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Some exact models of inhomogeneous dust collapse

E. P. T. Liang*†

Center for Relativity Theory, The University of Texas at Austin, Austin, Texas 78712
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Three classes of irrotational-dust-collapse models with high symmetries are studied. The interior collapsing solution is joined smoothly onto an exterior static vacuum. The spherical model is just a generalization of the Oppenheimer-Snyder model. A black hole is formed in this case. The plane-symmetric model turns out to either have negative gravitating mass and bounce, never reaching the singularity, or have no static exterior. In the latter case all we get for the exterior is a Kasner universe. In the cylindrical case, however, we succeed in constructing a model that, starting with initially regular conditions, collapses into the naked singularity of an external static field.

I. INTRODUCTION

It is the common opinion today that gravitational collapse of massive astronomical objects is a likely phenomenon. However, the exact nature of the collapse process is generally not yet understood. In particular, the ultimate fate of the collapsed object (black hole or naked singularity[1]) in generic situations remains unknown. Hopefully, the recent intensive studies of linear perturbations of the vacuum stationary Kerr-Newman[2] metrics may shed some light on this question. Yet the nonlinearity of the full Einstein equations may betray any conclusions from perturbation calculations near the strong-field region.

We tend to approach the problem from the opposite direction. In the hope that careful analyses of simple exact models may show the clue to some of the answers, we here study in detail three classes of highly symmetric, but inhomogeneous exact collapse models: spherical, plane, and cylindrical symmetric. We choose irrotational dust as the interior source because (a) it is the simplest, (b) it guarantees collapse to a physical singularity (Raychaudhuri's theorem[3]), and (c) experience indicates that many important features of the dust models may persist in general models.[4]

The spherical case is well known. It is included here for comparison. The plane and cylindrical cases are, of course, unrealistic. But we hope to see some important aspects of nonlinear collapse even in such models, especially the relation of the singularity structure to other properties of space-time. The choice of inhomogeneous interior metrics also has some merits: (a) We expect our models to be stable against metric perturbations to a certain extent, due to the larger degrees of freedom; (b) the singularity will in general appear to be nonsimultaneous to collapsing comoving observers; (c) as the matter density μ can be made to vanish arbitrarily smoothly (say, C^∞) across the boundary surface, the interior and exterior metrics form a single, smooth solution. This might be helpful in future works on perturbations.

In the vacuum region ($\mu=0$), the spherical-and plane-symmetric models automatically admit an additional Killing vector (Birkhoff-type theorems[5]). When this Killing vector is timelike, the metrics are static (Schwarzschild and Taub[5]). The cylindrical solutions, however, are in general radiative (Einstein-Rosen[6]). To make the mathematics manageable, however, we choose to assume a static exterior in this case too. This might appear unphysical, as it eliminates the possibility of radiation. However, as it turns out, even in this case we obtain a physical model collapsing to a naked singularity, and there seems to be no obvious reason why a similar model could not exist with radiation.

Section II is a brief review of spherical collapse. Section III is devoted to plane collapse. It turns out that the only plane static vacuum solution corresponds to repulsive gravity (negative gravitational mass). No collapse to the singularity of the static metric is possible. The cylindrical models are discussed in Sec. IV. In addition, a special class of exactly soluble Einstein-Rosen waves with high-frequency incoming radiation as source is exhibited to clarify some points. Some of the tedious computations are separately recorded in the Appendixes.

II. SPHERICAL COLLAPSE

A space-time is said to be spherical symmetric if it admits a three-parameter group of motion isomorphic to the symmetrics of the Euclidean 2-sphere and transitive on families of spacelike 2-surfaces. If dust is the matter source, the flow lines must be radial, irrotational geodesics. These are known as the Tolman-Bondi[7] models. Using comoving normal coordinates the metric can be written as[8]

$$ds^2 = -dt^2 + e^{2\psi(r,t)}dr^2 + R^2(r,t)d\Omega^2\,,$$
$$u^\mu = \delta_0^\mu \text{ (dust flow velocity)}\,. \tag{1}$$

