Dan. Vidensk. Selsk. Mat.–Fys. Medd. 29, No. 17 (1955)].

[12]The conclusions of this discussion are applicable in any gauge since $b_n(0)$ is gauge-invariant.

[13]The additional reduction from the azimuthal (ϕ) integration can also be anticipated from Eq. (3.4).

[14]T. D. Lee and M. Nauenberg, Phys. Rev. 133, B1549 (1964).

[15]T. Kinoshita, J. Math. Phys. 3, 650 (1962).

[16]We do not antisymmetrize the intermediate-state fermions because we are simultaneously ignoring antisymmetrization between the displayed fermions and those arising from cut vacuum bubbles. Dividing out vacuum bubbles and ignoring antisymmetrization is equivalent to properly antisymmetrizing states including vacuum-bubble contributions and then dividing out only the vacuum-bubble contributions from antisymmetrized states.

[17]While M is gauge-independent, M_a, M_b, and M_c are gauge-dependent. These expressions are for the Feynman gauge.

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Photon pair creation in intense magnetic fields*

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The vacuum polarization of photons in intense, homogeneous magnetic fields is recalculated, using a proper-time method presented by Schwinger. This result is applied to compute exactly, in closed form, the photon absorption coefficient due to pair creation, $\kappa_{\parallel,\perp}$, corresponding to the polarization of the photon parallel or perpendicular to the plane of the photon momentum $\vec{k}$ and the homogeneous magnetic field $\vec{H}$. Specializing this general expression to the high-frequency, weak-field limit yields

$$\kappa_{\parallel,\perp}(\omega)=\tfrac{1}{2}\alpha\sin\theta\;\omega_H\frac{4\sqrt{3}}{\lambda\pi}\int_0^1\frac{dv}{1-v^2}\left[(1-\tfrac{1}{3}v^2)_\parallel,\,(\tfrac{1}{2}+\tfrac{1}{6}v^2)_\perp\right]K_{2/3}\!\left(\frac{4}{\lambda(1-v^2)}\right),$$

where $|\vec{k}|=\omega$, $\lambda=\frac{3}{2}(eH/m^2)(\omega/m)\sin\theta$, $\omega_H=eH/m$, and θ is the angle between $\vec{k}$ and $\vec{H}$. Comparing this expression with those obtained in the prior computations, we find that ours is more compact and much simpler in form and that ours is a simplied version of theirs.

I. INTRODUCTION

The possible existence of magnetic fields of the order 10^{12}–10^{14} G in the vicinity of pulsars[1] has stimulated interest in the investigation of various quantum-electrodynamical processes in intense magnetic fields. The absorption coefficient for photon splitting,[2] the power spectrum for synchrotron radiation,[3] and the cross section for Compton scattering[4] have been calculated recently by using the proper-time technique.[5,6] The energy straggling and the radiation reaction for synchrotron radiation[7] and the probability for the transition of a relativistic electron to its ground state, i.e., the synchrotron-spectrum-"tip" problem,[8] have also been investigated by using the exact relativistic electron wave functions in the conventional approach.

Another interesting process which might be of great significance in an astrophysical context is the direct creation of electron-positron pairs in intense magnetic fields.[9] This possibility was first discussed about two decades ago by Robl[10] and Toll[11]; Toll's results were subsequently confirmed by a number of independent calculations.[12-17] However, their results were obtained under the restriction of high-energy photons in weak magnetic fields. These authors[11-17] based their calculations on the conventional method of first computing the amplitudes for pair creation using the exact wave function, and then squaring the amplitude, and summing over the appropriate final states to obtain the total conversion rate.

The purpose of the present paper is to use the proper-time method to calculate the vacuum polarization in homogeneous magnetic fields—thus avoiding the use of the exact electron wave function—and then to extract the total probability of pair creation $\kappa_{\parallel,\perp}$, by means of the optical theorem. In this way, we are able to obtain an exact result in closed form [Eqs. (46)–(49)], which becomes particularly simple in the high-frequency, weak-field situation [Eq. (58)]. Comparing this result [Eq. (58)] with those obtained in the prior calculations,[11-17] we find that ours is more compact and much simpler in form. In fact, we are able to show (in the Appendix) that ours is actually a simplified version of theirs.

Since the present discussion depends heavily on the Lagrangian for vacuum polarization, $\mathcal{L}^{(2)}$, we will first rederive it in Sec. II, using an approach slightly different from that employed by Minguzzi[18] and Adler.[2] The utilization of $\mathcal{L}^{(2)}$ in the calculation of the absorption coefficient and the index of refraction is then outlined in Sec. III.

II. VACUUM POLARIZATION IN INTENSE MAGNETIC FIELDS

Vacuum polarization, to order α in the radiation field, can be calculated exactly using the proper-time method.[2,18] Schwinger[5] has illustrated the calculation of this process, without external fields, by two methods: (1) by a perturbative expansion in the radiation field (Sec. VI of Ref. 5) and (2) by the extension of the proper-time technique (Appendix A of Ref. 5). If we wish to extend these calculations to include external fields, we still have a choice of either of the two methods. Adler[2] and Minguzzi[18] used the second method; here we will rederive their results using the first method.

