$. A similar thing happens in superconductivity, where the eigenstates of the Hamiltonian are not eigenstates of the charge operator.

One final question: Since gauge-invariant meson fields [Eq. (64)] and fermion fields [Eq. (74)] are available, why not write any old Lagrangian down in terms of these fields, and call it gauge-invariant? (Note, incidentally, that the symmetric Lagrangian $\mathcal{L}_S$ is invariant under the substitution of gauge-invariant fields for ordinary ones.) Instead of sticking to symmetry-breaking mass terms, one could also add symmetry-violating vertex terms and presumably retain renormalizability. While no firm conclusion can be drawn yet, it seems unlikely that such a shotgun approach will succeed. We demand that the symmetry-breaking terms all vanish asymptotically, and a preliminary investigation shows that it is difficult to get vertex symmetry breaking to vanish asymptotically when it appears in the Lagrangian. In the homogeneous-integral-equation approach, one would have a new set of homogeneous Dyson equations to solve, along with those that already exist for the propagators and Goldstone vertices. Each homogeneous equation imposes special constraints (akin to the positivity of δ_1, δ_2) on the field theory of a type not usually considered. It appears that the minimum number of such constraints is embodied in the program described here and earlier,[3,4] and it may well be that only this program can be consistently interpreted as spontaneous breakdown of a symmetric gauge theory. Whether or not this is really true clearly merits further investigation.

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APPENDIX: YANG-MILLS THEORY IN THE LIGHT-CONE GAUGE

For some time, it has been known that the Yang-Mills Lagrangian can be canonically quantized in the gauge $n_\mu B^\mu = 0$, where n_μ is a fixed spacelike or lightlike vector,[21,22] and that there are no Faddeev-Popov ghosts in these gauges. The absence of ghosts effects an astounding simplification of the Ward-Takahashi identities, which makes it well worth investigating such gauges. Unfortunately, as Mohapatra's papers[22] show, it is extremely tedious to do loop integrations in the spacelike gauges $n^2 < 0$. The purpose of this appendix is twofold: first, to show that loop integrations can be done quite simply and covariantly in the light-cone gauge $n^2 = 0$, which is quite often invoked for Abelian theories,[20] and second, to show that in a formal and heuristic way the light-cone gauge is useful for understanding the problem of spontaneous symmetry breakdown in Yang-Mills theory in a way analogous to the homogeneous-integral-equation approach for Abelian theories.[3,4] This heuristic utility rests directly on the simplified Ward identities, and can best be appreciated in comparison with Sarkar's[26] discussion of the Schwinger mechanism in conventional covariant gauges with ghosts. The light-cone gauge suffers from one great drawback: Loop integrals in momentum space sometimes lead to Feynman-parameter integration divergences of the type $\int_0 dZ/Z$ (in four dimensions). It remains to be seen whether this drawback is fatal or not.

There are two approaches to the light-cone gauge; one is to consider the Lagrangian with a gauge-breaking term added,

$$\mathcal{L} = \mathcal{L}_{\text{YM}} - \frac{1}{2\lambda}(n\cdot B)^2, \qquad \text{(A1)}$$

and pass to the limit $\lambda = 0$; the other is to impose the constraint $n\cdot B = 0$ on $\mathcal{L}_{\text{YM}}$ before writing down the field equations. The two approaches differ only trivially. The author knows of no published work on quantizing $\mathcal{L}_{\text{YM}}$ at equal times in the light-cone gauge, but Tomboulis's work,[23] in which he quantizes $\mathcal{L}_{\text{YM}}$ on the null plane in this gauge, can easily be transcribed to equal times with a few minor changes. In this second approach, the free vector propagator is

$$\Delta_{\mu\nu}(q) = -\frac{P_{\mu\nu}(q)}{q^2+i\epsilon}, \tag{A2}$$

where

$$P_{\mu\nu} = g_{\mu\nu} - \frac{(n_\mu q_\nu + n_\nu q_\mu)}{n\cdot q} + \frac{q^2 n_\mu n_\nu}{(n\cdot q)^2},$$
$$n^\mu P_{\mu\nu} = q^\mu P_{\mu\nu} = 0. \tag{A3}$$