$4\pi R^2(r,t)$ is the area of a 2-sphere at (r,t). The coordinate r is just a label for the mass shells. It is arbitrary to the extent that $r\to r'(r)$ leaves metric (1) form-invariant. With metric (1) the Einstein field equations (EFE) $G_{\mu\nu} = -\mu u_\mu u_\nu$ reduce to two essential equations for ψ and R:

$$e^{\psi(r,t)} = \frac{R'(r,t)}{k(r)}\,, \tag{2}$$

where the prime denotes $\partial/\partial r$,

$$\tfrac{1}{2}\dot{R}^2(r,t) - \frac{m(r)}{R(r,t)} = E(r)\,, \tag{3}$$

where the dot denotes $\partial/\partial t$ and

$$E(r) \equiv \tfrac{1}{2}(k^2-1) \geq -\tfrac{1}{2}\,,$$

where $k(r), m(r)$ are arbitrary integration functions. Equation (2) is still defined when both $k(r)$ and $R'(r,t)$ vanish, provided the ratio exists. When $k(r) = R'(r,t) = 0$ for *all* r, the solution reduces to the *homogeneous* closed Friedmann solution. However, in principle $k(r)$ and $R'(r,t)$ can vanish just locally, say, at $r = r_n$, but $k'(r_n) \neq 0$. Then we have in one single solution regions with $k(r)>0$ and $k(r)<0$. In each region non-negativity of e^ψ requires $R'(r,t)$ to be of the same sign as $k(r)$. Equation (3) is identical to the equation of motion of a particle with radial coordinate R in a central Newtonian gravitational potential, provided we identify $m(r)$ as the total gravitational mass inside $R(r)$ and $E(r)$ as its total energy. From the source equation $G_0^0 = \mu$ we obtain

$$\mu = m'(r)/4\pi R^2 e^{\psi(r,t)}k(r)$$
$$= m'(r)/4\pi R^2 R'\,,$$

which gives

$$m(r) = \int_0^r \mu 4\pi R^2 R' dr$$
$$= \int_0^r \mu k\, d^3 v\,, \tag{4}$$

where $d^3v \equiv 4\pi R^2 e^\psi dr$ is proper 3-volume. We now make the (physically reasonable) requirement that a non-negative μ gives a non-negative contribution to the "mass" $m(r)$. Thence we need $k \geq 0$ and $R' \geq 0$ for all r. So from now on we concentrate on the branch of solutions with $k \geq 0$ and for definiteness write $k(r) = +[1+2E(r)]^{1/2}$. The condition $R'(r,t)>0$ means that larger values of r denote "outer" shells. To make $r=0$ the topological origin we also need the boundary condition $R(0,t)=0$ for all t. Furthermore, for μ to be finite at $r=0$, we need[9]

$$\frac{m'(r)}{4\pi R^2 R'} \underset{r\to 0}{\sim} O(1)\,.$$

Any solution of $R(r,t)$ in Eq. (3) with given $m(r)$, $E(r)$ subject to the above boundary conditions defines a physical solution. Suppose $\mu = 0$ outside some exterior world tube of flow lines defined by $r = r_s$. In the vacuum exterior we have from Eq. (4) $m = M$, a constant. We can then define new coordinates

$$R = R(r,t), \quad T = T(r,t)$$

such that

$$\dot{T} = \frac{(1+\dot{R}^2-2M/R)^{1/2}}{1-2M/R}\,,$$
$$T' = \frac{\dot{R}R'}{(1-2M/R)(1+\dot{R}^2-2M/R)^{1/2}}\,. \tag{5}$$

T is integrable since $(\dot{T})' = (T')^{\boldsymbol{\cdot}}$ via the field equations. In (R,T) coordinates metric (1) is reduced to the Schwarzschild metric with mass M,

$$ds^2 = -(1-2M/R)dt^2 + \frac{dR^2}{(1-2M/R)} + R^2 d\Omega^2 \tag{6}$$

in agreement with the Birkhoff theorem.[5] We could could also make a spherical shell of dust by letting $\mu(r \leq r_i) = 0$ for some inner radius r_i. In this region we have $m(r \leq r_i) = 0$ and space-time is Minkowski. We can make the $\mu\to 0$ transitions across (r_s, r_i) arbitrarily smooth so that the exterior and interior metrics form a single C^∞ solution.