In quantum electrodynamics the processes with all "external" particles restricted to being photons can be described generally by the Lagrangian[5,6]

$$\mathcal{L}(\tilde{A}) = \tfrac{1}{2} i \int_0^\infty \frac{ds}{s} e^{-ism^2}(\mathrm{Tr}\, e^{-is\mathcal{H}} + \text{c.t.}), \tag{1}$$

where, for the interaction of a photon with an electron, we have

$$\begin{aligned} &\mathcal{H} = \tilde{\Pi}^2 - e\sigma\tilde{F}, \\ &\tilde{\Pi}_\mu = p_\mu - e\tilde{A}_\mu, \quad p_\mu = \frac{1}{i}\partial_\mu \\ &\sigma\tilde{F} = \tfrac{1}{2}\sigma_{\mu\nu}\tilde{F}^{\mu\nu}, \end{aligned} \tag{2}$$

and the trace, Tr, operates both on the spin indices and on the space-time coordinates. The contact terms (c.t.), which will be determined later, are designed to satisfy the appropriate physical normalization conditions. To discuss processes in external fields, we specialize $\tilde{F}$ to be the linear combinations of the external fields, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the radiation field, $f_{\mu\nu} = \partial_\mu \mathcal{a}_\nu - \partial_\nu \mathcal{a}_\mu$, and rewrite Eq. (2) in the form

$$\begin{aligned} &\tilde{\Pi}_\mu = \Pi_\mu - e\mathcal{a}_\mu, \quad \Pi_\mu = p_\mu - eA_\mu \\ &\sigma\tilde{F} = \sigma F + \sigma f, \end{aligned} \tag{3}$$

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \tag{4}$$

where

$$\mathcal{H}_0 = \Pi^2 - e\sigma F, \tag{5}$$

$$\mathcal{H}_1 = e(\Pi\mathcal{a} + \mathcal{a}\Pi + \sigma f) + e^2\mathcal{a}^2. \tag{6}$$

The Lagrangian for the vacuum polarization, which we shall designate by $\mathcal{L}^{(2)}$, can then be obtained by expanding $\mathrm{Tr}\, e^{-is\mathcal{H}}$ to quadratic power in $\mathcal{a}$, while treating the external field exactly, using the formula[5]

$$\begin{aligned} \mathrm{Tr}\, e^{-is\mathcal{H}} = \mathrm{Tr}\, e^{-is\mathcal{H}_0} - is\, \mathrm{Tr}(e^{-is\mathcal{H}_0}\mathcal{H}_1) \\ - \frac{s^2}{2}\int_{-1}^{+1}\frac{dv}{2}\mathrm{Tr}(e^{-is(1-v)\mathcal{H}_0/2}\mathcal{H}_1 \\ \times e^{-is(1+v)\mathcal{H}_0/2}\mathcal{H}_1) + \cdots . \end{aligned} \tag{7}$$

The resulting expression is

$$\mathcal{L}^{(2)} = -\tfrac{1}{4} i e^2 \int_0^\infty s\, ds\, e^{-ism^2}(I_a + I_b + \text{c.t.}), \tag{8}$$

where

$$I_a = \frac{2i}{s}\mathrm{Tr}(e^{-is\mathcal{H}_0}\mathcal{a}^2), \tag{9}$$

$$\begin{aligned} I_b = \int_{-1}^{+1}\frac{dv}{2}\mathrm{Tr}[e^{-is(1-v)\mathcal{H}_0/2}(\Pi\mathcal{a} + \mathcal{a}\Pi + \sigma f) \\ \times e^{-is(1+v)\mathcal{H}_0/2}(\Pi\mathcal{a} + \mathcal{a}\Pi + \sigma f)]. \end{aligned} \tag{10}$$

In the following, without loss of generality, we will choose the external magnetic field to be in the $+z$ direction, such that $F_{12} = -F_{21} = H$. The evaluation of I_a and I_b then follows closely the methods described in Sec. VI of Ref. 5. In particular, for I_a, we have

$$\begin{aligned} I_a &= \frac{2i}{s}\mathrm{Tr}\int (dx')\langle x'(s)|x'\rangle \mathcal{a}^2(x') \\ &= 4Jz\cos z \frac{2i}{s}\int (dk)\mathcal{a}^\mu(-k)\mathcal{a}_\mu(k), \end{aligned} \tag{11}$$

since[5]

$$\langle x'(s)|x'\rangle \equiv \langle x'|e^{-is\mathcal{H}_0}|x'\rangle = Je^{iz\sigma_3}, \tag{12}$$

$$J = -\frac{i}{(4\pi)^2}\frac{1}{s^2}\frac{z}{\sin z}, \quad z = seH. \tag{13}$$

