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Single-spectral forms for three-point functions: The general-mass situation*

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The establishment of spectral forms in source theory has been systematically studied, with it being a natural consequence of the causal methods employed therein that these results apply only for limited ranges of the particle masses. Here the three-point-function single-spectral form, considered in lowest nontrivial order and for scalar particles, is extrapolated to completely general values of the masses. The arguments employed continue in the physically oriented vein of source theory, not being founded on analyticity, and some consistency checks on these methods are included. In the case of stable particles the common spectral form with anomalous threshold is obtained, while for unstable particles the spectral mass squared can assume negative or complex values and the spectral weight function can take on an imaginary part. All the results are summarized in tabular form.

I. INTRODUCTION

The establishment of spectral forms in source theory has recently been studied in detail, with both three-[1] and four-point functions[2] being treated. Central to these developments was the consideration of a causal realization of the relevant amplitude, the so-called causal process. Figure 1 illustrates the process associated with the three-point-function single-spectral form. In order to guarantee the existence of the causal process, i.e., guarantee that certain internal particles be real (a and b in Fig. 1), it was necessary to impose certain mass restrictions. So, as directly obtained, the spectral forms are subject to such mass restrictions. It is the purpose of this present work to extend the three-point-function single-spectral form, considered in lowest nontrivial order and for scalar particles, to completely general values of all the masses involved—in fact, to arbitrary spacelike or timelike values of the external momenta not in spectral form.[3] The procedure so employed is referred to as mass extrapolation.

Some consideration of mass extrapolation has appeared in our two previous works. A general discussion of it was given in Ref. 1, but none was explicitly carried out.[4] In Ref. 2, though, some extrapolation was necessary just to obtain the four-point-function single- and double-spectral forms for a limited mass range. What is required in the present instance is somewhat more involved, but in the spirit of those previous works we are able to employ physically oriented arguments. The causal process is no longer as central, but it still figures importantly. First, the kinematics of the original causal process (Fig. 1) enter frequently in the extrapolation, while second, there now occurs a contribution to the spectral form with spectral-mass values below those originally specified by the causal process—the so-called anomalous-threshold contribution—and this is calculated from another causal process. So, unlike the conventional approaches, ours is not one founded on analyticity, although some consideration of the singularities of the spectral-weight function must enter because they can occur in the spectral-mass range upon extrapolation.

One specific example involving an anomalous threshold has already been studied in source theory by Schwinger, namely, the deuteron electromagnetic form factor.[5,6] A main aspect of that work which occurs in the present one is the manner of interpreting when there is an additional contribution below the normal threshold (although we go further here and relate that contribution to a causal process). For the general-mass situation within the realm of stable particles, we find that the main features of the spectral form are basically the same as in the deuteron case. But, for the situation with unstable particles, there are significant changes. In particular, the spectral mass squared can take on negative or complex values, singularities of the weight function occur within the spectral-mass range, and the weight function can assume an imaginary part (including when the spectral mass is positive).

In the situation of stable particles, anomalous thresholds have been considered in many places[7] by conventional means, and our spectral form is identical to what is obtained there. But, for the general-mass situation with unstable particles, we are aware of only one work, by Fronsdal and Norton.[8] We have not studied their work in detail, but it appears that our results are mathematically equivalent to theirs. (They perform a contour rotation in order to avoid complex values of the spectral mass squared.)

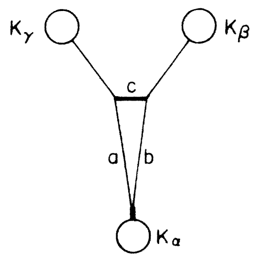

FIG. 1. Causal process leading to the spectral form with the initial mass restrictions. The long, thin lines refer to real particles and the short, heavy ones to virtual particles.

Quite apart, though, from any comparison with the results of analyticity studies, we should like some checks on our results that are phrased solely within the framework of source theory. For the case of stable particles a check is presented in Schwinger's work.[5,9] What he shows, working in a certain approximation, is that the single-spectral form with anomalous threshold can also be obtained by the reworking of a double-spectral form (with normal threshold). For the case where the external particles may be unstable, but the internal particles are stable, explicit consistency checks are presented below. This much consistency having been exhibited, we then turn things around when internal instability is also admitted, and employ the demand of consistency as an active part of the extrapolation procedure.

There are several further applications along the lines of this present work that should also be considered within the framework of source theory. First, there is the matter of extrapolation for three-point-function single-spectral forms when the particles are not scalar.[10] In particular, there is the source-theoretic work of Tsai, DeRaad, and Milton[11] which treats weak-boson electromagnetic form factors in a unified gauge theory of weak and electromagnetic interactions. They consider the bosons to be stable particles, and so an extrapolation is necessary to give these particles their physical masses. What must then be studied is the behavior of the new factors, related to the presence of charge and spin, that now occur in the spectral-weight function, because these factors can also be singular upon extrapolation. It would seem that the methods laid down here should be applicable to such work. On another example, the three-point-function double-spectral form was derived[1] only when the external momentum not in spectral form is spacelike. So perhaps the results and methods developed here would aid in extrapolating to a wider range of that variable. Eventually, also, the techniques developed here should be applied to four-point functions.[12]

In the ensuing work, in order to cover all possible values of the masses, it is necessary to consider separately several different mass regions. The sectioning of this paper is broken down according to these cases, with the single-spectral form of Ref. 1 being briefly reviewed at the beginning of Sec. II. In going through these sections the reader may often find it useful to refer to Fig. 2, which graphically depicts the various mass regions. Also, in Table I there is presented a summary of all the results.

II. STABLE EXTERNAL PARTICLES

A. Regions A, A', and A''

The vacuum-amplitude term referring to three external particles may generally be written as

$$\frac{i}{2\pi}\int \prod_{j=\alpha,\beta,\gamma}\left[\frac{(dp_j)}{(2\pi)^4}\,\phi_j(p_j)\right](2\pi)^4\delta(p_\alpha+p_\beta+p_\gamma)F\,, \tag{1}$$

where the ϕ_j and p_j are the fields and momenta of the particles. For the unextrapolated situation, F is expressible as the single-spectral form[1]

$$F=\int_{(m_a+m_b)^2}^{\infty} dM^2\frac{1}{p_\alpha{}^2+M^2-i\epsilon}\,\chi(M^2)\,. \tag{2}$$

And, as directly obtained from the lowest-order causal process (Fig. 1), the weight function is given by[1]

$$\chi=\int\frac{1}{2(\vec{p}_a{}^2+m_a{}^2)^{1/2}}\frac{(d\vec{p}_a)}{(2\pi)^3}\,\frac{1}{2(\vec{p}_b{}^2+m_b{}^2)^{1/2}}\frac{(d\vec{p}_b)}{(2\pi)^3}$$
$$\times(2\pi)^4\delta(p_\alpha-p_a-p_b)\left[(p_b+p_\beta)^2+m_c{}^2\right]^{-1}, \tag{3}$$

in which it is understood that $-p_\alpha{}^2=M^2$. In the α center-of-mass (c.m.) frame this expression reduces to an integral over the cosine of the angle between particles b and β:

$$\chi=(8\pi)^{-1}(|\vec{p}_b|/M)\int_{-1}^{1}dz\,(m_c{}^2-m_b{}^2-m_\beta{}^2-2p_b^0p_\beta^0+2\,|\vec{p}_b|\,|\vec{p}_\beta|\,z)^{-1}, \tag{4}$$

and the kinematic factors here may be expressed in terms of Lorentz invariants according to

$$2M^2(m_c{}^2-m_b{}^2-m_\beta{}^2-2p_b^0p_\beta^0)=f\equiv M^2(M^2-m_a{}^2-m_b{}^2+2m_c{}^2)-m_\beta{}^2(M^2+m_a{}^2-m_b{}^2)-m_\gamma{}^2(M^2-m_a{}^2+m_b{}^2)\,, \tag{5}$$

$$4M^2\vec{p}_b^{\,2}=g^2\equiv[M^2-(m_a+m_b)^2][M^2-(m_a-m_b)^2]\,, \tag{6}$$

and

$$4M^2\vec{p}_\beta^{\,2}=\Delta\equiv[M^2-(m_\beta+m_\gamma)^2][M^2-(m_\beta-m_\gamma)^2]\,. \tag{7}$$

The explicit expression of the weight function is thus

$$\chi=\frac{1}{8\pi}\frac{1}{\Delta^{1/2}}\ln\frac{f+g\Delta^{1/2}}{f-g\Delta^{1/2}}\,. \tag{8}$$

The mass restrictions to which this unextrapolated vacuum amplitude is subject are[1]

$$m_\beta<m_b+m_c\,, \tag{9a}$$

$$m_\gamma<m_a+m_c\,, \tag{9b}$$

and

$$m_\beta+m_\gamma<m_a+m_b\,; \tag{10}$$

the domain admitted by these restrictions is depicted by region A in Fig. 2. Extrapolation of the vacuum amplitude out of this mass domain is meaningful only if the vacuum amplitude remains well defined in doing such. Thus, we must study whether $f^2=g^2\Delta$ or $\Delta=0$ can be realized for $M\geq m_a+m_b$ as the restrictions (9) and (10) are removed, and, if so, how the vacuum amplitude is to be interpreted at such points.