The last term of $P_{\mu\nu}$ yields a seagull (term with no $q^2=0$ pole) in $\Delta_{\mu\nu}$. However, it is easy to show that there are seagulls in the interaction Lagrangian which just cancel this last term, so the propagator is, effectively,

$$\Delta_{\mu\nu} = \frac{Q_{\mu\nu}}{q^2+i\epsilon}, \tag{A4}$$

$$Q_{\mu\nu} = g_{\mu\nu} - \frac{(n_\mu q_\nu + n_\nu q_\mu)}{n\cdot q}, \quad n^\mu Q_{\mu\nu} = 0. \tag{A5}$$

In the first approach, based on the Lagrangian (A1), the propagator is

$$\Delta_{\mu\nu} = -\frac{Q_{\mu\nu}}{q^2+i\epsilon} - \lambda\frac{q_\mu q_\nu}{(n\cdot q)^2}. \tag{A6}$$

Note that (A4) and (A6) agree when $\lambda=0$, and that the resulting propagator is *homogeneous* in the components of n^μ. As we shall see, the loop integration rules do not destroy this homogeneity, so every Feynman graph of the theory is homogeneous in n^μ. This simple fact allows us to write down the complete kinematical dependence of self-energy graphs on n^μ without doing any calculation of the graphs, a property which is definitely not true for the spacelike gauges $n^2<0$. The kinematical dependence is governed by the Ward-Takahashi identities, just as in covariant gauges.

The Ward-Takahashi identities are easily derived by making an infinitesimal gauge transformation of the variables of integration B_μ in the Green's function generating functional $W(j)$[34]:

$$W(J) = \int [dB_\mu] \exp\left\{ i\int dx\left[\mathfrak{L}_{\text{YM}} - \frac{1}{2\lambda}(n\cdot B)^2 + J_\mu B^\mu\right]\right\}. \tag{A7}$$

Thus we find

$$\left\{-\frac{1}{\lambda}n\cdot\partial n^\mu\left[\frac{-i\delta}{\delta J^\mu}\right] + \partial_\mu J^\mu - gJ^\mu\times\left[\frac{-i\delta}{\delta J^\mu}\right]\right\} W(J) = 0. \tag{A8}$$

The corresponding expression for gauges with ghosts is considerably more complicated.[34] One simple consequence, found by differentiating (A8) once with respect to J^ν, is that the vector self-energy $\Pi_{\mu\nu}$ is conserved: $q^\mu\Pi_{\mu\nu}(q)=0$. There are two possible forms for $\Pi_{\mu\nu}$, instead of only one as in covariant gauges:

$$\Pi_{\mu\nu} = \theta_{\mu\nu}\Pi_1 - P_{\mu\nu}\Pi_2, \tag{A9}$$

where $\theta_{\mu\nu} = q_\mu q_\nu - q^2 g_{\mu\nu}$ and Π_1, Π_2 are scalar functions of q^2, independent of n^μ. In a covariant gauge, masslessness of the Yang-Mills vector is ensured kinematically in the symmetric case (no pole in Π_1), but here masslessness only occurs if also $\Pi_2(0)=0$. To show this, recall[34] the rules for transforming from one gauge to another, say from the Landau gauge to the light-cone gauge. One solves the equations $n\cdot B'(\theta)=0$, where $B'(\theta)$ is the gauge transform of the Landau gauge field B_μ [see (56)], for θ, with the result

$$B'_\mu = B_\mu - \partial_\mu\left(\frac{1}{n\cdot\partial}n\cdot B\right) + O(B^2). \tag{A10}$$