The evaluation of I_b proceeds as follows: By using the relations[5]

$$\Pi_\mu e^{-i\tau\mathcal{H}_0} = e^{-i\tau\mathcal{H}_0}(\Pi e^{-2eF\tau})_\mu, \tag{14}$$

$$\gamma_\mu e^{-i\tau\mathcal{H}_0} = e^{-i\tau\mathcal{H}_0}(\gamma e^{-2eF\tau})_\mu, \tag{15}$$

$$x(\tau) \equiv e^{i\tau\mathcal{H}_0} x e^{-i\tau\mathcal{H}_0} = x + D(\tau)\Pi, \tag{16}$$

$$D(\tau) = \frac{e^{2eF\tau} - 1}{eF}, \quad \tau = s\frac{1+v}{2} \tag{17}$$

$$\mathcal{a}_\mu(x) = \int \frac{(dk)}{(2\pi)^2} e^{ikx}\mathcal{a}_\mu(k), \tag{18}$$

we obtain

$$I_b=\int\frac{(dq)(dk)}{(2\pi)^4}\mathfrak{A}^\mu(q)\mathfrak{A}^\nu(k)$$
$$\times\int_{-1}^{+1}\frac{dv}{2}\mathrm{Tr}\left\{\int dx'\langle x'(s)|[(2\Pi e^{-2eF\tau}-q)_\mu+(\gamma e^{-2eF\tau})_\mu(\gamma e^{-2eF\tau}q)+q_\mu]e^{iqx(\tau)}[(2\Pi-k)_\nu+\gamma_\nu\gamma k+k_\nu]|x'\rangle\right\}. \tag{19}$$

The next step is to use the identities (for a and b commuting with $[a, b]$)

$$e^{a+b}=e^a e^b e^{-[a,b]/2}, \tag{20}$$

$$e^a b=(b+[a,b])e^a \tag{21}$$

to show that

$$e^{iqx(\tau)}=e^{iqdx(s)}e^{iq(1-d)x}e^{\delta}, \tag{22}$$

where

$$d_{\mu\nu}=[D(\tau)/D(s)]_{\mu\nu}, \tag{23}$$

$$\delta=-\tfrac{1}{2}iq(1-d)^T D(\tau)q. \tag{24}$$

Then we move $e^{iqdx(s)}$ to the far left and $e^{iq(1-d)x}$ to the far right by using Eq. (21). Next, with the help of the evaluation[5]

$$\langle x'(s)|\Pi_\mu|x'\rangle=0, \tag{25}$$

and

$$\langle x'(s)|\Pi_\mu\Pi_\nu|x'\rangle=\langle x'(s)|x'\rangle\left(-\frac{i}{D(s)}\right)_{\mu\nu}, \tag{26}$$

as well as the identity

$$e^{2eF\tau}[d+eFD(\tau)]^T=d, \tag{27}$$

we obtain, after taking the trace,

$$I_b=4J\cos z\int(dk)\mathfrak{A}^\mu(-k)\mathfrak{A}^\nu(k)$$
$$\times\int_{-1}^{+1}\frac{dv}{2}e^{\delta'}\left[-4i\left(\frac{e^{2eF\tau}}{D(s)}\right)_{\mu\nu}+[(1-2d)k]_\mu[k(1-2d)]_\nu+2\tan z\frac{\cos zv-\cos z}{\sin z}\tilde{k}_\mu\tilde{k}_\nu+R_{\mu\nu}\right], \tag{28}$$

where

$$R_{\mu\nu}=(ke^{-2eF\tau}k)(e^{2eF\tau})_{\mu\nu}-(ke^{-2eF\tau})_\mu(e^{-2eF\tau}k)_\nu$$
$$+\tan z\left[(\tilde{k}e^{2eF\tau}k)(e^{2eF\tau})_{\mu\nu}-(ke^{-2eF\tau}k)\left(e^{-2eF\tau}\frac{eF}{eH}\right)_{\mu\nu}-(\tilde{k}e^{-2eF\tau})_\mu(e^{-2eF\tau}k)_\nu+(ke^{-2eF\tau})_\mu(\tilde{k}e^{2eF\tau})_\nu\right], \tag{29}$$

$$\delta'=\delta(q\to-k)$$
$$=-is\left[\frac{1-v^2}{4}k^2+\left(\frac{\cos zv-\cos z}{2z\sin z}-\frac{1-v^2}{4}\right)\vec{k}_\perp{}^2\right], \tag{30}$$

$$\tilde{k}_\mu=k^\lambda\left(\frac{eF}{eH}\right)_{\lambda\mu},\quad \vec{k}_\perp{}^2=k_x{}^2+k_y{}^2. \tag{31}$$

