Conformal transformations for quantized fields*

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Finite conformal transformations for quantized fields are obtained. The apparent conflict with Einstein's causality principle is resolved. It is shown that in quantum field theory one is in general dealing with representations of the universal covering group of the conformal group.

I. INTRODUCTION

Considerable interest has been devoted lately to conformal invariance in quantum field theory and its possible relevance for the description of high-energy phenomena.[1-4] For a comprehensive review with extensive references to relevant literature see Todorov.[5]

The peculiar feature of conformal transformations being able to change spacelike into timelike separations has led to an ill feeling concerning its compatibility with local commutativity (Einstein's causality principle).[5-8] The causality problem has been avoided traditionally either by working only with the infinitesimal elements of the group or by discussing the conformal invariance of Green's functions in the Euclidean region. Nevertheless, the problem of obtaining the transformation law of the fields under the conformal group remained in general unsolved. For free fields those transformations were obtained in Ref. 7 and they turned out to be nonlocal, in this way resolving the apparent conflict with local commutativity. A similar picture emerges in the case of generalized free fields.[9,10] For an interacting field, however, the transformation law was not known.

In this note we show first that for (generalized) free fields the nonlocality of the transformation law is related to the fact that, as conjectured in Ref. 7, we are in general dealing with representations of the universal covering group of the conformal group.

We investigate next a simple two-dimensional model whose structure underlies all the soluble two-dimensional models.[11,12] The conformal transformations of the fields in this model are sufficiently nontrivial to allow one to conjecture a general transformation law for interacting fields. The detailed form of this law depends on the dimensions of both the fundamental and composite fields, being in this way inexorably linked to the precise nature of the dynamics of the theory.

II. FREE FIELDS

According to Ref. 7, the action of a special conformal transformation on a scalar zero-mass field in $D>2$ space-time dimensions is given by

$$U(b)\phi^{\pm}(x)U^{-1}(b)=\frac{1}{[\sigma_{\pm}(b,x)]^{d_\phi}}\phi^{\pm}(x_T) \tag{2.1}$$

with

$$d_\phi=\tfrac{1}{2}(D-2),$$

$$\sigma_{\pm}(b,x)=1-2b\cdot x+b^2x^2,$$

where the substitution $x_0\to x_0\pm i\epsilon, b_0\to b_0\pm i\epsilon$ is made. ϕ^- and ϕ^+ are the annihilation and creation parts, respectively, of the field, and

$$x_T^\mu=\frac{x^\mu-b^\mu x^2}{1-2b\cdot x+b^2x^2}.$$

From (2.1) we immediately see that for odd D (reverberating case) the transformation is nonlocal because of different phase factors picked up by the creation and annihilation parts of the field. Furthermore, a true representation of the whole conformal group $O(D,2)$ would imply[7]

$$\phi(x)\to\frac{1}{|\sigma(b,x)|^{d_\phi}}\phi(x_T), \tag{2.2}$$

which is only compatible with (2.1) in the case that $D=4l+2$, with l an integer. In all other cases we are dealing with a representation of the universal covering group of the conformal group. To see that, consider a general conformal transformation $U(C)$ canonically decomposed into a product of a dilation, a Lorentz transformation, a special conformal transformation, and a translation acting on the annihilation (creation) part of the field. Since the only difference between (2.1) and (2.2) is a phase factor we have

$$U^{-1}(C_2C_1)U(C_2)U(C_1)\phi^{\pm}(x)U^{-1}(C_1)U^{-1}(C_2)U(C_2C_1)=\exp[+id_\phi\varphi(x,C_2,C_1)]\,\phi^{\pm}(x). \tag{2.3}$$

By applying the D'Alembertian on both sides of (2.3) we obtain

$$[\Box \exp(\mp i d_\phi \varphi)]\, \phi^\pm + 2\nabla \exp(\mp i d_\phi \varphi)\nabla \phi^\pm = 0 . \tag{2.4}$$

Taking the matrix element of (2.4) between the vacuum and a one-particle state with momentum p,

$$\Box \exp(\mp i d_\phi \varphi) \mp i p \nabla \exp(\mp i d_\phi \varphi) = 0 ,$$

and using the arbitrariness of p,

$$\nabla \exp(\mp i d_\phi \varphi) = 0 ,$$

so that φ is independent of x.