The determination of the M^2 roots of $f^2=g^2\Delta$ is

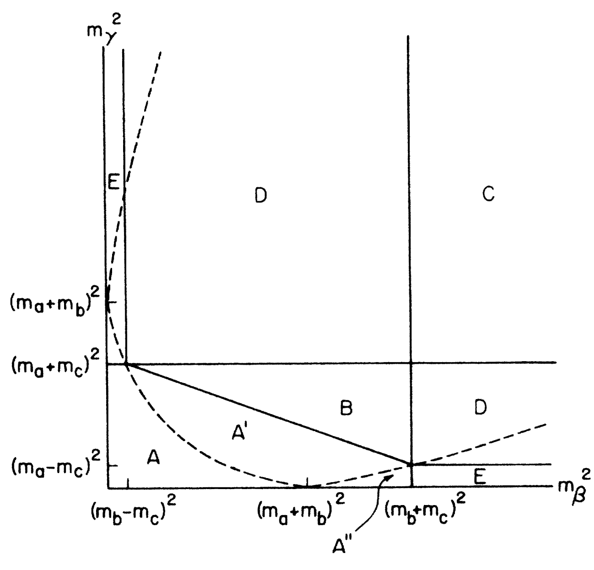

FIG. 2. Diagram illustrating the m_β^2-m_γ^2 regions into which the extrapolation study is divided. The area between the dashed curve and the two axes and satisfying $m_{\beta,\gamma}<m_a+m_b$ is where $m_a+m_b>m_\beta+m_\gamma$; the remaining two areas between this curve and the axes are where $|m_\beta-m_\gamma|>m_a+m_b$. In this diagram the ordering $m_b>m_c>m_a$ is assumed; for other possible orderings the only difference is the absence of region A'' or the occurrence of two such regions.

simplified by some physical considerations. The reworking of the denominator in Eq. (3) simply amounted to writing $p_c^2+m_c^2=f+g\Delta^{1/2}z$. Thus, since particles a and b are already on shell, $f^2=g^2\Delta$ corresponds to all three internal particles being on shell, subject to $z=\pm1$. In Ref. 1 we saw that a simple and convenient way to describe the reality[13] of the three internal particles was by a Gram-determinant statement:

$$\Delta_3(p_a,p_b,p_c)\leq 0\,. \tag{11}$$

As also noted in that reference, a natural set of variables for evaluating the determinant is

$$y=(M^2-m_a^2-m_b^2)/2m_am_b\,, \tag{12a}$$

$$y_\beta=(m_\beta^2-m_b^2-m_c^2)/2m_bm_c\,, \tag{12b}$$

$$y_\gamma=(m_\gamma^2-m_a^2-m_c^2)/2m_am_c\,, \tag{12c}$$

with the result being

$$\Delta_3=(m_am_bm_c)^2(y^2+y_\beta^2+y_\gamma^2+2yy_\beta y_\gamma-1)\,. \tag{13}$$

Alternately, from $\Delta_3=-(\epsilon_{\mu\nu\lambda}p_a^\mu p_b^\nu p_c^\lambda)^2$, there is in the α c.m. frame the evaluation

$$\Delta_3=-M^2\vec{p}_b^{\,2}\vec{p}_\beta^{\,2}(1-z^2)\,. \tag{14}$$

Thus, $f^2=g^2\Delta$ is equivalent to $\Delta_3=0$,[14, 15] and from Eqs. (12a) and (13) it follows that $f^2=g^2\Delta$ occurs at

$$M^2=M_\pm^2\equiv 2m_am_b[-y_\beta y_\gamma\pm(1-y_\beta^2)^{1/2}(1-y_\gamma^2)^{1/2}]+m_a^2+m_b^2\,. \tag{15}$$

In the extrapolation consider first only the removal of the restriction (10), maintaining for now the stability conditions (9) (which imply $y_{\beta,\gamma}<1$). Then we have that $M_-\leq M_+\leq(m_a+m_b)^2$, with the latter equality being reached only for $y_\beta=-y_\gamma$, or that $M_\pm$ are complex. So, as we start extrapolating from region A, $f^2=g^2\Delta$ does not occur in the weight function as long as $y_\beta+y_\gamma<0$ is maintained, which corresponds to the addition of regions A' and A'' in Fig. 2 (note the remarks in the caption of that figure concerning the existence of region A''). But before claiming the validity of the vacuum amplitude in these regions we must examine the behavior of the weight function at $\Delta=0$, which can now occur for $M\geq m_a+m_b$. In region A', $\Delta=0$ is realized in the spectral-mass domain only at $M=m_\beta+m_\gamma$, while in region A'', it is also realized at $M=|m_\beta-m_\gamma|$. An expansion of the logarithm in Eq. (8) shows that the weight function is well defined at $M=m_\beta+m_\gamma$, it only being necessary to rewrite it as[16]

$$\chi=\frac{1}{4\pi}\frac{1}{(-\Delta)^{1/2}}\left[\frac{\pi}{2}-\tan^{-1}\frac{f}{g(-\Delta)^{1/2}}\right]\begin{cases} m_\beta+m_\gamma>M\geq m_a+m_b\,, & \text{region } A' \\ m_\beta+m_\gamma>M\geq |m_\beta-m_\gamma|\,, & \text{region } A'' \end{cases} \tag{16}$$

in order to have a manifestly real expression. Likewise, for region A'' this expression simply reverts to the logarithmic form (8) for $|m_\beta - m_\gamma| > M \geq m_a + m_b$. The $\tan^{-1}$ expansions employed at $M=|m_\beta - m_\gamma|$ and $M = m_\beta + m_\gamma$ depended on the fact, easily shown [see Eqs. (50) and (52)], that for the masses presently under consideration, f is a positive function at those points.

B. Region B

Extrapolating from region A', we now consider the situation with $y_\beta + y_\gamma \geq 0$, region B. In such, the weight function, given by Eqs. (8) and (16), remains everywhere well defined, except for $M = m_a + m_b$ at $y_\beta + y_\gamma = 0$; there $f^2 = g^2\Delta$ is realized. And this in turn implies that f vanishes at $M = m_a + m_b$ since g does, so $f(M = m_a + m_b)$ changes sign as $y_\beta + y_\gamma$ goes through zero. In fact, one easily finds

$$f(M = m_a + m_b) = -4m_a m_b m_c(m_a + m_b)(y_\beta + y_\gamma). \tag{17}$$

The effect of the realization of $f^2 = g^2\Delta$ is thus that the weight function no longer vanishes at $M = m_a + m_b$ for $y_\beta + y_\gamma > 0$; explicitly, since $\tan^{-1}(\pm\infty) = \pm\frac{1}{2}\pi$,

$$\begin{aligned} &\chi(M = m_a + m_b) = 0, \quad y_\beta + y_\gamma < 0 \\ &\chi(M = m_a + m_b) = \tfrac{1}{4}(-\Delta)^{-1/2}, \quad y_\beta + y_\gamma > 0. \end{aligned} \tag{18}$$

This nonvanishing at the usual threshold is taken to imply that the spectral-mass range now extends below that point. So, to obtain the complete vacuum amplitude we must find and add on the contribution associated with this additional range.

