# Classification of black holes with electromagnetic fields

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Pseudostationary asymptotically flat axisymmetric black-hole solutions of the source-free Einstein-Maxwell equations are considered. It is shown that these solutions form discrete continuous families, each depending on at most four parameters, of which only one—the Kerr-Newman family—contains members with zero angular momentum.

It has been shown by Carter[1,2] that, provided a reasonable causality condition is satisfied, the class of pseudostationary solutions of Einstein's pure vacuum equations which may determine the geometry of the space-time manifold exterior to the boundary of an axisymmetric and topologically spherical black hole consists of a discrete set of continuous families. Each family depends on at least one and at most two parameters and the possibility of zero angular momentum is admitted by a unique family—the two-parameter family of Kerr black-hole solutions. The theorems of Hawking[3,4] state that in the pseudostationary case the topology of the black-hole boundary must be spherical as long as the dominant-energy condition is satisfied and that the exterior geometry of the black hole must be axisymmetric when matter fields which may be present obey well-behaved hyperbolic equations. Hence Carter's result holds for all possible pure vacuum exteriors of single pseudostationary black-hole systems.

These results together with further work by Carter and others[2,5,6] and the theorem of Israel[7] as extended by Muller Zum Hagen, Robinson, and Seifert[8] suggest that a similar result should hold for the exteriors of black holes which are sources of electromagnetic fields. In this paper it will be shown that this is in fact the case and by assuming exactly the same conditions of geometrical regularity and causal reasonableness (no closed timelike curves) as Carter the following theorem will be proven:

*The class of pseudostationary solutions of the source-free Einstein-Maxwell equations which determines the exteriors of isolated electromagnetic black holes consists of discrete sets of continuous families each depending on at least one and at most four parameters. The four-parameter Kerr-Newman black-hole family (including a magnetic monopole moment) is unique in admitting the possibility of zero angular momentum.*

It should be noted that although nonstationary (time-dependent) electromagnetic fields and electromagnetic fields with multivalued potentials are locally compatible with stationary geometries which correspond to solutions of the Einstein-Maxwell equations such fields have peculiar global properties which suggest that they are not compatible with the requirements of a regular horizon and asymptotic flatness.[8] Hence only stationary electromagnetic fields are considered here. The possibility that the black hole possesses a magnetic monopole moment is included for completeness. Clearly, the available physical evidence favors a restatement of the above theorem in terms of *three* rather than *four* parameters. Formally, since only the source-free Maxwell equations are considered, the magnetic monopole can always be eliminated by carrying out a duality rotation of the electric and magnetic field quantities.

The following considerations will be developed by using the formalism and notation of Carter's extended paper.[2] Although a brief summary of the salient results of his work will be given here the reader is referred to that paper for the detailed analysis of the underlying assumptions and boundary conditions which ensure that the appropriate regularity conditions are satisfied in the axisymmetric, asymptotically flat systems under consideration.

Carter has shown that, apart from a degeneracy on the symmetry axis, the space-time region external to the boundary of an axisymmetric pseudostationary electromagnetic black hole may be globally covered by a coordinate system $t, \phi, \lambda, \mu$ in which the metric takes the form

$$ds^2 = Q\left(\frac{d\lambda^2}{\lambda^2 - c^2} + \frac{d\mu^2}{1-\mu^2}\right) + Xd\phi^2 + 2Wd\phi dt - Vdt^2$$

while the electromagnetic field form is given by

$$F = 2dA ,$$

where $A = Bd\phi + \Phi dt$. The nonignorable coordinates are $\lambda$ and $\mu$ and $c < \lambda < \infty$, $-1 < \mu < 1$, where $c$ is a strictly positive constant which fixes the scale of the manifold. The two axisymmetry axis components are characterized by $\mu = \pm 1$, respectively, and orthogonally intersect the horizon of the black hole which corresponds to $\lambda = c$.

The quantities $X$, $W$, $\Phi$, and $B$ can be regarded as scalars on a two-dimensional manifold with metric

$$ds^2 = \frac{d\lambda^2}{\lambda^2 - c^2} + \frac{d\mu^2}{1-\mu^2},$$

and boundary with components corresponding to $\lambda = c$ and $\mu = \pm 1$, respectively. These four quantities determine $Q$ and $V$ via the Einstein-Maxwell equations.