These expressions have been simplified by using the fact that the integrand should be symmetric in v. Substituting Eqs. (11) and (28) into Eq. (8), and utilizing integration by parts,

$$\frac{2i}{s}g_{\mu\nu}+\int_{-1}^{+1}\frac{dv}{2}e^{\delta'}\left[-4i\left(\frac{e^{2eF\tau}}{D(s)}\right)_{\mu\nu}\right]$$
$$=\int_{-1}^{+1}\frac{dv}{2}e^{\delta'}2k(1-2d)kd_{\mu\nu}, \tag{32}$$

we obtain the result

$$\mathfrak{L}^{(2)}=-\tfrac{1}{2}\int(dk)\mathfrak{A}^\mu(-k)\mathfrak{A}^\nu(k)M_{\mu\nu}(k), \tag{33}$$

where

$$M_{\mu\nu}(k)=\frac{\alpha}{2\pi}\int_0^\infty\frac{ds}{s}e^{-ism^2}\int_{-1}^{+1}\frac{dv}{2}\left[z\cot z\,e^{\delta'}\left([(1-2d)k]_\mu[k(1-2d)]_\nu+2k(1-2d)k\,d_{\mu\nu}\right.\right.$$
$$\left.\left.+2\tan z\frac{\cos zv-\cos z}{\sin z}\tilde{k}_\mu\tilde{k}_\nu+R_{\mu\nu}\right)+\mathrm{c.t.}\right]. \tag{34}$$

The structure of the contact term can now be inferred from the photon propagator normalization—specifically, that as $H\to0$ and $k^2\to0$, $M_{\mu\nu}$ must vanish. The explicit result is

$$\mathrm{c.t.}=-(k^2g_{\mu\nu}-k_\mu k_\nu)(1-v^2). \tag{35}$$

Equations (33)–(35) are the exact expressions for the vacuum polarization in homogeneous magnetic fields.

Before proceeding to discuss the applications of Eq. (33), we make the following remarks: First, $M_{\mu\nu}$ is gauge-invariant, since

$$k^\mu M_{\mu\nu}(k) \propto \int_{-1}^{+1} \frac{dv}{2} e^{\delta'} k(1-2d)k\, k_\nu = 0\,, \tag{36}$$

by the requirement that the integrand be symmetric in v. (Note that

$$k(1-2d)k = -\left[vk^2 + \left(\frac{\sin zv}{\sin z} - v\right)\vec{k}_\perp{}^2\right] \tag{37}$$

is odd under $v \to -v$.) Second, Eq. (33) is valid for arbitrary values of k^2; this is a useful starting point for discussing processes mediated by virtual photons. In the absence of the external field, i.e., $H=0$, we have

$$\begin{aligned} M_{\mu\nu}(k) &= (g_{\mu\nu}k^2 - k_\mu k_\nu)\frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_0^1 dv(1-v^2)e^{-ism^2}(e^{-is(1-v^2)k^2/4}-1) \\ &= (g_{\mu\nu}k^2 - k_\mu k_\nu)\left(-\frac{\alpha}{4\pi}k^2\right)\int_0^1 dv \frac{v^2(1-\frac{1}{3}v^2)}{m^2+(1-v^2)k^2/4}\,, \end{aligned} \tag{38}$$

which is the familiar result obtained in Ref. 5. Finally, we note that Eq. (33) is valid for general values of ω. The change of contour, $s \to -is$, is not permissible unless the range of photon frequencies is restricted to lie below the pair creation threshold, $\omega < 2m$.[19] The results of Minguzzi[18] and Adler[2] were obtained by making this change of contour and therefore are valid only in the region $0<\omega<2m$, as they have cautioned.[20] Even though our result, Eq. (33), differs from theirs by a simple change of contour $s \to -is$, the range of physical situations encompassed is quite different and requires careful treatment.

III. PHOTON PAIR CREATION

We now proceed to show that the photon absorption coefficient corresponding to pair creation can easily be extracted from the imaginary part of Eq. (33).

Without loss of generality, we choose the coordinate system such that $\vec{H}$ coincides with the $+z$ direction and $\vec{k}$ lies in the x-z plane. The polarization vector of the photon, $\vec{\epsilon}$, can be resolved into

$$\vec{\epsilon} = a\vec{\epsilon}_\parallel + b\vec{\epsilon}_\perp\,, \quad a^2+b^2=1 \tag{39}$$

corresponding to the directions parallel and perpendicular to the plane containing $\vec{k}$ and $\vec{H}$. If we denote by θ the angle between $\vec{k}$ and $\vec{H}$, then we have

$$\begin{aligned} &\vec{\epsilon}_\perp = \hat{j}\,, \\ &\vec{\epsilon}_\parallel = -\cos\theta\,\hat{i} + \sin\theta\,\hat{k}\,, \\ &\vec{k} = \omega(\sin\theta\,\hat{i} + \cos\theta\,\hat{k})\,, \\ &F_{12} = -F_{21} = H\,, \end{aligned} \tag{40}$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors in the x, y, and z directions. It can be easily shown that the matrix elements of $M_{\mu\nu}$ between the $\vec{\epsilon}_\parallel$ and $\vec{\epsilon}_\perp$ states are zero; therefore, we may consider only

$$M_\parallel \equiv \epsilon^\mu_\parallel M_{\mu\nu}\epsilon^\nu_\parallel\,, \quad M_\perp \equiv \epsilon^\mu_\perp M_{\mu\nu}\epsilon^\nu_\perp\,. \tag{41}$$