The complete vacuum amplitude refers to one basic coupling (think, for example, in terms of the noncausal representation of the vacuum amplitude[17]). The causal process of Fig. 1 provided a causal realization of this coupling that allowed its contribution for $M \geq m_a + m_b$ to be calculated. Now we need another causal realization of the coupling that will provide the contribution for $M < m_a + m_b$.[18] To this end we consider the causal process illustrated in Fig. 3.[19] The mass restrictions necessary for the existence of this process are $m_\gamma < m_a + m_c$ and p_β spacelike. The latter, of course, must be removed in order to have a result applicable to region B. And that we shall accomplish by use of a technique, developed in the work on four-point functions,[2] in which the extrapolation is carried out with the weight function still in the original (unintegrated) form provided by the causal process. Loosely speaking, this technique can be said to employ causal processes with unphysical momenta.

For the causal process of Fig. 3, the usual method for the calculation of single-spectral forms leads to an expression of the form (2) (apart from the spectral-mass range), with the weight function χ replaced by

$$i\bar{p} = \frac{i}{2\pi}\int (dp)\delta((p-p_\alpha)^2 + m_a^2)\,\delta(p^2 + m_b^2) \times \delta((p+p_\beta)^2 + m_c^2), \tag{19}$$

in which it is understood that $-p_\alpha^2 = M^2$. In the α c.m. frame with the x direction chosen along $\vec{p}_\beta$, this expression becomes

$$i\bar{p} = \frac{i}{2\pi}\int (dp)\delta(-m_b^2 + 2p^0p_\alpha^0 + p_\alpha^2 + m_a^2) \times \delta(-m_b^2 - 2p^0p_\beta^0 + 2p_x p_{\beta x} + p_\beta^2 + m_c^2) \times \delta(-p^{02} + p_x^2 + \vec{p}_\perp^{\,2} + m_b^2). \tag{20}$$

When p_β and p_γ are brought on shell to values associated with region B, the external momenta appearing in $\bar{p}$ refer to an unphysical situation since $M < m_a + m_b$, with $m_\beta + m_\gamma > m_a + m_b$; nor can M refer to a physical scattering, since, as will be seen, $M > |m_\beta - m_\gamma|$.[20] The occurrence of the unphysical situation is signaled, in the α c.m. frame, by $p_{\beta x}$ becoming imaginary. So, bringing p_β and p_γ on shell, we have

$$p_{\beta x} \to i\bar{p}_{\beta x}, \quad \bar{p}_{\beta x} \text{ real}, \tag{21}$$

and to keep the second δ function in Eq. (20) meaningful, we match this with[21]

$$p_x \to -i\bar{p}_x, \quad \bar{p}_x \text{ real}. \tag{22}$$

The integral (20) then involves only real quantities, and is evaluated to give

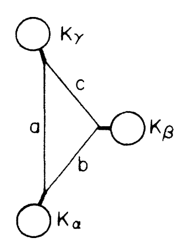

FIG. 3. Causal process leading to the spectral-form contribution with $M < m_a + m_b$.

$$\rho \equiv i\bar{\rho}(-p_{\beta,\gamma}{}^2 = m_{\beta,\gamma}{}^2) = (8p_\alpha^0 \bar{p}_{\beta x})^{-1} . \tag{23}$$

By use of $(p_\alpha^0 \bar{p}_{\beta x})^2 = -M^2 \vec{p}_\beta{}^2$ and Eq. (7), the weight function is finally presented as

$$\rho = \tfrac{1}{4}(-\Delta)^{-1/2} . \tag{24}$$

Note the continuity of this result at $M = m_a + m_b$ with the weight function χ [Eq. (18)].

Concerning the spectral-mass domain associated with ρ, consider first the range of M for which the weight-function expression (20), with the substitutions (21) and (22), is nonvanishing. This range is determined by the conditions that the δ-function arguments in Eq. (20) can vanish. The only argument for which this is not always possible is the last one, thereby providing the condition $\vec{p}_\perp{}^2 \geq 0$. In the α c.m. frame there is the evaluation [see Eq. (14)]

$$\Delta_3(p_a, p_b, p_c) = -M^2 p_{\beta x}{}^2 \vec{p}_\perp{}^2 , \tag{25}$$

and for $|m_\beta - m_\gamma| < M < m_\beta + m_\gamma$, $p_{\beta x}{}^2$ is negative. So, recalling the roots of $\Delta_3 = 0$ [Eq. (15)], we have that the weight function is nonvanishing for $M_0 \equiv M_+ \leq M \leq m_\beta + m_\gamma$ and for $M_- \leq M \leq |m_\beta - m_\gamma|$ or $|m_\beta - m_\gamma| \leq M \leq M_-$, depending on whether $|m_\beta - m_\gamma|$ or M_- is larger. Since we are looking for the additional vacuum-amplitude contribution below $M = m_a + m_b$, and since we take it that the entire additional piece is connected to the original one,[22] the ρ contribution to the vacuum amplitude thus has the spectral-mass domain $M_0 \leq M < m_a + m_b$, where

$$M_0{}^2 = 2m_a m_b[-y_\beta y_\gamma + (1 - y_\beta{}^2)^{1/2}(1 - y_\gamma{}^2)^{1/2}] + m_a{}^2 + m_b{}^2 \quad (\text{region } B). \tag{26}$$

Also, it is not difficult to show that $M_0 > |m_\beta - m_\gamma|$, so that ρ is well defined in this domain.

In summary, then, for $y_\beta + y_\gamma > 0$ with particles β and γ stable, the vacuum amplitude is of the form (1) and (2) with the weight function being given by Eqs. (8), (16), and (24), these expressions applying, respectively, for $M \geq m_\beta + m_\gamma$, $m_\beta + m_\gamma > M \geq m_a + m_b$, and $m_a + m_b > M \geq M_0$.

III. UNSTABLE EXTERNAL PARTICLES, STABLE INTERNAL PARTICLES

A. Regions *C* and *D*

Consider next the cases where one or both of particles β and γ are unstable. When only one of the two is unstable, we presently restrict ourselves to situations of internal stability, e.g., if γ is the stable particle, then $m_a < m_\gamma + m_c$ and $m_c < m_\gamma + m_a$ (which means $y_\gamma > -1$). The regions being considered in Fig. 2 are then regions C and D. The approach here is simply as follows: Take the result for region B and directly extrapolate it into these new mass regions. What must be investigated to carry out this extrapolation is whether the weight function remains well defined, and this divides into two tasks: examination of the weight function for $M^2 \geq (m_a + m_b)^2$ and for $(m_a + m_b)^2 > M^2 \geq M_0{}^2$, χ and ρ, respectively.

As earlier noted, the study for χ requires investigation of this weight function at those points $M \geq m_a + m_b$ for which $f^2 = g^2\Delta$ or $\Delta = 0$. From Eq. (15) it follows for the masses of regions C and D that $M_\pm{}^2 \leq (m_a - m_b)^2$ or that $M_\pm{}^2$ are complex, so $f^2 = g^2\Delta$ cannot occur for the relevant spectral-mass range. The realization of $\Delta = 0$ at $M = m_\beta + m_\gamma$ is already taken into account by the transition of χ from the expression (8) to (16). However, in parts of regions C and D $|m_\beta - m_\gamma|$ exceeds $m_a + m_b$ (see Fig. 2), so there we must also consider the realization of $\Delta = 0$ at $M = |m_\beta - m_\gamma|$, as was not the case in region B. It is easily seen that, in these parts of regions C and D, f is negative at $M = |m_\beta - m_\gamma|$. Thus, considering χ in its power-series form as M approaches $|m_\beta - m_\gamma|$ from above, extending this result through $M = |m_\beta - m_\gamma|$, and then resuming, we get

$$\chi = \frac{1}{8\pi\Delta^{1/2}}\left(\ln\frac{f + g\Delta^{1/2}}{f - g\Delta^{1/2}} + 2\pi i\right) \quad \begin{cases} \text{regions } C \text{ and } D, & |m_\beta - m_\gamma| > m_a + m_b ; \\ |m_\beta - m_\gamma| > M \geq m_a + m_b . \end{cases} \tag{27}$$

In obtaining this result it was necessary to choose between $(-\Delta)^{1/2} = \pm i\Delta^{1/2}$ for $|m_\beta - m_\gamma| > M \geq m_a + m_b$. Extending through $M = |m_\beta - m_\gamma|$ by means of $M^2 - i\epsilon$, as suggested by the spectral denominator,[23] we got the $-i\Delta^{1/2}$ result. For the situation under consideration, then, the weight function χ becomes singular at $M = |m_\beta - m_\gamma|$, although integrably so, as $\epsilon \to 0$. Below, there is a consistency check involving a similar singularity.