From Eq. (3) we see that for $m(r)>0$, and with appropriate initial conditions [e.g., $\dot{R}(r,0) \leq 0$], $R(r,t)\to 0$ for each r at some t in the future, say $t = {}_0t(r)$, and space-time is singular at $t = {}_0t(r)$ (curvature invariants and matter density blow up). In the exterior there is the well-known Schwarzschild singularity. We now see that it joins smoothly onto the interior singularity. The whole singularity is spacelike[10] and is, in fact, velocity-dominated.[11] It is in general nonsimultaneous with respect to the proper time of the dust particles. Now it is true that even for initial conditions such that $R'(r,0)>0$, in general $R'(r,t)\to 0$ at some later t say ${}_0\bar{t}(r)$. Since $k(r)\neq 0$, this implies $e^\psi \to 0$ and $\mu\to\infty$ and space-time is also singular at ${}_0\bar{t}(r)$. This other type of singularity is usually known as "shell crossing." However, detailed study shows that in the collapsing solutions we can always choose initial conditions such that ${}_0\bar{t}(r) > {}_0t(r)$, so that "shell crossing" occurs only *in the future of* the "Schwarzschild-type" singularity. Hence they lie outside of the physically reachable region of space-time. For our discussions we can therefore assume $R'(r,t)>0$ always.

The norm of the normals to $R=$ const hypersurfaces are given by

$$R_{,\mu}R^{,\mu}=1-\frac{2m(r)}{R}\Rightarrow R_{,\mu}R^{,\mu}=0\Longleftrightarrow R=2m(r)\,. \quad (7)$$

Consider an $R=$ const hypersurface with $R<2M$ (M is now the external total mass). For $r>r_s$ (in vacuum region) it is spacelike. However, since $m(r)$ decreases in the interior with r, $R=$ const ultimately becomes null and then timelike. The boundary where $R=$ const becomes null precisely coincides with the boundary of trapped surfaces.[1] $R=2M$ is null and coincides with the event horizon[1] in the exterior region, but becomes timelike for $r<r_s$. In the interior, the event horizon extrapolates into smaller and smaller values of R until it reaches the central (regular) $r=0$ world line. These and other features are illustrated in Fig. 1.

The above spherical model is the inhomogeneous generalization of the Oppenheimer-Snyder[12] model, who use the closed Friedmann solution as the interior solution, a specialization of the Tolman-Bondi solution. The important point to note here is that the existence of trapped surfaces near the singularity is associated with the spacelike nature of the singularity.

III. PLANE COLLAPSE

A space-time is called plane-symmetric if it admits a three-parameter group of motion isomorphic to the symmetries of the Euclidean 2-plane[5] and transitive on families of spacelike 2-surfaces. The dust flow lines are again contrained to be irrotational geodesics. In comoving normal coordinates the metric is

$$ds^2=-dt^2+e^{2\psi(z\,t)}dz^2+A^2(z,t)\,(dx^2+dy^2)\,,$$
$$u^{\mu}=\delta^{\mu}_{0}\,(\text{dust flow velocity})\,. \quad (8)$$

For easy visualization it is more convenient to compactify the 2-planes into unit area flat tori by identifying $x=0$ with $x=1$ and $y=0$ with $y=1$. $A^2(z,t)$ is then the area of the torus at (z,t). Such an assumption does not affect any of the local properties we are studying. Again, z is just a label for the mass shells. Metric (8) is form invariant under $z\to z'(z)$. The EFE's reduce to