The eigenvalue equation is

$$(\vec{k}_{\parallel,\perp})^2 + M_{\parallel,\perp} = (k^0)^2\,, \tag{42}$$

which for a given frequency of the (real) photon, $k^0=\omega$, gives

$$|\vec{k}_{\parallel,\perp}| \approx \omega - \frac{1}{2\omega}M_{\parallel,\perp}\,. \tag{43}$$

The index of refraction is defined by

$$\tilde{n}_{\parallel,\perp} = \frac{|\vec{k}_{\parallel,\perp}|}{\omega} = 1 - \frac{1}{2\omega^2}M_{\parallel,\perp}\,. \tag{44}$$

The real part of Eq. (44) corresponds to the usual interpretation of the index of refraction,

$$n_{\parallel,\perp}(\omega) = 1 - \frac{1}{2\omega^2}\mathrm{Re}M_{\parallel,\perp}\,, \tag{45}$$

while the imaginary part is related to the absorption coefficient by

$$\kappa_{\parallel,\perp}(\omega) = -\frac{1}{\omega}\mathrm{Im}M_{\parallel,\perp}\,. \tag{46}$$

The evaluation of $M_\parallel$ and $M_\perp$ can be carried out by using the explicit representations of Eq. (40); this leads to the parametric integrals

$$M_{\parallel,\perp} = \frac{\alpha}{2\pi}\omega^2\sin^2\theta\int_0^\infty \frac{dz}{z}e^{-izm^2/eH}\int_{-1}^{+1}\frac{dv}{2}\exp\left[-iz\frac{\omega^2\sin^2\theta}{eH}\left(\frac{\cos zv - \cos z}{2z\sin z} - \frac{1-v^2}{4}\right)\right]N_{\parallel,\perp}\,, \tag{47}$$

where

$$N_{\parallel} = -z\cot z\left(1 - v^2 + \frac{v\sin zv}{\sin z}\right) + z\frac{\cos zv}{\sin z}, \quad (48)$$

and

$$N_{\perp} = -\frac{z\cos zv}{\sin z} + \frac{zv\cot z\sin zv}{\sin z} + \frac{2z(\cos zv - \cos z)}{\sin^3 z}. \quad (49)$$

These expressions for $N_{\parallel}$ and $N_{\perp}$ coincide with the results obtained by Adler,[2,21] corresponding to his $J_{\perp}$ and $J_{\parallel}$, respectively. As noted previously, the shift of integration contour, $s \to -is$, restricts the application of Adler's results to the region of photon frequency below the pair creation threshold. Our equations are valid for all values of ω, and Eqs. (46)–(49) constitute the general expression for the photon absorption coefficient due to pair creation.

The physical significance of these results can be displayed in terms of some particular cases. In the low-frequency $[(\omega/m)\sin\theta \ll 1]$ and weak-field $(eH/m^2 \ll 1)$ limit, the main contributions arise from the region where $z \ll 1$. Therefore, we may expand the integrand of Eq. (47) in power series in z,

$$N_{\parallel} = \tfrac{1}{2}(1 - v^2)(1 - \tfrac{1}{3}v^2)z^2, \quad (50)$$

$$N_{\perp} = \tfrac{1}{2}(1 - v^2)(\tfrac{1}{2} + \tfrac{1}{6}v^2)z^2. \quad (51)$$

This leads to the evaluation

$$M_{\parallel,\perp} = \frac{\alpha}{4\pi}\omega^2\sin^2\theta\int_0^\infty dz\, z\, e^{-izm^2/eH} \times \int_0^1 dv(1 - v^2)(1 - \tfrac{1}{3}v^2, \tfrac{1}{2} + \tfrac{1}{6}v^2)$$

$$= -\frac{\alpha}{4\pi}\omega^2\sin^2\theta\left(\frac{eH}{m^2}\right)^2[(\tfrac{28}{45})_{\parallel}, (\tfrac{16}{45})_{\perp}]. \quad (52)$$

This is purely real as expected, since below the threshold, $\omega < 2m$, no (single photon) attenuation due to pair creation can occur and $M_{\parallel,\perp}$ must be real. The corresponding result for the index of refraction is

$$n_{\parallel,\perp}(\omega) - 1 = \frac{\alpha}{4\pi}\sin^2\theta\left(\frac{eH}{m^2}\right)^2[(\tfrac{14}{45})_{\parallel}, (\tfrac{8}{45})_{\perp}], \quad (53)$$

which agrees with previous work.[11,2,13]