Turning to the weight function ρ, we consider first just region C. It is not difficult to show that $M_0{}^2 < (m_\beta - m_\gamma)^2$, where, from Eq. (26),

$$M_0{}^2 = 2m_a m_b[-y_\beta y_\gamma - (y_\beta{}^2 - 1)^{1/2}(y_\gamma{}^2 - 1)^{1/2}] + m_a{}^2 + m_b{}^2 \quad (\text{region } C) \tag{28}$$

(which can be negative). So, in at least part of its domain, ρ has become imaginary, and the sign of the square root is determined as in the previous paragraph. The weight function ρ for region C is thus given as

$$\begin{aligned} &i\tfrac{1}{4}\Delta^{-1/2}, \quad M_0^2 \le M^2 < (m_\beta - m_\gamma)^2 \\ &\tfrac{1}{4}(-\Delta)^{-1/2}, \quad (m_\beta - m_\gamma)^2 \le M^2 < (m_a + m_b)^2 \,; \end{aligned} \tag{29}$$

when $|m_\beta - m_\gamma| \ge m_a + m_b$, the latter of these contributions is absent and the former terminates at $(m_a + m_b)^2$. It is the singularity of this weight function at $M = |m_\beta - m_\gamma|$, when $|m_\beta - m_\gamma| \ge m_a + m_b$, that will enter in the consistency check below.

Now consider region D. Here M_0^2 has become complex, and the sign of the square root referring to the unstable particle, say, β, is determined by use of $y_\beta^2 + i\epsilon$ [i.e., $p_\beta^2 - i\epsilon$ (Ref. 24)]; thus

$$M_0^2 = 2m_a m_b[-y_\beta y_\gamma - i(|y_\beta^2 - 1|)^{1/2}(|y_\gamma^2 - 1|)^{1/2}] + m_a^2 + m_b^2 \quad (\text{region } D)\,. \tag{30}$$

For the spectral-mass domain associated with ρ, one may take, for example, the straight line from M_0^2 to $(m_a + m_b)^2$, or any other contour with the same endpoints obtained by deformation from this one without going through the points giving rise to $\Delta = 0$. If the contour is taken to have a portion along the real axis, one should distinguish between the two possible expressions for ρ as done in Eq. (29). For the portion of the contour in the complex M^2 plane, the square-root branch used in ρ is that defined by extension from either of these real expressions.

This completes the establishment of the single-spectral forms appropriate to regions C and D; the results are summarized in Table I. Note in these regions, for any $|m_\beta - m_\gamma|$, that ρ is continuous at $M = m_a + m_b$ with χ.

B. Consistency checks

In this section we present some consistency checks on our results. These checks are comparisons of these results against others (obtained in terms of different spectral variables) that required little or no mass extrapolation in their establishment. That is, we choose the spectral variable $-p_\alpha^2$ as some particular m_α^2, making also some simplifying choices for the other masses, and investigate whether this result is equal to the other evaluated for the same set of masses. Also, as they are the easiest to explicitly evaluate, our efforts will be confined to comparing the imaginary parts of the spectral forms. These checks may be viewed, primarily, as tests on the correctness of our methods, particularly the extrapolation procedures. But, alternatively, as discussed in Refs. 23 and 24, they may be viewed as means to avoid the necessity of using $i\epsilon$ prescriptions to determine the signs of the square roots in Eqs. (27), (29), and (30).

First, we will compare the single-spectral form of region C, for $p_\alpha^2 = 0$ and $m_\beta = m_\gamma$, $m_a = m_b = m_c$, with the result from a double-spectral form. For this choice of masses and $-p_\alpha^2 < 4m_a^2$, the imaginary part of the single-spectral form comes solely from its ρ contribution:

$$F_\rho^C(p_\alpha^2) = \int_{M_0^2}^{4m_a^2} dM^2 \frac{1}{p_\alpha^2 + M^2 - i\epsilon}\,\rho; \tag{31}$$

here, from Eqs. (28) and (29),

$$M_0^2 = -4m_\beta^2(m_\beta^2/4m_a^2 - 1) \tag{32}$$

and

$$\begin{aligned} \rho &= i\tfrac{1}{4}[-M^2(4m_\beta^2 - M^2)]^{-1/2} \equiv \rho_{\rm II}\,, \quad M_0^2 \le M^2 < 0 \\ &= \tfrac{1}{4}[M^2(4m_\beta^2 - M^2)]^{-1/2} \equiv \rho_{\rm I}\,, \quad 0 \le M^2 < 4m_a^2\,. \end{aligned} \tag{33}$$

Because of the behavior of ρ at $M^2 = 0$, we cannot just set $p_\alpha^2 = 0$ in F_ρ^C for that portion around $M^2 = 0$; rather, we take p_α^2 small and positive, perform the evaluation, and then let p_α^2 be zero (the same result ensues if p_α^2 is taken as small and negative). So, with $p_\alpha^2 \to +0$ understood, we have

$$F_\rho^C(0) = \int_{M_0^2}^{-\eta^2} dM^2 \frac{\rho_{\rm II}}{M^2} + \int_{-\eta^2}^{0} dM^2 \frac{\rho_{\rm II}}{p_\alpha^2 + M^2 - i\epsilon} + \int_0^{\eta'^2} dM^2 \frac{\rho_{\rm I}}{p_\alpha^2 + M^2} + F_0\,, \tag{34}$$

where η and η' are arbitrarily small, except $\eta^2 \ge p_\alpha^2$, and F_0 refers to the contribution from η'^2 to $4m_a^2$, which is real and insensitive to the p_α^2 limit. One then calculates

$$\int_{M_0^2}^{-\eta^2} dM^2 \frac{\rho_{\rm II}}{M^2} = i\,\frac{1}{8m_\beta^2}\,\frac{(1 - M_0^2/4m_\beta^2)^{1/2}}{(-M_0^2/4m_\beta^2)^{1/2}} - i\,\frac{1}{4m_\beta}\,\frac{1}{\eta} + O(\eta)\,, \tag{35}$$

$$\int_{-\eta^2}^{0} dM^2 \frac{\rho_{\rm II}}{p_\alpha^2 + M^2 - i\epsilon} = -\frac{\pi}{8m_\beta}\,\frac{1}{(p_\alpha^2)^{1/2}} + i\,\frac{1}{4m_\beta}\,\frac{1}{\eta} + O((p_\alpha^2)^{1/2}) + O(\eta)\,, \tag{36}$$

and

$$\int_0^{\eta'^2} dM^2 \frac{\rho_{\rm I}}{p_\alpha^2 + M^2} = \frac{\pi}{8m_\beta}\,\frac{1}{(p_\alpha^2)^{1/2}} + O((p_\alpha^2)^{1/2}) + O(\eta')\,. \tag{37}$$

Thus, the imaginary part of the single-spectral form at $p_\alpha^2 = 0$ is

$$\mathrm{Im}F(0) = (16m_a m_\beta)^{-1}(m_\beta^2/4m_a^2 - 1)^{-1/2}\,, \tag{38}$$

where use was made of Eq. (32).