We can therefore compute $\varphi(C_2, C_1)$ by considering (2.3) with $x=0$, obtaining

$$d_\phi \varphi(C_2, C_1) = \arg(\sigma_+(b_2, L_2 e^{-\delta_2} a_1))^{d_\phi} , \tag{2.5}$$

which can assume values $\pm(D-2)\pi/2$ or 0, the latter possibility being always the case for C_1 and C_2 sufficiently close to the identity.

By restricting oneself to sectors with a given number of particles one sees clearly from (2.3) and (2.5) that in general we are dealing with a ray representation of the conformal group and therefore[13] with a representation of the universal covering group of the conformal group.[14]

It is readily seen that $Z(C_2, C_1) = U^{-1}(C_2 C_1) \times U(C_2) U(C_1)$ commutes with all $U(C)$ and is therefore a central element of the covering group of the conformal group. We can rewrite (2.3) as

$$Z\phi(x)Z^{-1} = \exp(-i d_\phi \varphi N)\phi(x)\exp(+i d_\phi \varphi N) \tag{2.6}$$

so

$$Z(C_2, C_1) = \exp[-i d_\phi \varphi(C_2, C_1) N] , \tag{2.7}$$

where N is the number operator.

If $D \neq 4l+2$ there is a nontrivial Z (together with its inverse), and if D is odd (reverberating case) the transformation (2.6) is obviously nonlocal. From the above considerations we learn that the reason that (2.1) does not lead to a true representation of the conformal group has a much simpler origin than the one conjectured in Ref. 8. We have instead a representation of the universal covering group of the conformal group.

The above discussion can be extended to generalized free fields. The crucial point is the validity of the analog of (2.1), i.e.,

$$U(b)\phi_d^\pm(x)U^{-1}(b) = \frac{1}{[\sigma_\pm(b, x)]^d}\phi_d^\pm(x_T) , \tag{2.8}$$

where d is the dimension of the generalized free field. For $d = \frac{1}{2}(D-2+l)$ $(l = 0, 1, 2, \ldots)$, Eq. (2.8) follows immediately from (2.1) by regarding the generalized free field as a zero-mass free field in $D+l$ dimensions. More precisely,

$$\phi_d(x_0, x_1, \ldots, x_{D-1}) = \phi(x_0, x_1, \ldots, x_{D-1}, 0, \ldots, 0) ,$$

with

$$\left(\frac{\partial^2}{\partial x_0{}^2} - \sum_{i=1}^{D+l-1} \frac{\partial^2}{\partial x_i{}^2} \right) \phi(x_0, x_1, \ldots, x_{D+l-1}) = 0 .$$

For arbitrary values of d, (2.8) has to be established independently.[10] The main step, as in Ref. 7, consists in proving the self-adjointness of the generators of special conformal transformations.

From (2.8) and using the conformal group composition law we infer, as in (2.3),

$$Z(C_2, C_1)\phi_d^\pm(x)Z^{-1}(C_2, C_1) = \exp[\mp i d \varphi(C_2, C_1)]\, \phi_d^\pm(x) , \tag{2.9}$$

with φ given by (2.5). Again we see that we are dealing with a representation of the universal covering group of the conformal group with

$$Z(C_2, C_1) = \exp[-i d \varphi(C_2, C_1) N] \tag{2.10}$$

and the particle number defining the sectors transforming under different ray representations of the conformal group.

Despite its triviality, the case of (generalized) free fields already illustrates the point that conformal invariance in quantum field theory is linked to the details of the dynamics: In the free case we see its unexpected connection with particle number conservation, (2.10). Such a connection is not visible in purely group-theoretical approaches as in Ref. 9, where the separate consideration of irreducible (ray) representation of the conformal group makes the distinction between generalized free fields and interacting fields impossible. It is through the fact that in quantum-field theory we have (infinitely) many irreducible representations in the Hilbert space that the conformal transformation of a quantized field acquires a dynamical content.

III. A SIMPLE MODEL

We shall investigate in this section the conformal transformations in a simple two-dimensional model whose structure is sufficiently rich to allow a conjecture on the general transformation law for interacting fields. The model is formulated in terms of exponentials of free two-dimensional fields, which are the main building blocks of all two-dimensional soluble models.[11,12] Being interested in conformal invariance, we shall be concerned with exponentials of zero-mass free fields. As is well known, zero-mass free fields in two space-time dimensions require, because of infrared singularities, either an indefinite metric in the "Hilbert space" or a restricted class of test functions. In Ref. 7 the latter possibility was considered in order to study conformal transformations. Here we will adopt the indefinite-metric formulation, which allows us in a straightforward manner to compute the transformation law of the exponentials from the transformation properties of the field.