The propagation of a high-frequency photon in a weak field is characterized by the conditions $(\omega/m)\sin\theta \gg 1$ and $eH/m^2 \ll 1$. The relevant parameter in this case is

$$\lambda = \frac{3}{2}\frac{eH}{m^2}\frac{\omega}{m}\sin\theta, \quad (54)$$

which can become large for sufficiently high frequencies or in intense magnetic fields (although $eH/m^2 \ll 1$), and therefore we will treat it without approximation. Under these conditions, the principal contributions in Eq. (47) still come from the regions of the z integration where $z \ll 1$, and the approximations (50) and (51) are pertinent. However, the exponential factor has to be treated carefully, viz.,

$$\exp\left[-i\frac{m^2}{eH}z - iz\frac{\omega^2}{eH}\sin^2\theta\left(\frac{\cos zv - \cos z}{2z\sin z} - \frac{1 - v^2}{4}\right)\right] \approx e^{-i\Theta}, \quad (55)$$

where

$$\Theta = \frac{m^2}{eH}\left(z + \frac{\omega^2}{m^2}\sin^2\theta\frac{(1 - v^2)^2}{48}z^3\right) = \tfrac{3}{2}\xi(y + \tfrac{1}{3}y^3), \quad (56)$$

with

$$y = \frac{1 - v^2}{4}\frac{\omega}{m}\sin\theta z, \quad \xi = \frac{4}{\lambda}\frac{1}{1 - v^2}. \quad (57)$$

The final result for $\kappa_{\parallel,\perp}$ is

$$\kappa_{\parallel,\perp}(\omega) = -\frac{\alpha}{2\pi}\omega\sin^2\theta\,\mathrm{Im}\int_0^\infty\frac{dy}{y}\int_0^1 dv\, e^{-i\Theta}N_{\parallel,\perp}$$

$$= \frac{4\alpha}{\pi}\frac{m^2}{\omega}\int_0^1 dv(1 - v^2)^{-1}[(1 - \tfrac{1}{3}v^2)_{\parallel}, (\tfrac{1}{2} + \tfrac{1}{6}v^2)_{\perp}]\int_0^\infty dy\, y\sin\Theta$$

$$= \tfrac{1}{2}\alpha\sin\theta\;\omega_H\frac{4\sqrt{3}}{\pi\lambda}\int_0^1 dv(1 - v^2)^{-1}[(1 - \tfrac{1}{3}v^2)_{\parallel}, (\tfrac{1}{2} + \tfrac{1}{6}v^2)_{\perp}]K_{2/3}\left(\frac{4}{\lambda}\frac{1}{1 - v^2}\right), \quad (58)$$

where we have used the Airy integral

$$\int_0^\infty dy\, y\sin[\tfrac{3}{2}\xi(y + \tfrac{1}{3}y^3)] = \frac{1}{\sqrt{3}}K_{2/3}(\xi).$$

Comparing this result with those obtained previously[11-17], we see that our expression is more compact and much simpler. Further comparisons can be made by considering the various limits. In the limit $\lambda \gg 1$, we obtain

$$\kappa_{\parallel,\perp}(\omega)=\tfrac{1}{2}\alpha\sin\theta\,\omega_H\lambda^{-1/3}\frac{2^{1/3}3^{1/2}}{\pi}\Gamma(\tfrac{2}{3})\int_0^1 dv(1-v^2)^{-1/3}[(1-\tfrac{1}{3}v^2)_{\parallel},(\tfrac{1}{2}+\tfrac{1}{6}v^2)_{\perp}]$$
$$=\tfrac{1}{2}\alpha\sin\theta\,\omega_H\lambda^{-1/3}\frac{2^{1/3}3^{1/2}}{7\pi^{1/2}}\frac{[\Gamma(\frac{2}{3})]^2}{\Gamma(\frac{7}{6})}[(3)_{\parallel},(2)_{\perp}]$$
$$=\tfrac{1}{2}\alpha\sin\theta\,\omega_H\lambda^{-1/3}[(1.04)_{\parallel},(0.69)_{\perp}]. \tag{59a}$$

The other limiting case, $\lambda\ll1$, leads to

$$\kappa_{\parallel,\perp}(\omega)=\alpha\sin\theta\,\omega_H\left(\frac{3}{2\pi\lambda}\right)^{1/2}\int_0^1 dv(1-v^2)^{-1/2}e^{-4/\lambda(1-v^2)}[(1-\tfrac{1}{3}v^2)_{\parallel},(\tfrac{1}{2}+\tfrac{1}{6}v^2)_{\perp}]$$
$$=\tfrac{1}{2}\alpha\sin\theta\,\omega_H\left(\frac{3}{2\pi\lambda}\right)^{1/2}\int_1^\infty du(u-1)^{-1/2}\left[\left(\frac{2}{3u}+\frac{1}{3u^2}\right)_{\parallel},\left(\frac{2}{3u}-\frac{1}{6u^2}\right)_{\perp}\right]e^{-(4/\lambda)u}$$
$$=\tfrac{1}{2}\alpha\sin\theta\,\omega_H e^{-4/\lambda}(\tfrac{3}{2})^{1/2}[(\tfrac{1}{2})_{\parallel},(\tfrac{1}{4})_{\perp}]. \tag{59b}$$

These are precisely the same results obtained in Refs. 11–17. This strongly indicates that our result might be a simplified version of theirs. Indeed, in the Appendix, we are able to show that this is true.