Now consider the double-spectral representation of the vacuum amplitude.[1] This vacuum amplitude (derived for p_α^2 spacelike) has the structure (1), but the single-spectral form F is replaced by

$$G=\int dM^2 dM'^2 \frac{1}{p_\beta^2+M^2-i\epsilon}\frac{1}{p_\gamma^2+M'^2-i\epsilon}\times\sigma(M^2,M'^2)\,, \tag{39}$$

where

$$\sigma=(8\pi)^{-1}(M^4+M'^4+p_\alpha^4-2M^2M'^2+2M^2p_\alpha^2+2M'^2p_\alpha^2)^{-1/2} \tag{40}$$

and, upon specialization to the choice of internal masses made above, the spectral-mass domain is specified by

$$p_\alpha^2[(M^2-2m_a^2)(M'^2-2m_a^2)-m_a^2(p_\alpha^2+4m_a^2)]\geq m_a^2(M^2-M'^2)^2\,,$$
$$M^2, M'^2\geq 4m_a^2\,. \tag{41}$$

The check is thus to see if

$$\mathrm{Im}F(0)=\mathrm{Im}G(-p_\beta^2=-p_\gamma^2=m_\beta^2,\ p_\alpha^2=0)\equiv\mathrm{Im}G(0) \tag{42}$$

is true.

The calculation of $\mathrm{Im}G(0)$ is made quite simple by use of the x, v variables introduced by Schwinger in his deuteron work[5,25] and employed for purposes somewhat similar to our present ones:

$$M^2-2m_a^2=m_a(p_\alpha^2+4m_a^2)^{1/2}x+m_a(p_\alpha^2)^{1/2}(x^2-1)^{1/2}v\,,$$
$$M'^2-2m_a^2=m_a(p_\alpha^2+4m_a^2)^{1/2}x-m_a(p_\alpha^2)^{1/2}(x^2-1)^{1/2}v\,. \tag{43}$$

The first of the restrictions (41) then reduces to an expression in terms of v only, so x and v range independently, their domains being, respectively, 1 to ∞ and -1 to 1. So, $G(-p_\beta^2=-p_\gamma^2=m_\beta^2,\ p_\alpha^2)$ is expressed in terms of these new variables, the specialization to $p_\alpha^2=0$ is then made, and the v integration is performed. Upon the change of variables $t^2=\frac{1}{2}(x-1)$ and an integration by parts, we obtain

$$G(0)=(4\pi)^{-1}\int_0^\infty dt(t^2+1)^{-1/2}\times(4m_a^2t^2+4m_a^2-m_\beta^2)^{-1}; \tag{44}$$

consideration then of an unstable β particle is reflected in the use of $m_\beta\to m_\beta^2+i\epsilon$ [i.e., again, $p_\beta^2-i\epsilon$ (Ref. 26)]. From this expression immediately follows

$$\mathrm{Im}G(0)=(16m_am_\beta)^{-1}(m_\beta/4m_a^2-1)^{-1/2}\,, \tag{45}$$

thus exhibiting the consistency check.

Next we compare the single-spectral form of region D for $-p_\alpha^2=(m_\beta-m_\gamma)^2$ with the result from another single-spectral form. For definiteness, let β be the unstable particle and γ the stable one. Then, for the latter spectral form we use one with $-p_\beta^2$ as the spectral variable; that is, we proceed from a "rotated" version of Fig. 1. In this result $-p_\beta^2$ is set equal to m_β^2 and the masses of the two outgoing external particles are $m_\alpha=m_\beta-m_\gamma$ and m_γ. In order that α be a stable particle, we impose $m_\beta-m_\gamma<m_a+m_b$; the latter spectral form is then one associated with regions A' or B.

The imaginary part of the spectral form of region D, with $-p_\alpha^2=(m_\beta-m_\gamma)^2<(m_a+m_b)^2$, comes entirely from its ρ contribution, which ranges over complex values of M^2:

$$F_\rho^D=\frac{1}{4}\int_{M_0^2}^{(m_a+m_b)^2}dM^2\frac{1}{M^2-(m_\beta-m_\gamma)^2}\times\{-[M^2-(m_\beta+m_\gamma)^2]\times[M^2-(m_\beta-m_\gamma)^2]\}^{-1/2}, \tag{46}$$

where M_0^2 is given by Eq. (30). This integration is simply performed, and omitting the contribution from the upper limit because it is real, we have

$$F_\rho^D=\frac{1}{8m_\beta m_\gamma}\frac{(m_\beta+m_\gamma)^2-M_0^2}{M_0^2-(m_\beta-m_\gamma)^2}\,. \tag{47}$$

The other single-spectral form is given by the results of Sec. II with the interchanges appropriate to the "rotated" causal process and with $m_\alpha=m_\beta-m_\gamma$; for this spectral form we use the notations of that section with primes affixed. Concerning its imaginary part, when $-p_\beta^2=m_\beta^2$ that comes solely from the contribution associated with the vanishing of the spectral denominator:

$$\mathrm{Im}F'(-p_\beta^2=m_\beta^2)=\pi\chi_0'\,, \tag{48}$$

where 0 designates evaluation at $M'=m_\beta$. For this particular value of the spectral mass we have $\Delta'=0$, and so the weight function reduces to

$$\pi\chi_0'=\tfrac{1}{4}g_0'/f_0'=\frac{1}{8}\frac{\{[m_\beta^2-(m_b+m_c)^2][m_\beta^2-(m_b-m_c)^2]\}^{1/2}}{m_\beta^2(m_a^2-m_c^2-m_\gamma^2)-m_\beta m_\gamma(m_b^2-m_c^2-m_\beta^2)}\,. \tag{49}$$

The check is thus to see if the imaginary part of Eq. (47) equals Eq. (49). There is little to be gained by carrying this out in algebraic generality, and it suffices just to treat a specific case within the range of masses being considered. A choice that leads to especially simple evaluations is $m_a = m_b = m_c = m$, $\frac{1}{2} m_\beta = m_\gamma = \sqrt{2}\, m$. Then it is easily shown that the check is met, with the common value of the imaginary parts being $(16\sqrt{2}\, m^2)^{-1}$.

IV. UNSTABLE EXTERNAL AND INTERNAL PARTICLES

A. Region E

Region E corresponds to one stable external particle and one unstable one, with the former being coupled to an unstable internal particle. To carry out the work for this region, it is necessary to distinguish which of the external particles is unstable, and then which of the possible two internal particles is unstable. We shall explicitly treat the case where particles β and a are the unstable ones. The case where c is instead the unstable internal particle is treated quite similarly, while the results for the situations with γ unstable are simply obtained by interchange from those with β unstable. Table I contains the results for all possible cases.

As above, the M points at which we must be concerned with the definition of the weight function are $M_\pm$ and $m_\beta \pm m_\gamma$. The former two now are real and lie above $m_a + m_b$, as is easily seen from Eq. (15). Physically, this is because for the masses of region E and $M \geq m_a + m_b$ it is possible that all three internal particles may be real. That is, when β and a are the unstable particles, there may exist the physical process of Fig. 4(a),[27] and the condition for the existence of such is $\Delta_3 \propto (M - M_+)(M - M_-) \leq 0$, as discussed in Sec. II A. Furthermore, since the existence of this process of course requires $M \geq m_\beta + m_\gamma$, we have the ordering $m_\beta - m_\gamma \leq m_\beta + m_\gamma \leq M_+ \leq M_-$. [In the instance where c is the unstable internal particle, the physical process with the three real internal particles is that of Fig. 4(b), and one has the ordering $M_+ \leq M_- \leq m_\beta - m_\gamma \leq m_\beta + m_\gamma$.]

It will be necessary to know when the equality $M_+ = m_\beta + m_\gamma$ occurs. Since $M = M_+$ corresponds to $f^2 = g^2 \Delta$ and $M = m_\beta + m_\gamma$ to $\Delta = 0$, such occurs when $f = 0$ at $M = m_\beta + m_\gamma$, which is expressed in terms of the simple evaluation

$$f(M = m_\beta + m_\gamma) \equiv f^+ = -4 m_\beta m_c m_\gamma (m_\beta + m_\gamma) \times (y_a + y_b), \tag{50}$$

where

$$y_a = (m_a{}^2 - m_c{}^2 - m_\gamma{}^2)/2 m_c m_\gamma, \quad y_b = (m_b{}^2 - m_c{}^2 - m_\beta{}^2)/2 m_c m_\beta. \tag{51}$$

It is easily seen that for region E, at any fixed but arbitrary m_β, f^+ passes through zero once as m_γ varies across the region. More specifically, f^+ decreases as m_γ decreases at fixed m_β, and the line $f^+ = 0$ occurs for intermediate values of m_γ except at $m_\beta = m_b + m_c$, where it occurs for $m_\gamma = m_a - m_c$. [When c is the unstable internal particle, $M_- = m_\beta - m_\gamma$ occurs when

$$f(M = m_\beta - m_\gamma) = -4 m_\beta m_c m_\gamma (m_\beta - m_\gamma)(y_a - y_b) \tag{52}$$

vanishes. Note the similarity of Eqs. (50) and (52) to Eq. (17).]