Then the propagator in the light-cone gauge, $\Delta_{\mu\nu}$, is given in terms of the Landau-gauge propagator, $\Delta^{(\text{L})}_{\mu\nu}$, by

$$\Delta^{(\text{LC})}_{\mu\nu} = R_{\mu\alpha}R_{\nu\beta}\Delta^{(\text{L})}_{\alpha\beta} + \cdots, \tag{A11}$$

where

$$R_{\mu\alpha} = g_{\mu\alpha} - \frac{q_\mu n_\alpha}{n\cdot q}, \quad n^\mu R_{\mu\alpha} = 0, \quad q^\alpha R_{\mu\alpha} = 0 \tag{A12}$$

and the omitted terms in (A11) which come from the $O(B^2)$ terms in (A10) have no pole at $q^2=0$. It is then simple algebra to show that $\Pi_2(0)=0$. [In quantum electrodynamics, where the $O(B^2)$ terms in (A10) are missing, one can show $\Pi_2(q)\equiv 0$.]

This furnishes an amusing illustration of the breakdown of the Goldstone theorem when manifest covariance is given up (for further discussion, see the original work of Higgs.[1] It is no longer required that self-energy parts, vertices, etc., have zero-mass poles in order for spontaneous breakdown to take place; here $\Pi_2(0)\neq 0$ is enough. For another example, suppose we add (to the free theory) a gauge-conserving mass term of the type (13). The free propagator now becomes

$$\Delta_{\mu\nu} = -\frac{Q_{\mu\nu}}{q^2-M_{B0}^2+i\epsilon} - \frac{\lambda q_\mu q_\nu}{(n\cdot q)^2} \tag{A13}$$

instead of (A6). There are no zero-mass poles in this propagator, but there are in the corresponding covariant gauge free-field theory [Eq. (17)].

The Ward-Takahashi identity for the proper three-meson vertex is derived from (A8) by two differentiations and then stripping off some propagators; the result is

$$p^\lambda\Gamma^{abc}_{\lambda\mu\nu}(p,q,r) = \epsilon_{abc}[\Delta^b_{\mu\nu}(q)^{-1} - \Delta^c_{\mu\nu}(r)^{-1}], \tag{A14}$$

where $\Gamma^{abc}_{\lambda\mu\nu}$ is the proper vertex for the three vectors (p_λ, a), (q_μ, b), (r_ν, c), and the outgoing momenta obey $p+q+r=0$. This is indeed satisfied by the free-field Green's functions:

$$\Gamma^{abc}_{\lambda\mu\nu}=\epsilon_{abc}[(p-q)_\nu g_{\mu\lambda}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda}], \tag{A15}$$

$$\Delta_{\mu\nu}{}^{-1}(q)=-q^2 g_{\mu\nu}+q_\mu q_\nu+\frac{1}{\lambda}n_\mu n_\nu, \tag{A16}$$

and can be formally verified "by hand" in higher orders.

We may now copy the homogeneous-integral-equation approach[3,4] reviewed in Sec. II for the case of spontaneous breakdown. If, in the exact $\Delta_{\mu\nu}{}^{-1}$ formed from (A9), either Π_1 has a pole or $\Pi_2(0)\neq 0$, the identity (A14) implies that $\Gamma^{abc}_{\lambda\mu\nu}$ is singular at $p=0$, $q=r$. But now there are two types of singular symmetry-breaking vertices:

$$\left(\frac{p_\lambda}{p^2}\text{ or }\frac{n_\lambda}{n\cdot p}\right)\epsilon^{abc}[\Delta^{bV}_{\mu\nu}(q)^{-1}-\Delta^{cV}_{\mu\nu}(q)^{-1}]+\cdots, \tag{A17}$$

where the omitted terms are not singular at $p=0$, and $\Delta^{bV}_{\mu\nu}(q)^{-1}$ is the symmetry-violating inverse propagator. Again, there need be no Goldstone pole at $p^2=0$; $n\cdot p$ singularity will do. If we calculate the two-vector contribution to the vector self-energy in analogy to the calculation of the fermion-antifermion contribution,[3,4] it is clear that the Goldstone pole in (A17) contributes to the pole in Π_1, while the other term contributes to $\Pi_2(0)\neq 0$. However, in fact there is no contribution, because at the other end of the loop there is an ordinary vertex ϵ^{abc} which when multiplied into (A17) and summed over the intermediate-state group labels b, c gives zero. (This also happens in the Landau gauge; see Sec. III.)