We shall not justify a number of formal manipulations at the level of the indefinite-metric theory, since our final results referring to the transformation law of the exponentials in a positive-metric Hilbert space can be directly checked by inspecting the Wightman functions of the theory, and we therefore regard the first part of this section as an economical way of deriving this transformation law.

A conveniently normalized zero-mass two-dimensional field has a two-point function given by

$$\langle 0|\phi(x)\phi(y)|0\rangle = -\ln[(x-y)_-^2], \qquad (3.1)$$

with

$$z_-^2 = (z_0 - i\epsilon)^2 - \vec{z}^2,$$

all higher truncated n-point functions vanishing.

We can have a pseudounitary realization of the dilations by

$$U(\delta)\phi(x)U^{-1}(\delta) = \phi(e^{-\delta}x) + (a + a^\dagger)\delta, \qquad (3.2)$$

with a and $a^\dagger$ auxiliary annihilation and creation operators, respectively, satisfying

$$\begin{aligned}&[a, a^\dagger] = 0,\\ &[a, \phi^+(x)] = [\phi^-(x), a^\dagger] = -1.\end{aligned} \qquad (3.3)$$

For special conformal transformations we have, analogously to (2.8),

$$U(b)\phi^\pm(x)U^{-1}(b) = \phi^\pm(x_T) + a^\pm \ln\sigma_\pm(b,x), \qquad (3.4)$$

and for the Poincaré group ϕ transforms conventionally,

$$U(P)\phi(x)U^{-1}(P) = \phi(Px). \qquad (3.5)$$

It is readily seen that (3.2), (3.4), and (3.5) leave the two-point function (3.1) invariant, which shows the pseudounitarity of our U operators with $U|0\rangle = |0\rangle$.

The properly defined exponentials of the free field are given by

$$e_\lambda(x) = \exp[i\lambda\phi^+(x)]\exp[i\lambda\phi^-(x)]. \qquad (3.6)$$

The general n-point function of the e_λ's is easily obtained:

$$\left\langle 0\left|\prod_i e_{\lambda_i}(x_i)\right|0\right\rangle = \exp\left\{\sum_i \sum_{j>i} \lambda_i\lambda_j \ln[-(x_i-x_j)_-^2]\right\} = \prod_i \prod_{j>i} [-(x_i-x_j)_-^2]^{\lambda_i\lambda_j}. \qquad (3.7)$$

From (3.4), (3.5) we get the transformation law of the e_λ's under the special conformal group

$$U(b)e_\lambda(x)U^{-1}(b) = \exp[i\lambda a^\dagger \ln\sigma_+(b,x)]\, e_\lambda(x_T) \times \exp[i\lambda a \ln\sigma_-(b,x)]. \qquad (3.8)$$

From (3.8) we get, by reordering all the a's to the right and the $a^\dagger$'s to the left,

$$\left\langle 0\left|\prod_i e_{\lambda_i}(x_i)\right|0\right\rangle = \exp\left[\sum_i \lambda_i \ln\sigma_+(b,x_i)\sum_{j<i}\lambda_j\right] \times \left\langle 0\left|\prod_i e_{\lambda_i}(x_{iT})\right|0\right\rangle \times \exp\left[\sum_i \lambda_i \ln\sigma_-(b,x_i)\sum_{j>i}\lambda_j\right]. \qquad (3.9)$$

Up to now we have been working in the indefinite-metric space. We now go over to a positive-definite theory by introducing new Wightman functions,

$$W(x_1,\lambda_1,x_2,\lambda_2,\ldots,x_n,\lambda_n) = \delta_{\Sigma\lambda_i,0}\left\langle 0\left|\prod_i e_{\lambda_i}(x_i)\right|0\right\rangle. \qquad (3.10)$$

One can convince oneself that those Wightman functions satisfy the positivity requirement [all other linear properties[15] are obvious from (3.10)] by regarding the massless theory as a limit of a massive one as in Ref. 11. In the known soluble models[11,12] a property like (3.10) is automatically fulfilled by fermion conservation. We can now regard the new Wightman functions (3.10) as vacuum expectation values of new operators $\tilde{e}_\lambda(x)$ acting in a positive-definite Hilbert space,[15]

$$(\Omega, \tilde{e}_{\lambda_1}(x_1)\cdots\tilde{e}_{\lambda_n}(x_n)\Omega) = \delta_{\Sigma\lambda_i,0}\langle 0|e_{\lambda_1}(x_1)\cdots e_{\lambda_n}(x_n)|0\rangle. \qquad (3.11)$$

The λ plays now the role of a charge. If we consider arbitrary real λ's, we are led to a nonseparable Hilbert space with a nondenumerable number of charge sectors. We can also restrict ourselves to separable subspaces by considering for instance only λ's which are integral multiples of a given one.