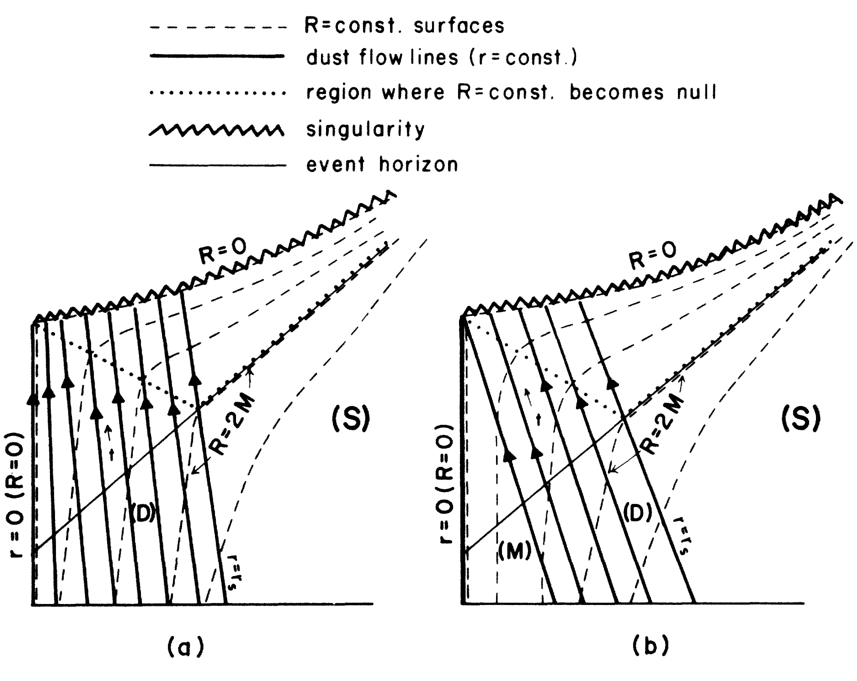

FIG. 1. (a) Space-time diagram for inhomogeneous dust collapse. (D) is the dust-filled interior region. (S) is exterior Schwarzschild metric. $R=0$ is the central particle world line. The triangular region bounded by the dotted lines and $R=0$ singularity is the region of trapped surfaces. Note that the event horizon coincides with $R=2M$ in the exterior but extrapolates to smaller and smaller values of R in the interior. (b) A similar diagram for a dust shell. Space-time is Minkowski (M) inside $r=r_i$. The double solid line $R=0$ is the regular "center" of the Minkowski region. (R is the Schwarzschild coordinate.)

$$e^{\psi(z,t)} = \frac{A'(z,t)}{k(z)}, \tag{9}$$

where the prime indicates $\partial/\partial z$,

$$\tfrac{1}{2}\dot{A}^2(z,t) - \frac{m(z)}{A(z,t)} = E(z), \tag{10}$$

where the dot indicates $\partial/\partial t$ and

$$E(z) \equiv \tfrac{1}{2}k^2(z) \geq 0,$$

where $k(z)$,$m(z)$ are arbitrary integration functions. The situation is similar to the spherical case. Equation (9) is still defined if $k(z)$ and $A'(z,t)$ both vanish, provided the ratio exists. The *flat* Friedmann model results if $k(z) = A'(z,t) = 0$ for *all* z. But locally, $k(z)$ can vanish at, say, $z = z_n$, with $k'(z_n) \neq 0$. $A'(z,t)$ must have the same sign as k everywhere. From the source equation we similarly obtain

$$\mu = m'(z)/A^2 e^{\psi} k(z) = \frac{m'(z)}{A^2 A'},$$

which gives

$$m(z) = \int_{z_0}^{z} \mu A^2 A'\, dz + m(z_0)$$
$$= \int_{z_0}^{z} \mu k\, d^3v + m(z_0), \tag{11}$$

where $d^3v = dxdydz\ e^{\psi}A^2$ is proper 3-volume. Unlike the spherical case, however, there is no compelling physical reason for the existence of an "origin", such that $A(0,t) = 0$. So the lower limit in integral (11) can be taken as the first z value where μ starts to become nonzero. Note, however, that even in this case $m(z_0)$ need not vanish, as illustrated by the following case. Although Eq.(10) again suggests interpretation of $m(z)$ as some "mass" function, we need not impose $k \geq 0$ and $A' \geq 0$ because there is no "origin." In fact, it is possible to have solutions with, say, $k(0) = 0$, $k(z) < 0$ for $z < 0$ and $k(z) > 0$ for $z > 0$ such that m varies from $m(z_0) > 0$ at $z_0 < 0$ to $m = 0$ at $z = 0$ and $m(z) > 0$ for $z > 0$, assuming $\mu \geq 0$ always. These correspond in some sense to models with a plane of symmetry at $z = 0$, and have local Newtonian analogs. In the following, however, to simplify matters we concentrate on local solutions with $k(z) > 0$ and $A'(z,t) \geq 0$. Solutions for $k(z) \leq 0$ are similar as long as $m(z) \geq 0$.