The light-cone gauge, with its simple Ward-Takahashi identities, will be useful if loop integrals can be done with no more effort than in covariant gauges. We give here an integration theorem which is very useful and covers all one-loop graphs. We do not discuss the question of overlapping divergences which might occur in multiloop graphs.

Theorem. Let

$$F(p;q_i,Q_j)\equiv\prod_j^K n\cdot(p-Q_j)\left[\prod_i^N n\cdot(p-q_i)\right]^{-1} \tag{A18}$$

be a general rational function of scalar products of n^μ with the integration variable p^μ and other vectors q^μ, Q^μ. Then, in d-dimensional space (in the sense of 't Hooft and Veltman[25]),

$$I(q,Q,k)\equiv\int d^dp\,\frac{F(p;q_i,Q_j)}{[(p-k)^2-M^2]^l}=F(k;q_i,Q_j)\int\frac{d^dp}{[(p-k)^2-M^2]^l}\qquad[\mathrm{Re}(l-\tfrac{1}{2}d)>0]. \tag{A19}$$

In short, scalar functions involving n^μ can be factored out of every loop integral.

Proof. First let $K\leq N$ in (A18). Then by partial fractions F can be written as a sum of different terms like (A18), in each of which the numerator is 1 (i.e., $K=0$). So it suffices to show (A19) for $K=0$ to cover all $K\leq N$. Use the identity

$$x^{-1}=\tfrac{1}{2}i\int_{-\infty}^{\infty}d\alpha\,e^{-i\alpha x}\epsilon(\alpha) \tag{A20}$$

to write (A19) (with $K=0$) as

$$I=(\tfrac{1}{2}i)^N\int\prod d\alpha_i\epsilon(\alpha_i)\int d^dp\,\frac{\exp[-in\cdot\sum\alpha_i(p-q_i)]}{[(p-k)^2-M^2]^l}. \tag{A21}$$

By translation of variables $p\to p-k$,

$$I=(\tfrac{1}{2}i)^N\int\prod d\alpha_i\epsilon(\alpha_i)\exp[-in\cdot\sum\alpha_i(k-q_i)]\times\int d^dp\,\frac{\exp[-ip\cdot\sum\alpha_i n]}{(p^2-M^2)^l}. \tag{A22}$$

The integral over p is an invariant function of argument $y^\mu\equiv\sum\alpha_i n^\mu$, thus a function of y^2 only, but $y^2=0$. (A19) follows at once. It is clear from this proof that $n^2=0$ plays a vital role in achieving the simple result.

The proof for the case where $K-N$ is any fixed positive integer can be established by means of a sequence of formulas which are used when the integrand is a tensor of rank $K-N$. These are all established by differentiating the fundamental result (A19) with respect to k^μ. Although a general formula can be given, it is rather unwieldy. Differentiate (A19) with respect to k^μ and rearrange:

$$\int d^dp\,p_\mu\frac{F(p;q_i,Q_j)}{D^l}=F(k;q_i,Q_j)\int d^dp\,\frac{p_\mu}{D^l}+\frac{1}{2(l-1)}\left[\frac{\partial}{\partial k^\mu}F(k;q_i,Q_j)\right]\int\frac{d^dp}{D^{l-1}}, \tag{A23}$$

where $D=(p-k)^2-M^2$. By dotting this formula into n^μ, (A19) is immediately proved for $K=N+1$, since the derivative in the last term of (A23) has n^μ as a factor. By further differentiations one proves not only (A19) for all $K>N$, but also all of its derivatives such as (A23). It should be noted that if the integrand is homogeneous in M of degree J so is the integral (A19) or any of its k derivatives.