It is well known that when the variables $\Phi$ and $W$ are replaced by the Ernst-type potentials[9] $E$ and $Y$, which are determined by the relations

$$\begin{aligned}
(1-\mu^2)E_{,\mu} &= X\Phi_{,\lambda} - WB_{,\lambda}, \\
-(\lambda^2 - c^2)E_{,\lambda} &= X\Phi_{,\mu} - WB_{,\mu}, \\
(1-\mu^2)Y_{,\mu} &= XW_{,\lambda} - WX_{,\lambda} \\
&\quad + 2(1-\mu^2)(BE_{,\mu} - EB_{,\mu}), \\
-(\lambda^2 - c^2)Y_{,\lambda} &= XW_{,\mu} - WX_{,\mu} \\
&\quad - 2(\lambda^2 - c^2)(BE_{,\lambda} - EB_{,\lambda}),
\end{aligned}$$

the independent Einstein-Maxwell equations reduce to four nonlinear partial differential equations in the four unknowns $X, Y, E, B$. These four quantities uniquely determine both the Maxwell field and the four-dimensional geometry for each possible black-hole exterior. As long as the four scalars are well-behaved functions of $\lambda$ and $\mu$ and their derivatives satisfy certain simple conditions on each of the components of the boundary of the two-dimensional manifold regularity of the four-dimensional geometry on the axisymmetry axis and horizon is ensured. It follows from the causality requirement of no closed timelike lines that $X \geq 0$, with equality only on the axisymmetry axis, and asymptotic flatness is ensured by demanding that as $\lambda \to \infty$

$$\begin{aligned}
\lambda^{-2}X &= 1 - \mu^2 + O(\lambda^{-1}), \\
Y &= 2h\mu(3-\mu^2) + O(\lambda^{-1}), \\
E &= -e\mu + O(\lambda^{-1}), \\
B &= -b\mu + O(\lambda^{-1}),
\end{aligned}$$

where $h, e, b$ are, respectively, the asymptotically conserved angular momentum, charge, and magnetic monopole moment of the system.

The independent Einstein-Maxwell equations are given by

$$E_1 = E_2 = E_3 = E_4 = 0, \tag{1}$$

where

$$\begin{aligned}
E_1 &\equiv \nabla\cdot(\rho X^{-2}\nabla X) \\
&\quad + \rho X^{-3}[|\nabla X|^2 + |\nabla Y + 2(E\nabla B - B\nabla E)|^2] \\
&\quad + 2\rho X^{-2}(|\nabla E|^2 + |\nabla B|^2), \\
E_2 &\equiv \nabla\cdot\{\rho X^{-2}[\nabla Y + 2(E\nabla B - B\nabla E)]\}, \\
E_3 &\equiv \nabla\cdot(\rho X^{-1}\nabla E) \\
&\quad - \rho X^{-2}\nabla B\cdot[\nabla Y + 2(E\nabla B - B\nabla E)], \\
E_4 &\equiv \nabla\cdot(\rho X^{-1}\nabla B) \\
&\quad + \rho X^{-2}\nabla E\cdot[\nabla Y + 2(E\nabla B - B\nabla E)],
\end{aligned} \tag{2}$$

$\rho = (\lambda^2 - c^2)^{1/2}(1-\mu^2)^{1/2}$ and $\nabla$ is defined with respect to the two-dimensional metric

$$ds^2 = \frac{d\lambda^2}{\lambda^2 - c^2} + \frac{d\mu^2}{1-\mu^2}.$$

If two distinct families of solutions of (1) which satisfy the boundary conditions can bifurcate from some given black-hole solution, say $\{X, Y, E, B\}(h, c, e, b)$, then a nonzero tangent vector in the space of functions, $\{\dot{X}, \dot{Y}, \dot{E}, \dot{B}\}(h, c, e, b)$, which satisfies the linearized form of the boundary conditions must exist and satisfy the linearized field equations

$$\dot{E}_1 = \dot{E}_2 = \dot{E}_3 = \dot{E}_4 = 0, \tag{3}$$

where

$$\begin{aligned}
\dot{E}_1 &\equiv \nabla\cdot[\rho(X^{-2}\nabla\dot{X} - 2X^{-3}\dot{X}\nabla X)] - 3\rho X^{-4}\dot{X}|\nabla X|^2 - 3\rho X^{-4}\dot{X}|\nabla Y + 2(E\nabla B - B\nabla E)|^2 \\
&\quad + 2\rho X^{-3}\nabla\dot{X}\cdot\nabla X + 2\rho X^{-3}[\nabla Y + 2(E\nabla B - B\nabla E)]\cdot[\nabla\dot{Y} + 2(\dot{E}\nabla B + E\nabla\dot{B} - \dot{B}\nabla E - B\nabla\dot{E})] \\
&\quad + 4\rho X^{-2}(\nabla B\cdot\nabla\dot{B} + \nabla E\cdot\nabla\dot{E}) - 4\rho X^{-3}\dot{X}(|\nabla E|^2 + |\nabla B|^2), \\
\dot{E}_2 &\equiv \nabla\cdot\{\rho X^{-2}[\nabla\dot{Y} + 2(E\nabla\dot{B} - B\nabla\dot{E} + \dot{E}\nabla B - \dot{B}\nabla E)] - 2\rho X^{-3}\dot{X}[\nabla Y + 2(E\nabla B - B\nabla E)]\}, \\
\dot{E}_3 &\equiv \nabla\cdot[\rho(X^{-1}\nabla\dot{E} - X^{-2}\dot{X}\nabla E)] - \rho[X^{-2}\nabla\dot{B} - 2X^{-3}\dot{X}\nabla B]\cdot[\nabla Y + 2(E\nabla B - B\nabla E)] \\
&\quad - \rho X^{-2}\nabla B\cdot[\nabla\dot{Y} + 2(\dot{E}\nabla B + E\nabla\dot{B} - \dot{B}\nabla E - B\nabla\dot{E})], \\
\dot{E}_4 &\equiv \nabla\cdot[\rho(X^{-1}\nabla\dot{B} - X^{-2}\dot{X}\nabla B)] + \rho(X^{-2}\nabla\dot{E} - 2X^{-3}\dot{X}\nabla E)\cdot[\nabla Y + 2(E\nabla B - B\nabla E)] \\
&\quad + \rho X^{-2}\nabla E\cdot[\nabla\dot{Y} + 2(\dot{E}\nabla B + E\nabla\dot{B} - \dot{B}\nabla E - B\nabla\dot{E})].
\end{aligned} \tag{4}$$