From Eq. (10) we again see that for $m(z) > 0$ and appropriate initial conditions [e.g., $\dot{A}(z,0) \leq 0$ etc.] $A(z,t) \to 0$ for each z at some time in the future, say, $t = {}_0t(z)$ and space-time is singular there. This singularity is also spacelike and velocity-dominated. Just as in the spherical case, we can ignore the singularities of "shell-crossing" $A'(z,t) \to 0$ ($\Rightarrow e^{\psi} \to 0$, $\mu \to \infty$) by choosing initial conditions such that they occur only in the future of $t = {}_0t(z)$, and are physically unreachable.

If we let $\mu = 0$ for, say, $z \geq z_s$, $m(z) = M$, const. Then in the vacuum region we can again define new coordinates

$$A = A(z,t),$$
$$T = T(z,t)$$

such that

$$\dot{T} = \frac{A}{2M}\left(\dot{A}^2 - \frac{2M}{A}\right)^{1/2},$$
$$T' = \frac{A\dot{A}A'}{2M}\left(\dot{A}^2 - \frac{2M}{A}\right)^{-1/2}. \tag{12}$$

T is integrable since $(\dot{T})' = (T')^{\cdot}$ via the field equations. In (T, A) coordinates the metric becomes

$$ds^2 = \frac{2M}{A} dT^2 - \frac{A}{2M} dA^2 + A^2(dx^2 + dy^2). \tag{13}$$

Contrary to our expectation, this metric is time-dependent (A is timelike) for $M > 0$ and is, in fact, a special case of the Kasner metric.[13] To get back the static metric of Taub[5] we have to assume $M < 0$. But from the above derivation such a metric can only represent the external field of a plane-symmetric distribution of *negative* total gravitational "mass". In this case $k\mu$ must be negative for some range of z in the interior. Moreover, from Eq. (10) we see that if for a particular z, $m(z) < 0$, then $A(z,t)$ cannot reach zero, but has a minimum at

$$A_{\min}(z,t) = -m(z)/E(z).$$

For the total space-time picture we thus have three possibilities. (a) If we insist that the model have $m(z) > 0$ everywhere, the only allowed vacuum solution outside the dust is a Kasner cosmological model [with $\vec{p} = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$]. Space-time has a global spacelike singularity in both the interior and exterior. (b) If we let $m(z) < 0$ everywhere, then we have a plane-symmetric dust distribution with negative gravitating mass bouncing off a minimum (of the plane area) while leaving a repulsive static metric outside all the time. There is no singularity in this space-time. (c) If we allow $m(z) > 0$, say, for $z < z_0$, $m(z) = 0$ at $z = z_0$, and $m(z) < 0$ for $z > z_0$, space-time then consists of two universes, one with positive mass and collapsing to a physical singularity, the other with negative mass and bouncing, separated by a thin layer of $m = 0$ space-time at $z = z_0$ [metric (8) is still singular where $m = 0$ but $m' \propto \mu \neq 0$]. The above possibilities are illustrated in Fig. 2.