Now we turn explicitly to the extrapolation into region E (with β and a as the unstable particles), and for the moment consider $m_\beta + m_\gamma \geq m_a + m_b$. At $m_\gamma = m_a - m_c$, the boundary of region E, the D results give for the weight function of the spectral form the expressions

$$\begin{aligned}
&\frac{1}{8\pi}\frac{1}{\Delta^{1/2}} \ln \frac{f + g\Delta^{1/2}}{f - g\Delta^{1/2}}, \quad M:\ m_a + m_b \to m_\beta - m_\gamma \\
&\frac{1}{4\pi}\frac{1}{(-\Delta)^{1/2}}\left[\frac{\pi}{2} + \tan^{-1}\frac{f}{g(-\Delta)^{1/2}}\right], \quad m_\beta - m_\gamma \to m_\beta + m_\gamma \\
&\frac{1}{8\pi}\frac{1}{\Delta^{1/2}}\left(\ln\frac{f + g\Delta^{1/2}}{f - g\Delta^{1/2}} + 2\pi i\right), \quad m_\beta + m_\gamma \to M_0 \\
&\frac{1}{8\pi}\frac{1}{\Delta^{1/2}} \ln \frac{f + g\Delta^{1/2}}{f - g\Delta^{1/2}}, \quad M_0 \to \infty;
\end{aligned} \tag{53}$$

when $m_\beta - m_\gamma \leq m_a + m_b$, the first contribution here is deleted and the second is started at $m_a + m_b$. These expressions we extrapolate into region E. But, since $M_- = M_+ \equiv M_0$ at $m_\gamma = m_a - m_c$, with $M_\pm$ then separating as one moves into region E, the weight function on the interval between these points must be determined. Also, as discussed in the

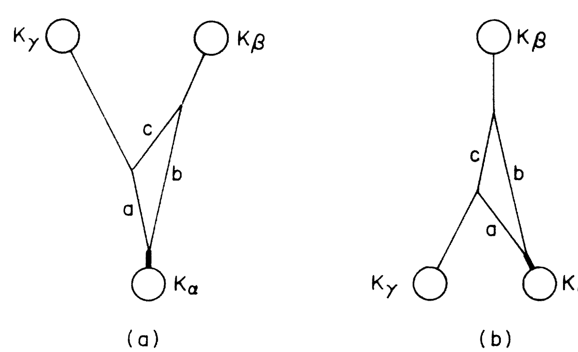

FIG. 4. Physical processes that can occur when particle β and (a) particle a or (b) particle c are unstable.

previous paragraph, M_+ and $m_\beta + m_\gamma$ can coalesce and then separate again as one extrapolates to smaller values of m_γ. And after this coalescence we cannot claim the weight function on the interval between these two points to be the same as before, so this weight-function expression must also be determined. One way for us to make these determinations is to take the weight-function expressions for $M \geq M_-$ and $M \leq m_\beta + m_\gamma$, which remain unchanged throughout region E, and extrapolate them in M through M_- and $m_\beta + m_\gamma$, respectively, using $i\epsilon$ prescriptions and knowing the sign of f at M_- and $m_\beta + m_\gamma$ [cf. the derivation of Eq. (27)]. In this way we immediately see that it is the imaginary part of the weight function on the stated intervals that needs to be determined, the real part always being

$$(8\pi\Delta^{1/2})^{-1}\ln|(f+g\Delta^{1/2})(f-g\Delta^{1/2})^{-1}|.$$

The explicit execution of this is, however, more involved than the simple procedures entailing $i\epsilon$ prescriptions employed earlier. There is, though, a simpler approach, more in the spirit of source theory, which promotes self-consistency to an active role.[28] Namely, the two undetermined imaginary parts continue to occur in the single-spectral form at $m_\gamma = 0$, and at that point there is available the double-spectral representation of the vacuum amplitude, from which it is simple to extract imaginary parts. So, by demanding the consistency of the two spectral representations, we easily obtain the undetermined imaginary parts.

To be specific, the double-spectral representation[1] has the structure (1) with F replaced by

$$G' = \int_{(m_a+m_b)^2}^{\infty} dM^2 \frac{1}{p_\alpha{}^2 + M^2 - i\epsilon} \times \int_{M'_+(M)}^{M'_-(M)} dM'^2 \frac{1}{p_\beta{}^2 + M'^2 - i\epsilon}\,\sigma'(M^2, M'^2). \tag{54}$$

The M, M' integration domain is the region where $-\Delta_3$, evaluated in terms of M, M', and $p_\gamma{}^2$, is non-negative (subject to $M \geq m_a + m_b$), $M'_\pm(M)$ thus being the M' roots of $\Delta_3 = 0$. And σ' is given by Eq. (40), with $p_\alpha{}^2 \to p_\gamma{}^2$. When evaluated at $p_\gamma{}^2 = 0$ (which is the lower limit of the $p_\gamma{}^2$ values for which the double-spectral form was derived) and $-p_\beta{}^2 = m_\beta{}^2$, G' must be equal to the single-spectral form with $m_\gamma = 0$, because both give representations of the vacuum amplitude referring to the same specification of sources. In G' so evaluated, the coefficient of $(p_\alpha{}^2 + M^2 - i\epsilon)^{-1}$ is then identified as the single-spectral weight function. And it is immediate that such a quantity has a nonvanishing imaginary part only on the M interval $M_+ - M_-$, with such being given as

$$\pi\sigma'(M^2, m_\beta{}^2) = (8\pi)^{-1}\Delta^{-1/2}. \tag{55}$$

We conclude, then, that the single-spectral weight-function expressions for region E not determined by the direct extrapolation of Eq. (53) are given by

$$\begin{aligned} &\frac{1}{8\pi}\,\frac{1}{\Delta^{1/2}}\ln\frac{f+g\Delta^{1/2}}{f-g\Delta^{1/2}}, \quad m_\beta + m_\gamma - M_+,\ y_a + y_b \geq 0 \\ &\frac{1}{8\pi}\,\frac{1}{\Delta^{1/2}}\left(\ln\left|\frac{f+g\Delta^{1/2}}{f-g\Delta^{1/2}}\right| + \pi i\right), \quad M_+ - M_-. \end{aligned} \tag{56}$$

The full result thus obtained is summarized in Table I.

Now we consider the situation with $m_\beta + m_\gamma < m_a + m_b$. Here the coalescence of M_+ and $m_\beta + m_\gamma$ of course cannot occur, but our concern with that must now be transferred to the coalescence of M_+ and $m_a + m_b$, which can now occur. At $m_\beta + m_\gamma = m_a + m_b$, $y_a + y_b$ is easily seen to be positive in region E. Thus, as we extrapolate into the domain $m_\beta + m_\gamma < m_a + m_b$, the weight-function expression on the interval[29] $m_a + m_b - M_+$ is given by the first term in Eq. (56). Upon further extrapolation the coalescence occurs; to be specific, it follows from Eq. (17) [or Eq. (15)] that such occurs when $y_\beta + y_\gamma = 0$. After the coalescence it is the imaginary part of the weight function that must be determined, and comparison with the above double-spectral form immediately shows that such is zero. Thus, the weight-function expression in question remains unchanged upon the coalescence.

B. One of p_β, p_γ spacelike

To complete the establishment of the single-spectral form for all possible values of $p_\beta{}^2$ and $p_\gamma{}^2$ not treated in Ref. 1, we must consider the region in which one of p_β, p_γ is spacelike, while the other refers to an unstable particle. To be specific, we consider p_γ spacelike; the results for p_β spacelike follow by simple interchange.