The conclusion that no such tangent vector exists follows from an identity deducible from (2) and (4):

$$
\begin{aligned}
&(X^{-1}Q^2 - X^{-1}\dot{X}^2 + 2\dot{E}^2 + 2\dot{B}^2)E_1 - 2X^{-1}\dot{X}QE_2 - 4X^{-1}(\dot{B}Q + \dot{X}\dot{E})E_3 \\
&\quad + 4X^{-1}(\dot{E}Q - \dot{X}\dot{B})E_4 - \dot{X}\mathring{E}_1 - Q\dot{E}_2 - 4\dot{E}\dot{E}_3 - 4\dot{B}\dot{E}_4 + \tfrac{1}{2}\nabla\cdot\{\rho\nabla[X^{-2}(\dot{X}^2 + Q^2) + 4X^{-1}(\dot{E}^2 + \dot{B}^2)]\} \\
&\qquad = \rho\,|\nabla(X^{-1}\dot{X}) + X^{-2}Q[\nabla Y + 2(E\nabla B - B\nabla E)] + 2X^{-1}(\dot{E}\nabla E + \dot{B}\nabla B)|^2 \\
&\qquad\quad + \rho\,|\nabla(X^{-1}Q) - X^{-2}\dot{X}[\nabla Y + 2(E\nabla B - B\nabla E)] + 2X^{-1}(\dot{E}\nabla B - \dot{B}\nabla E)|^2 \\
&\qquad\quad + 2\rho X^{-3}|X\nabla\dot{E} - \dot{B}[\nabla Y + 2(E\nabla B - B\nabla E)] - \dot{X}\nabla E|^2 + 2\rho X^{-3}|X\nabla\dot{B} + \dot{E}[\nabla Y + 2(E\nabla B - B\nabla E)] - \dot{X}\nabla B|^2 \\
&\qquad\quad + 12\rho X^{-2}|\dot{E}\nabla B - \dot{B}\nabla E|^2 + 2\rho X^{-1}|\nabla\dot{E} - X^{-1}\dot{E}\nabla X - X^{-1}Q\nabla B|^2 \\
&\qquad\quad + 2\rho X^{-1}|\nabla\dot{B} - X^{-1}\dot{B}\nabla X + X^{-1}Q\nabla E|^2 + \rho X^{-2}|X^{-1}\dot{X}[\nabla Y + 2(E\nabla B - B\nabla E)] - QX^{-1}\nabla X - 2(\dot{E}\nabla B - \dot{B}\nabla E)|^2,
\end{aligned}
\tag{5}
$$

where

$$Q = \dot{Y} + 2E\dot{B} - 2B\dot{E}.$$

Integration of (5) over the two-dimensional manifold, conversion of the integral of the divergence term to a boundary integral and application of the boundary conditions leads to the conclusion that the left-hand side is zero when (1) and (3) are satisfied. Each of the non-negative terms on the right hand side is therefore zero everywhere and the following differential equation is immediately obtained

$$\nabla Q = 2QX^{-1}\nabla X.$$

The only solution of this equation which is consistent with the boundary conditions is $Q = 0$. It then follows that

$$\nabla\dot{E} = \dot{E}X^{-1}\nabla X$$

and

$$\nabla\dot{B} = \dot{B}X^{-1}\nabla X,$$

with $\dot{E} = \dot{B} = 0$ being the only solutions of these equations consistent with the boundary conditions. It immediately follows that $\mathring{X} = 0$.

In order to conclude the proof of the theorem it is sufficient to note that Carter has shown that when the angular momentum is zero $Y$ must be zero and the four geometry must be *static*. It then follows from the generalization of the theorem of Israel that the only black-hole solutions of (1) are members of the Kerr-Newman family with $h = 0$ and $c \geq 0$.


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