Although these formulas are formally simple, they are troublesome in two aspects. The first is that one loses convergence of integrals at a faster than usual rate by adding tensor indices in the numerator of (A19). Consider (A23) with $d=4$, $l=2$, $F\equiv 1$. The integral with or without the p^μ is logarithmically divergent (barring triangle anomalies). However, if F is a nontrivial function of n^μ, the last term on the right of (A23) is quadratically divergent. In order that Yang-Mills theory be renormalizable in this gauge, it is necessary that such quadratic divergences be absent. Fortunately, gauge invariance does seem to remove them. These more-divergent-than-usual terms always have the most factors of n^μ absorbing the tensorial index structure. Consider the $P_{\mu\nu}\Pi_2$ part of the self-energy (A9). We have already argued that $\Pi_2(q)$ vanishes (at least in the symmetric case) as $q^2\to 0$, and the expression for $P_{\mu\nu}$ in (A3) shows that an extra factor of q^2 must be supplied in the $n_\mu n_\nu$ part. The requirement that the coefficient of $n_\mu n_\nu$ vanish like q^4 can be used to reduce what would naively be a quartic divergence to a "logarithmic" one.

We have used quotation marks here because of the second troublesome aspect, which might be far more serious. The vectors k, q_i, Q_j in (A19) are linear combinations of external momenta with Feynman parameters. In four dimensions, it is quite possible to produce divergent integrals over Feynman parameters of the type $\int_0 dZ/Z$. These are not spurious divergences, but real ones; they appear in the imaginary part of integrals and can be directly traced to the fact that $n\cdot p$ can vanish for an *on-shell* ($p^2=0$) vector meson. They would thus appear in any application of the Cutkosky rules. (At least they would appear in Cutkosky rules applied to off-shell Green's functions; the S matrix should be independent of n^μ and have no such problems.) One might be tempted to call them infrared divergences, because $n\cdot p$ can never vanish for a massive on-shell meson, as long as the components of p are finite. But they are really ultraviolet divergences, because as the components of p become infinite with $p^2=M^2$ fixed, $n\cdot p$ is of $O(p_0^{-1})$ and not $O(p_0)$ in certain kinematical configurations.

All is not necessarily lost, though. The extra divergences appear to be no worse than logarithmic, and they can be regulated by staying away from $d=4$. (The parameter integral will look something like $\int dZ\,Z^{3-d}$.) So the vector-meson self-energy behaves like $\ln^2\Lambda^2$ instead of $\ln\Lambda^2$, in second order. These extra logarithms never affect the number or kind of renormalizations necessary to make the theory finite, nor do they interfere with making gauge-invariant renormalizations. Sometimes (but not always) a factor of Z appears in the numerator to cancel out the Z^{-1} coming from the loop integral. However, in view of these new and peculiar difficulties, it is only fair to say that the light-cone gauge may not exist for non-Abelian gauge theories. It does seem to exist for Abelian gauge theories; for example, we have already remarked that in quantum electrodynamics the photon self-energy is the same function in the light-cone gauge that it is in any covariant gauge. If this gauge does exist, it will have interesting implications for the CS equations, a subject to which the author will return in another publication.

After the work reported in this appendix was completed, the author was informed of work by Chakrabarti and Darzens, covering some of the same points.[24] These authors have given a special case of the integration formula (A19) which they have derived by other means.

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Classical electrodynamics of a nonlinear Dirac field with anomalous magnetic moment

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The classical electromagnetic interactions of a nonlinear spinor field are studied in perturbation theory. When Pauli terms are included, the model describes with reasonable accuracy (within the assumed approximations) such properties of the nucleons as spin, charge, magnetic moment, and the proton mass. With no other information one can calculate the proton-neutron mass difference, which comes out of the wrong sign and of the same size as in quantum electrodynamics.

I. INTRODUCTION

The purpose of this paper is to explore the classical electrodynamics of a nonlinear spinor field as a possible model of elementary particles.

Since the work by Rosen[1] the interaction of electromagnetism with other classical fields has been studied by many authors. These attempts have not been in general very successful, one of the reasons probably being the lack of satisfactory solutions for the "free" (noninteracting) classical fields. The absence of free solutions invalidates the use of perturbation methods, since the free zero-order states are a necessary first step for the perturbative procedure.

It has been shown, however,[2] that the classical theory of a spinor field with a positive $(\bar{\psi}\psi)^2$ self-interaction provides a satisfactory model for a