By comparing (3.9) with (3.11) we get

$$\left(\Omega, \prod_i e_{\lambda_i}(x_i)\Omega\right) = \left(\Omega, \prod_i \left\{\frac{[\sigma_-(b,x)]^{\lambda_i(\Sigma_{j>i}\lambda_j)}}{[\sigma_+(b,x)]^{\lambda_i^2+\lambda_i(\Sigma_{j>i}\lambda_j)}}\right\}\tilde{e}_{\lambda_i}(x_{iT})\Omega\right). \qquad (3.12)$$

From (3.12) we read off the transformation law of $\tilde{e}_\lambda$ when applied to a state of charge $q = \sum\lambda_i$. Writing

$$\tilde{e}_\lambda(x) = \sum_q \tilde{e}^q_\lambda(x), \qquad \tilde{e}^q_\lambda(x) = e_\lambda(x) P(q), \tag{3.13}$$

with $P(q)$ the projector on the charge q sector we have

$$\tilde{U}(b)\tilde{e}^q_\lambda(x)\tilde{U}^{-1}(b) = \exp\left\{ -\left[(\lambda^2 + 2\lambda q) \ln \frac{\sigma_+(b,x)}{|\sigma(b,x)|}\right]\right\} \times \frac{1}{|\sigma(b,x)|^{\lambda^2}} \tilde{e}^q_\lambda(x_T), \tag{3.14}$$

with $\tilde{U}$ now a truly unitary operator. Expression (3.14) is the analog of (2.1) and (2.8) for this model. Notice that λ^2 is the dimension of $\tilde{e}_\lambda(x)$ in the sense that

$$\tilde{U}(\delta)\tilde{e}_\lambda(x)\tilde{U}^{-1}(\delta) = e^{-\lambda^2\delta}\tilde{e}_\lambda(xe^{-\delta}), \tag{3.15}$$

and, just as in the free and generalized free case, (3.14) differs from the true conformal transformation law (2.2) by a phase factor.

As in the previous section, we are dealing with a representation of the universal covering group of the conformal group. To see this, consider as in (2.3) the action of

$$\tilde{U}^{-1}(C_2C_1)\tilde{U}(C_2)\tilde{U}(C_1) = Z(C_2, C_1)$$

on the $\tilde{e}^q_\lambda$:

$$Z(C_2,C_1)\tilde{e}^q_\lambda(x)Z^{-1}(C_2,C_1) = \exp[-i(\lambda^2 + 2\lambda q)\varphi(C_2,C_1)]\,\tilde{e}^q_\lambda(x). \tag{3.16}$$

To convince oneself that $\varphi(C_2, C_1)$ is x-independent here also, it suffices to notice that from (3.14) the phase is a multiple of the one occurring in a generalized free field of dimension $\frac{1}{2}$ and therefore φ is the same in both cases. On the other hand, a generalized free field in two space-time dimensions with $d=\frac{1}{2}$ can be viewed as in Sec. II as the appropriate restriction of a free zero-mass field in three space-time dimensions,

$$\phi_{1/2}(x_0, x_1) = \phi(x_0, x_1, 0).$$

Since the two-dimensional conformal group is a subgroup of the three-dimensional one that leaves x_2 unchanged, and since we know that for a free field the phase factor in (2.3) is x-independent, we conclude that also in (3.16) φ is x-independent and

$$\varphi(C_2,C_1) = \arg(\sigma_+(b_2, L_2e^{-\delta_2}a_1)) \tag{2.5$'$}$$

assuming values 0 or $\pm\pi$. We can rewrite (3.16) as

$$Z(C_2,C_1)\tilde{e}_\lambda(x)Z^{-1}(C_2,C_1) = \exp[-iQ^2\varphi(C_2,C_1)]\,\tilde{e}_\lambda(x)\exp[iQ^2\varphi(C_2,C_1)], \tag{3.17}$$

so

$$Z(C_2,C_1) = \tilde{U}^{-1}(C_2C_1)\tilde{U}(C_2)\tilde{U}(C_1) = \exp[-iQ^2\varphi(C_2,C_1)], \tag{3.18}$$

with Q the "charge" operator. In what follows we shall abbreviate $Z \equiv Z(\varphi=\pi)$.