At $m_\gamma = 0$, the boundary of the region under consideration, the results for region E give for the weight function of the single-spectral form the expressions

$$\begin{aligned} &\frac{1}{8\pi}\,\frac{1}{\Delta^{1/2}}\ln\frac{f+g\Delta^{1/2}}{f-g\Delta^{1/2}}, \quad M\!:\ m_a + m_b - M_+ \\ &\frac{1}{8\pi}\,\frac{1}{\Delta^{1/2}}\left(\ln\left|\frac{f+g\Delta^{1/2}}{f-g\Delta^{1/2}}\right| + \pi i\right), \quad M_+ - M_- \\ &\frac{1}{8\pi}\,\frac{1}{\Delta^{1/2}}\ln\frac{f+g\Delta^{1/2}}{f-g\Delta^{1/2}}, \quad M_- - \infty. \end{aligned} \tag{57}$$

These we directly extrapolate into the new region. The only question that arises concerns the coalescence of M_+ and $m_a + m_b$, which occurs

TABLE I. Summary of results.

Define

$$\chi_{\rm I}=\frac{1}{8\pi}\frac{1}{\Delta^{1/2}}\ln\left|\frac{f+g\,\Delta^{1/2}}{f-g\,\Delta^{1/2}}\right|,$$

$$\chi_{\rm II}=\frac{1}{4\pi}\frac{1}{(-\Delta)^{1/2}}\left[\frac{\pi}{2}-\tan^{-1}\frac{f}{g\,(-\Delta)^{1/2}}\right],\quad \chi_{\rm II'}=\frac{-1}{4\pi}\frac{1}{(-\Delta)^{1/2}}\left[\frac{\pi}{2}+\tan^{-1}\frac{f}{g\,(-\Delta)^{1/2}}\right],$$

$$\rho_{\rm I}=\frac{1}{4}\frac{1}{(-\Delta)^{1/2}},\quad \rho_{\rm II}=i\frac{1}{4}\frac{1}{\Delta^{1/2}},$$

and

$$\eta^+=\eta(-y_a-y_b),\qquad \eta^-=\begin{cases}\eta(y_a-y_b), & \text{particle } \beta \text{ unstable}\\ \eta(y_b-y_a), & \text{particle } \gamma \text{ unstable}\end{cases}$$

η being the unit step function of positive argument. Also define

$$M_0^2=2m_a m_b[-y_\beta y_\gamma+\kappa(|y_\beta^2-1|)^{1/2}(|y_\gamma^2-1|)^{1/2}]+m_a^2+m_b^2$$

and

$$M_\pm^2=2m_a m_b[-y_\beta y_\gamma\mp(y_\beta^2-1)^{1/2}(y_\gamma^2-1)^{1/2}]+m_a^2+m_b^2;$$

for regions B, C, and D, κ is respectively $+1$, -1, and $-i$. Here f, g, and Δ are given by Eqs. (5)–(7), with m_β^2 (m_γ^2) replaced by $-p_\beta^2$ ($-p_\gamma^2$) when p_β (p_γ) is spacelike; also $y_{\beta,\gamma}$ and $y_{a,b}$ are given by Eqs. (12) and (51). Lastly, the m_β^2-m_γ^2 regions are defined in Fig. 2. The vacuum amplitude is then of the form of Eqs. (1) and (2) with the following weight functions.

Region	Weight function	Spectral-mass range
A	$\chi_{\rm I}$	$(m_a+m_b)^2\to\infty$
A'	$\chi_{\rm II}$	$(m_a+m_b)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}$	$(m_\beta+m_\gamma)^2\to\infty$
A''	$\chi_{\rm I}$	$(m_a+m_b)^2\to(m_\beta-m_\gamma)^2$
	$\chi_{\rm II}$	$(m_\beta-m_\gamma)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}$	$(m_\beta+m_\gamma)^2\to\infty$
B	$\rho_{\rm I}$	$M_0^2\to(m_a+m_b)^2$
	$\chi_{\rm II}$	$(m_a+m_b)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}$	$(m_\beta+m_\gamma)^2\to\infty$
C, $\lvert m_\beta-m_\gamma\rvert\le m_a+m_b$	$\rho_{\rm II}$	$M_0^2\to(m_\beta-m_\gamma)^2$
	$\rho_{\rm I}$	$(m_\beta-m_\gamma)^2\to(m_a+m_b)^2$
	$\chi_{\rm II}$	$(m_a+m_b)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}$	$(m_\beta+m_\gamma)^2\to\infty$
$\lvert m_\beta-m_\gamma\rvert>m_a+m_b$	$\rho_{\rm II}$	$M_0^2\to(m_a+m_b)^2$
	$\chi_{\rm I}+\rho_{\rm II}$	$(m_a+m_b)^2\to(m_\beta-m_\gamma)^2$
	$\chi_{\rm II}$	$(m_\beta-m_\gamma)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}$	$(m_\beta+m_\gamma)^2\to\infty$
D, $\lvert m_\beta-m_\gamma\rvert\le m_a+m_b$	$\rho_{\rm I}$ [a]	$M_0^2\to(m_a+m_b)^2$ [a]
	$\chi_{\rm II}$	$(m_a+m_b)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}$	$(m_\beta+m_\gamma)^2\to\infty$
$\lvert m_\beta-m_\gamma\rvert>m_a+m_b$	$\rho_{\rm I}$ [a]	$M_0^2\to(m_a+m_b)^2$ [a]
	$\chi_{\rm I}+\rho_{\rm II}$	$(m_a+m_b)^2\to(m_\beta-m_\gamma)^2$
	$\chi_{\rm II}$	$(m_\beta-m_\gamma)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}$	$(m_\beta+m_\gamma)^2\to\infty$
E, β,a or γ,b unstable		
$m_\beta+m_\gamma\le m_a+m_b$	$\chi_{\rm I}$	$(m_a+m_b)^2\to M_+^2$
	$\chi_{\rm I}+\frac{1}{2}\rho_{\rm II}$	$M_+^2\to M_-^2$
	$\chi_{\rm I}$	$M_-^2\to\infty$
$\lvert m_\beta-m_\gamma\rvert\le m_a+m_b<m_\beta+m_\gamma$	$\chi_{\rm II'}$	$(m_a+m_b)^2\to(m_\beta+m_\gamma)^2$
	$\chi_{\rm I}+\eta^+\rho_{\rm II}$	$(m_\beta+m_\gamma)^2\to M_+^2$
	$\chi_{\rm I}+\frac{1}{2}\rho_{\rm II}$	$M_+^2\to M_-^2$
	$\chi_{\rm I}$	$M_-^2\to\infty$

Table I *(Continued)*

Region	Weight function	Spectral-mass range
$\lvert m_\beta - m_\gamma \rvert > m_a + m_b$	χ_{I}	$(m_a + m_b)^2 \to (m_\beta - m_\gamma)^2$
	$\chi_{\mathrm{II}'}$	$(m_\beta - m_\gamma)^2 \to (m_\beta + m_\gamma)^2$
	$\chi_{\mathrm{I}} + \eta^+ \rho_{\mathrm{II}}$	$(m_\beta + m_\gamma)^2 \to M_+^2$
	$\chi_{\mathrm{I}} + \frac{1}{2}\rho_{\mathrm{II}}$	$M_+^2 \to M_-^2$
	χ_{I}	$M_-^2 \to \infty$
β, c or γ, c unstable	χ_{I}	$(m_a + m_b)^2 \to M_+^2$
	$\chi_{\mathrm{I}} + \frac{1}{2}\rho_{\mathrm{II}}$	$M_+^2 \to M_-^2$
	$\chi_{\mathrm{I}} + \eta^- \rho_{\mathrm{II}}$	$M_-^2 \to (m_\beta - m_\gamma)^2$
	χ_{II}	$(m_\beta - m_\gamma)^2 \to (m_\beta + m_\gamma)^2$
	χ_{I}	$(m_\beta + m_\gamma)^2 \to \infty$
p_β, p_γ spacelike	χ_{I}	$(m_a + m_b)^2 \to \infty$
one of p_β, p_γ spacelike, the other stable	χ_{I}	$(m_a + m_b)^2 \to \infty$
one of p_β, p_γ spacelike, the other unstable	χ_{I}	$(m_a + m_b)^2 \to M_+^2$
	$\chi_{\mathrm{I}} + \frac{1}{2}\rho_{\mathrm{II}}$	$M_+^2 \to M_-^2$
	χ_{I}	$M_-^2 \to \infty$

[a] See text below Eq. (30).

at $y_\beta + y_\gamma = 0$ as noted above. But since both signs of $y_\beta + y_\gamma$ occur at $m_\gamma = 0$ in region E, no indeterminacy arises upon extrapolation in the present situation, as compared to that in region E (nor is there any change in the weight function upon coalescence). The weight function for the new region is thus completely given by the expressions (57).