For practical applications it is convenient to restore all the dimensional factors and to average over the photon polarization. The attenuation coefficient, per unit distance, for pair production by unpolarized photons with energy $\hbar\omega$ propagating across a magnetic field H in a direction perpendicular to the lines of flux then is

$$\alpha(\omega)=\frac{1}{2}\frac{\alpha}{\lambda\!\!\!^{-}_c}\frac{H}{H_{cr}}S(\chi), \tag{60a}$$

where

$$S(\chi)=\frac{\sqrt{3}}{2\pi}\chi\int_0^1 dv\frac{9-v^2}{1-v^2}K_{2/3}\left(\frac{2\chi}{1-v^2}\right), \tag{60b}$$

and

$$\chi=\frac{4}{3}\frac{mc^2}{\hbar\omega}\frac{H_{cr}}{H},$$
$$H_{cr}=\frac{m^2c^3}{e\hbar}\sim4.41\times10^{13}\text{ G},$$
$$\alpha\cong\tfrac{1}{137},$$
$$\lambda\!\!\!^{-}_c=\frac{\hbar}{mc}.$$

All the other symbols have their conventional meanings.

The remaining quadrature in (60b) may be carried out in terms of generalized hypergeometric functions, but the exact analytical forms are too cumbersome to provide any useful insight. As noted previously[13] the empirical approximation

$$S(\chi)\simeq0.24\chi K_{1/3}{}^2(\chi)$$

is adequate for numerical orientation; particularly in connection with megagauss-accelerator experiments. However, astrophysical situations, such as those considered by Sturrock,[9] require evaluation of the exact expressions (46)–(49).

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APPENDIX

In this appendix, we will show that our result, Eq. (58) or Eq. (60), is a simplified version of those obtained in the previous calculations.[11–17] The result of Toll was quoted by Adler in Ref. 2, Eqs. (30)–(33), as

$$\kappa_{\parallel,\perp}=\tfrac{1}{2}\alpha\sin\theta\,\omega_H T_{\parallel,\perp}(\lambda),$$
$$\text{for }\frac{\omega}{m}\sin\theta\gg1,\ \frac{eH}{m^2}\ll1, \tag{A1}$$

where

$$T_{\parallel,\perp}(\lambda)=\frac{9}{\lambda}\int_{(6/\lambda)^{2/3}}^\infty dv\left\{\frac{(1-3\eta_{\parallel,\perp}/2\lambda v^{3/2})[-\partial A(v)/\partial v]}{v^{5/4}(v^{3/2}-6/\lambda)^{1/2}}+\frac{(v^{3/2}-6/\lambda)^{1/2}}{3v^{3/4}}A(v)\right\}, \tag{A2}$$

$$\eta_{\parallel}=1,\quad \eta_{\perp}=3,$$
$$A(v)=\frac{1}{2\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dt\,e^{i(vt+t^3)}. \tag{A3}$$

The results of Refs. 12–15 have the same form. For the unpolarized case, the attenuation coefficient $\alpha(\omega)$ was given by[13]

$$\alpha(\omega)=\frac{1}{2}\frac{\alpha}{\lambda\!\!\!^{-}_c}\frac{H}{H_{\rm cr}}T(\chi), \tag{A4}$$

where

$$T(\chi)=3\left(\frac{\chi}{\pi}\right)^2\int_0^\infty\int_0^\infty du\,dw[(2\cosh^2 w\cosh^5 u-\sinh^2 u\cosh^3 u)K_{1/3}{}^2(\chi\cosh^2 w\cosh^3 u)$$
$$+(2\cosh^2 w-1)\cosh^5 u\,K_{2/3}{}^2(\chi\cosh^2 w\cosh^3 u)]. \tag{A5}$$

The equivalence between Toll's result and that of Refs. 12–15 was shown by Rassbach[16] on page 35 of his thesis. Therefore, all we have to prove here is the equivalence between Eq. (60a) and Eq. (A4), i.e.,

$$S(\chi)=T(\chi). \tag{A6}$$

One method of procedure is to use *Carleman's theorem*[22,13]: Suppose that $S(\chi)$ and $T(\chi)$ are non-negative for $\chi>0$, and that an infinite sequence of moments

$$C_n^{(1)}=\int_0^\infty d\chi\,\chi^n S(\chi), \tag{A7a}$$

$$C_n^{(2)}=\int_0^\infty d\chi\,\chi^n T(\chi) \tag{A7b}$$

are equal:

$$C_n^{(1)}=C_n^{(2)},\quad n=0,1,2,\ldots\,. \tag{A7c}$$

If these moments satisfy the divergence criterion

$$\sum_n^\infty C_n{}^{-1/2n}\to\infty, \tag{A7d}$$

then $S(\chi)=T(\chi)$.