In each charge sector we have a ray representation of the conformal group. That is, (3.18) restricted to a definite charge reads

$$\tilde{U}(C_2)\tilde{U}(C_1) = \exp[-iq^2\varphi(C_2,C_1)]\,\tilde{U}(C_2C_1) \tag{3.19}$$

with a nontrivial phase for suitably large C_2, C_1, as seen from (2.5$'$). It now follows from Bargmann's analysis[13] that any such representation can be viewed as a true representation of the universal covering group of the conformal group. The infinitely sheeted nature of this covering group reflects itself in that for nonrational q^2

$$(e^{\pm q^2\pi})^n \neq 1, \quad n = 0, 1, \ldots .$$

With respect to the representation problem the situation here is quite analogous to the one found in Sec. II for (generalized) free fields. However, in the nature of the (nontrivial) Z, for instance $\exp(-id\pi N)$ in the (generalized) free field and $\exp(-i\pi Q^2)$ in our model, relevant dynamical information is contained. In particular, as will become clearer in the following section, the additivity of dimensions for composite fields in the (generalized) free case, namely

$$\mathrm{Dim}\!:[\phi_d(x)]^N\!: = N\,\mathrm{Dim}\,\phi_d(x),$$

and its nonadditivity in our model where

$$:[\tilde{e}_\lambda(x)]^N: = \tilde{e}_{N\lambda}(x),$$

$$\mathrm{Dim}\!:[\tilde{e}_\lambda(x)]^N\!: = N^2\lambda^2 \neq N\,\mathrm{Dim}\,\tilde{e}_\lambda(x) = N\lambda^2$$

(which shows its nontrivial dynamical nature), are closely related to the different structure of the corresponding Z operators.

As a typical case which illustrates the connection of the phases in (3.14) and (3.16) with the dimensions of (composite) fields consider the special model generated by a complex field $\tilde{e}_\rho(x)$. The composite fields

$$:\tilde{e}_\rho{}^n(x): = \tilde{e}_{n\rho} \quad (n = 0, \pm 1, \ldots),$$

which may be obtained by a short-distance limiting procedure, form, together with their derivatives (inside and outside the Wick product), a basis of local fields in the sense that any product of two elements has an operator expansion in terms of this basis. The "charge" in this model can only assume values which are multiples of ρ.

It is clear that the Z phase in (3.16) for the q component of any field $A(x)$ in this basis carrying charge λ is the difference between the dimension of the composite field $B(x) = \tilde{e}_{\lambda+q}(x)$ and the dimension of the composite field $C(x)=\tilde{e}_q(x)$, so that

$$\varphi_A=\lambda^2+2\lambda q=d_b-d_c\,.$$

IV. THE GENERAL CASE

In the previous sections we obtained in a number of examples the proper interpretation of what is meant by a *Minkowski* conformally invariant *quantum field theory*. The main lessons we can extract are the following:

(a) The apparent conflict with local commutativity is removed by the fact that fields transform nonlocally under the special conformal group.

(b) This nonlocality reflects itself in that instead of having a true representation of the whole conformal group O(D, 2) we have in general a representation of its infinitely sheeted universal covering group. The relevant representation is specified by means of a unitary Z operator on whose eigenspaces we have ray representations of the conformal group. The precise form of the Z operator depends on the theory considered and cannot be obtained from purely group-theoretical arguments. It does contain truly dynamical information.

(c) Corresponding to the decomposition of the Hilbert space into sectors (eigenspaces of Z) we can decompose the field operators into a number of pieces each one of which has a simple transformation law under the special conformal group.

We take (a), (b), and (c) as being characteristic of any conformally invariant quantum field theory. We should have in general a nontrivial Z, the free and generalized fields with even scale dimension being obvious and typical exceptions.