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[1] R. J. Ivanetich, Phys. Rev. D 6, 2805 (1972).

[2] R. J. Ivanetich, Phys. Rev. D 8, 4545 (1973).

[3] In the developments of the text, timelike p_β^2 and p_γ^2 values always refer to real particles. But, according to a simple argument given in Ref. 1, the spectral forms may be extended so that either or both of the particles are virtual ones with the given values.

[4] Actually, some simple extrapolation was effected by considering causal processes of the type of Fig. 1 but with different K_β, K_γ specifications. In this way we obtained the spectral forms for one of p_β, p_γ spacelike and the other referring to a stable particle, and for both spacelike.

[5] J. Schwinger, *Particles, Sources, and Fields*, Vol. 2 (Addison-Wesley, Reading, Mass., 1973).

[6] Another example involving an anomalous threshold, but somewhat different in approach, is given by R. J. Ivanetich [Harvard University thesis, 1969 (unpublished)]. One further example employing extrapolation, but not referring to a situation in which an anomalous threshold occurs, is given by K. A. Milton [Phys. Rev. D 4, 3579 (1971)].

[7] A recent work, with some references to earlier literature, is by J. S. Frederiksen and W. S. Woolcock [Nucl. Phys. B28, 605 (1971)].

[8] C. Fronsdal and R. E. Norton, J. Math. Phys. 5, 100 (1964).

[9] The fact that the system treated in Ref. 5 is not one composed solely of scalar particles, as in the present discussion, is of little consequence on the structure of the check.

[10] Schwinger's deuteron work (Ref. 5) is a specific example involving particles that are not scalar (as is Milton's work, Ref. 6), but what we are particularly getting at here are systems involving unstable particles, where one must contend with singularities of the weight function within the spectral-mass domain.

[11] W.-y. Tsai, L. L. DeRaad, Jr., and K. A. Milton, Phys. Rev. D 8, 1887 (1973); and unpublished work.

[12] In a different vein, one should also give some thought as to whether the present results can be derived directly from some sort of causal-process scheme, i.e., without the use of extrapolation. The discussion of Ref. 1 concerning the necessity of imposing mass restrictions to guarantee the existence of causal processes, makes quite suspect any *a priori* direct treatment of the present mass cases by causal processes. But now that we have established the results in a firm manner, we are free to speculate about direct causal-process schemes and compare their results with those obtained here. To obtain such a scheme would further our understanding of the physical basis of the related spectral forms, and would probably be quite useful in progressing to more complicated examples.

[13]Although $f^2 = g^2\Delta$ corresponds to all three particles being on shell, it does not necessarily correspond to their being real, i.e., the on-shell conditions can be realized with unphysical momenta. Thus, to apply Eq. (11) and the steps following it, we really should first consider a kinematic situation which indeed corresponds to real particles (region E below). Then, having arrived at the result, Eq. (15), we may extend it to all kinematic situations because it is just a simple algebraic expression.

[14]Guided by this conclusion, one can do some algebra to find

$$f^2 - g^2\Delta = 16(m_a m_b m_c)^2 M^2 \times (y^2 + y_\beta^2 + y_\gamma^2 + 2yy_\beta y_\gamma - 1),$$

which otherwise might have been a little difficult to come upon.

[15]Observe then, as might well have been expected, that the spectral-mass points at which we must be concerned with the definition of χ are those at which the causal process of Fig. 1 ceases to exist. That is, $f^2 = g^2\Delta$ corresponds to the onset of particle c also being allowed to be real, while $\Delta = 0$ corresponds to the onset of particle β no longer having a physical momentum. Now, in Ref. 1 we presented some considerations about the structure of general-order causal processes; in fact, we were able to employ such to argue the existence of three-point-function single-spectral forms in arbitrary order, subject to the generalizations of the mass restrictions (9) and (10). Based on those considerations, it should be possible to specify the conditions of nonexistence for general-order causal processes. Then, turning around the implication of the above observation, we might be able to argue some about the structure of three-point-function single-spectral forms in arbitrary order for any mass values.

[16]The term in square brackets is of course equivalent to $\tan^{-1}[g(-\Delta)^{1/2}/f]$, but the expression used in the text is more convenient in that one does not have to change the branch on which $\tan^{-1}$ is defined in order to maintain a continuous weight function as f goes through the value zero, as it will in the studies below. (Tacit, then, in that later work is this weak assumption of continuity.)

[17]Briefly stated, a noncausal representation of the vacuum amplitude is one in which each internal particle is associated with a propagation function; see Ref. 5 and also Vol. 3 of that work (in preparation).

[18]There has been one other example in source theory, although somewhat different from the present, in which it was necessary to consider two causal realizations of one basic coupling in order to obtain the complete spectral form; see Ref. 2.

[19]This process differs from the similar looking one employed in Ref. 1 to obtain a double-spectral form in that here p_γ^2, in not being associated with a spectral variable, has a fixed value.

[20]The external momenta in the vacuum amplitude apart from those in $\bar{p}$ remain physical since they are not associated with M, but rather with the generalized $-p_\alpha^2$.

[21]The choice $+i\bar{p}_x$ is of course also possible, but demanding that the weight function be continuous at $M = m_a + m_b$, we rule in favor of the given choice. Alternately, one may use the consistency check in Schwinger's deuteron work, discussed in Sec. I, to decide the sign.

[22]Another reason for excluding the contribution associated with M_- is that it does not vanish for $y_\beta + y_\gamma = 0$, as would be necessary for consistency between the results of regions A' and B. (Also, when $M_- < |m_\beta - m_\gamma|$, $p_{\beta x}$ imaginary, the input to the whole scheme, is not true for the M_- contribution.)

[23]If one wishes to avoid the use of an $i\epsilon$ prescription, it is not difficult to do so. First, the region-C consistency check given in Sec. III B can be used to fix the undetermined sign that enters in the establishment of Eq. (29). And then, with continuity of the weight function at $M = m_a + m_b$ required, Eq. (29) serves to immediately fix the undetermined sign that would appear in Eq. (27).

[24]This is just the $-i\epsilon$ that is associated with momenta by virtue of the Euclidean hypothesis; see J. Schwinger, *Particles, Sources, and Fields*, Vol. 1 (Addison-Wesley, Reading, Mass., 1970). Alternately, as in Ref. 23, the sign can be determined without invoking an $i\epsilon$ prescription by use of the second consistency check in Sec. III B.

[25]One might inquire if x and v have any simple physical interpretation. We have found that, in the frame where $\vec{p}_b = -\vec{p}_c$ (the so-called brick-wall or Breit frame), they are, respectively, the energy of particle a divided by m_a, and the cosine of the angle between particles a and c.

[26]More correctly, one should admit instability at the beginning of the calculation and obtain the $i\epsilon$ in Eq. (44) from those in the propagators in the original double-spectral form. That requires not setting $m_\beta = m_\gamma$ immediately since the product of spectral denominators is then not well defined. This complicates somewhat the simple steps leading to Eq. (44), but the details are not worth recording here.

[27]Such a process differs from a causal process in that the causal stipulations depicted in the figure—in particular, those associated with the γac vertex—are not guaranteed to exist.

[28]Self-consistency has played an important part in the previous developments of source theory by serving to determine certain unresolved elements there. Most notably, there is the matter of determining contact terms, as is illustrated, e.g., in Ref. 5 and by R. J. Ivanetich [Phys. Rev. D 8, 4564 (1973)]. In addition, it has figured in more than one way in the extrapolation work carried out in Ref. 2.

[29]Since $m_\beta - m_\gamma$ provides the lower limit for one of the weight-function expressions, it might be suggested that as $m_\beta - m_\gamma$ decreases in value below $m_a + m_b$ that it should perhaps become the lower limit of the spectral-mass domain. However, comparison with the double-spectral form immediately reinstates $m_a + m_b$ in that role.