The application of this theorem to (A6) is straightforward; the non-negative character of the functions is easily checked. The moments $C_n^{(1)}$ can be determined by combining (60b) with (A7a) and introducing the auxiliary variable

$$\tilde{\chi}=\frac{2\chi}{1-v^2}. \tag{A8}$$

Absolute convergence then permits the reduction of the repeated integrals to a simple product, viz.,

$$C_n^{(1)}=(\sqrt{3}\,2^{n+3}\pi)^{-1}\int_0^1 dv\frac{9-v^2}{1-v^2}(1-v^2)^{n+2}\times\int_0^\infty d\tilde{\chi}\,\tilde{\chi}^{n+1}K_{2/3}(\chi), \tag{A9a}$$

or

$$C_n^{(1)}=2^{-3}(3\pi)^{-1/2}\left(\frac{9n+22}{2n+5}\right)\frac{\Gamma(\frac{1}{2}n+\frac{2}{3})\Gamma(\frac{1}{2}n+\frac{4}{3})\Gamma(n+2)}{\Gamma(n+2+\frac{1}{2})}. \tag{A9b}$$

The computation of $C_n^{(2)}$ is slightly more tedious. We now combine (A5) and (A7b), and split the integrations by introducing the new variable

$$\xi=\chi\cosh^2 w\cosh^3 u. \tag{A10}$$

One can easily check the absolute convergence of the triple integrals. We then find

$$C_n^{(2)}=\frac{3}{\pi^2}[I_1(n)+I_2(n)], \tag{A11a}$$

where

$$I_1(n)=\int_0^\infty du\int_0^\infty dw\,\frac{(2\cosh^2 w-1)\cosh^5 u+\cosh^3 u}{\cosh^{2n+6}w\cosh^{3n+9}u}\times\mathcal{J}_1(n), \tag{A11b}$$

and

$$\mathcal{J}_1(n)=\int_0^\infty d\xi\,\xi^{n+2}K_{1/3}{}^2(\xi); \tag{A11c}$$

furthermore,

$$I_2(n)=\int_0^\infty du\int_0^\infty dw\frac{(2\cosh^2 w-1)\cosh^5 u}{\cosh^{2n+6}w\cosh^{3n+9}u}\,\mathcal{J}_2(n), \tag{A11d}$$

with

$$\mathcal{J}_2(n)=\int_0^\infty d\xi\,\xi^{n+2}K_{2/3}{}^2(\xi). \tag{A11e}$$

These integrals can easily be carried out. We find

$$C_n^{(2)}=3\pi^{-2}2^{6n+5}\frac{\Gamma(n+2)[\Gamma(\frac{3}{2}n+2)\Gamma(\frac{1}{2}(n+3))]^2}{(n+2)(2n+5)\Gamma(2n+4)\Gamma(3n+4)}$$
$$\times\left[\left(\frac{6n^2+24n+23}{3n+5}\right)\Gamma(\tfrac{1}{2}(n+3)+\tfrac{1}{3})\Gamma(\tfrac{1}{2}(n+3)-\tfrac{1}{3})+(n+3)\Gamma(\tfrac{1}{2}(n+3)+\tfrac{2}{3})\Gamma(\tfrac{1}{2}(n+3)-\tfrac{2}{3})\right]. \tag{A11f}$$

Elementary computations lead to the special values

$$C_0^{(1)} = C_0^{(2)} = \frac{2\times 11}{3^3\times 5},$$
$$C_1^{(1)} = C_1^{(2)} = \frac{31}{3^{7/2}\times 7}. \qquad (A12)$$

The identity for arbitrary n (A7c) follows easily by induction. This is facilitated by the reduction[13]

$$\frac{3j+5}{3j+7} = \frac{\Gamma(\frac{1}{2}(j+3)+\frac{1}{3})\Gamma(\frac{1}{2}(j+3)-\frac{1}{3})}{\Gamma(\frac{1}{2}(j+3)+\frac{2}{3})\Gamma(\frac{1}{2}(j+3)-\frac{2}{3})}, \quad j=0,1,\ldots . \qquad (A13)$$

Finally, we note that

$$C_n \sim \frac{9}{8}\left(\frac{\pi}{3}\right)^{1/2} \frac{n^{n+1/2}}{2^{n+1}e^n} \text{ for } n \gg 1, \qquad (A14a)$$

and this implies

$$\sum_n C_n{}^{-1/2n} \to \sum n^{-1/2} \to \infty, \qquad (A14b)$$

in consonance with (A7d).

The moment representations, (A9b) and (A11f), can be continued to complex values of n. In fact one can easily show that the inverse Mellin transform corresponding to (A7a) exists, i.e.,

$$S(\chi) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, C_n^{(1)} \chi^{-n}. \qquad (A15)$$

This leads to an explicit albeit lengthy representation of $S(\chi)$ in terms of generalized hypergeometric functions.

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