Given a local field $A_d(x)$ of scale dimension d we decompose it as

$$A_d(x)=\int_0^1 d\xi\, A_d^\xi(x)\,, \tag{4.1}$$

with $A_d^\xi(x)$ formally given by

$$A_d^\xi(x)=\sum_{n=-\infty}^{\infty} Z^n A_d(x) Z^{-n}\exp[in\pi(d-2\xi)]\,. \tag{4.2}$$

From (4.2) we get

$$ZA_d^\xi(x)Z^{-1}=\exp[-i\pi(d-2\xi)]A_d(x)\,. \tag{4.3}$$

In agreement with (4.3) a scalar (and type I in the notation of Ref. 16) field behaves under special conformal transformations as

$$U(b)A_d^\xi(x)U^{-1}(b)=\frac{1}{[\sigma_+(b,x)]^{d-\xi}[\sigma_-(b,x)]^\xi}\times A_d^\xi(x_T)\,. \tag{4.4}$$

Although we concentrate on scalar fields, our results are generalizable to higher tensor fields.

From (4.4) we immediately get, applying the spectrum condition,

$$A_d(x)|0\rangle=A_d^0(x)|0\rangle\Rightarrow A_d^0(x)\neq 0\,, \tag{4.5}$$

$$A_d^*(x)|0\rangle=[A_d^{d\,\mathrm{mod}(1)}(x)]^*|0\rangle\Rightarrow A_d^{d\,\mathrm{mod}(1)}(x)\neq 0\,. \tag{4.6}$$

For free and generalized free fields those two are the only components in the decomposition (4.1). Already the Wick product of (generalized) free fields contains a discrete sum involving a finite number of terms in (4.1), equal in general to the power in the Wick product plus 1. In the model studied in Sec. III we had infinitely many terms. One expects the latter to be a typical situation of an interacting theory.

One can obtain information on the allowed ξ's by considering the three-point function of the field A with two arbitrary local fields and its transformation under Z. Since

$$\langle 0|C_{d_c}(x)A_{d_a}(y)B_{d_b}(z)|0\rangle=\int_0^1 d\xi\,\langle 0|C_{d_c}(x)A_{d_a}^\xi(y)B_{d_b}(z)|0\rangle\,,$$

we get with (4.3), (4.5), and (4.6) in the scalar case

$$\langle 0|C_{d_c}(x)A_{d_a}^\xi(y)B_{d_b}(z)|0\rangle=\exp[-i\pi(d_a+d_b-d_c-2\xi)]\times\langle 0|C_{d_c}(x)A_{d_a}^\xi(y)B_{d_b}(z)|0\rangle\,.$$

So we conclude that the nonvanishing of the three-point function implies

$$\xi=\tfrac{1}{2}(d_a+d_b-d_c)\ \mathrm{mod}(1)\,. \tag{4.7}$$

In the more general case of nonscalar fields one should substitute the dimensions in (4.7) by the twist ($t=d-s$), making the same replacement in (4.2) and (4.3). Operator-product expansions[4,17,18] do suggest that all ξ's can be obtained from three-point functions, so the generic form of the ξ of an operator A should be

$$\xi_a=\tfrac{1}{2}(t_a+t_b-t_c)\ \mathrm{mod}(1)\,. \tag{4.8}$$

Through (4.8), (4.1), and (4.4) the transformation law of a given field under the special conformal group depends on the dimensions of all other fields in the theory to which the original one couples. It is readily seen that (4.8) is equivalent to the statement that the eigenvalues of Z are of the form $\exp(-i\pi t_a)$, with t_a being the twist of a generic local field in the theory.

The A_d^ξ enjoy a number of properties which make them suitable objects for investigating conformal invariance directly in the Minkowski region. We first note that in an interacting field theory it is not to be expected that composite fields exist whose dimension is the sum of the dimensions of its constituents. This will lead to identities such as

$$A^{0}_{d_a}(x)B^{0}_{d_b}(y)=0\,. \tag{4.9}$$

Furthermore, the A^{ξ}_{d}, although nonlocal, have a well-defined scale dimension and transform simply under the conformal group. It is very natural therefore to investigate operator-product expansions in terms of them. In particular, the Minkowski analog of Mack's[18] global Euclidean conformal expansion which is not consistent when written in terms of the fields themselves can be meaningful in terms of the A^{ξ}_{d}. Those points will be developed in a forthcoming paper.

We conclude by remarking that since conformal invariance is not a physical symmetry (in the same sense that Poincaré invariance is) the sectors corresponding to different eigenvalues of Z are not associated with a superselection rule.[19] This is quite clear since even physical operators such as the energy-momentum tensor are not in general invariant under Z.

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