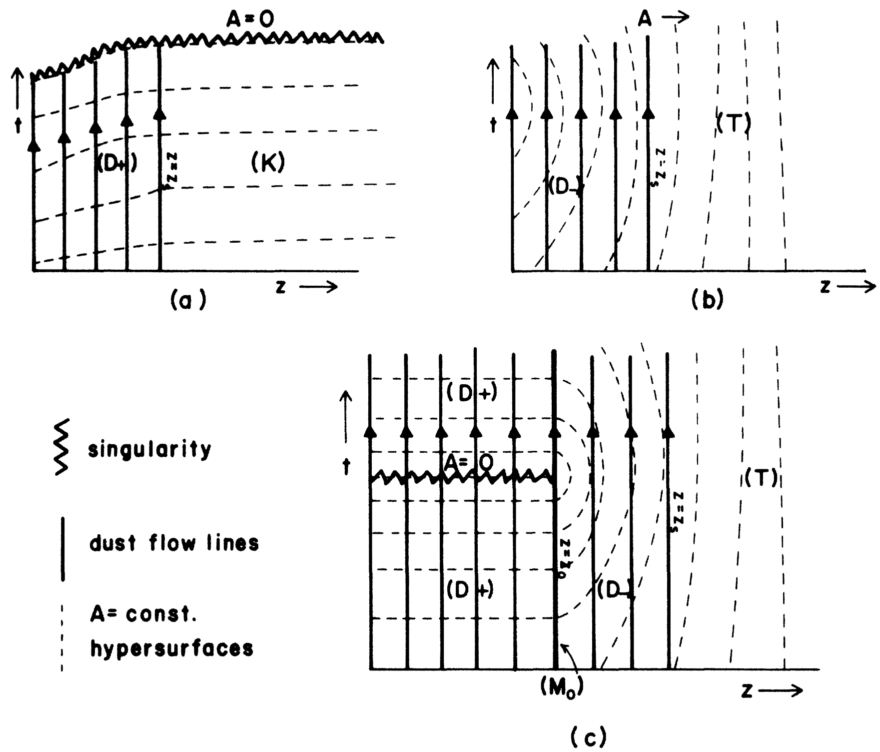

FIG. 2. (a) Space-time diagram for a plane-symmetric model with $m(z) \geq 0$ everywhere. Both exterior and interior collapse to a spacelike singularity ($A = 0$). (D_+) is the dust-filled interior and (K) is the Kasner universe outside. (b) A similar diagram for the case $m(z) \leq 0$. Here the interior world lines never hit a singularity, but bounce off a minimum (in A) and then reexpand. In this case one could have a static exterior (the Taub solution) (T). (c) Another plane-symmetric space-time where we allow m be both >0 and <0. In the regions where $m(z) > 0$ (D_+), space-time collapses to a physical singularity ($A = 0$). But for the regions where $m(z) < 0$, no singularity is reached and the flow lines are complete (D_-). Between the two we must have a region where $m(z) = 0$ which we denote by (M_0). One can of course assume $\mu(z > z_s) = 0$ and we have a Taub static solution. The intriguing thing, however, is that completeness of space-time in the future requires us to have another (D_+) space-time to be "born" after the older universe had collapsed. All these are in fact visible to observers in $z \geq z_0$.

To summarize, we cannot construct a physical model of collapsing dust with plane symmetry which has a static exterior. The important point to note here is that the singularity of the Taub static metric, which is timelike, is repulsive and therefore unreachable by timelike geodesics. A case similar to this is that of charged dust spherical collapse, which has been studied in detail by various authors.[14] The external field, which is just the Reissner-Nordström[15] solution, has a timelike singularity at $R=0$. But the charged dust distribution cannot collapse into this singularity because the repulsive electrostatic force ultimately overcomes the attractive gravitational force as R decreases. Rather, the shells of matter reach a minimum inside the inner horizon R_- and reexpand into another asymptotically flat universe. Here, again, the timelike structure is associated with repulsive gravity and there is no trapped surface in the immediate vicinity of the singularity.

IV. CYLINDRICAL COLLAPSE

Let us now look at models with cylindrical symmetry. For the exterior metric of the model we choose the static Einstein-Rosen solution[6] (or, equivalently, the cylindrical Weyl–Levi-Civita[16]) which is believed to represent the external field of a mass cylinder. For the interior solution we choose, again, irrotational dust. Aside from cylindrical symmetry, the model is not quite realistic, because we eliminate the possibility of radiation. In principle, some kind of outgoing wave solution should be added on to the static one in the exterior vacuum. But then the mathematics become unmanageable. The hope is that our model may be a good approximation to a radiative one so that the essential features are unchanged in the radiative case.

The static Einstein-Rosen solution has the metric

$$ds^2 = -e^{2\chi - 2\psi}(dT^2 - dR^2) + R^2 e^{-2\psi} d\varphi^2 + e^{2\psi} dz^2, \tag{14}$$

with

$$\psi = C \ln R + \ln D,$$

$$\chi = C^2 \ln R + \ln E, \quad C, D, E \text{ consts.}$$

Without loss of generality we can set $D = 1$. This metric has a timelike singularity at $R = 0$ which is naked[1]. For future discussions we shall assume $C < 0$ so that the metric is nonflat ($C = 0, 1 \Longleftrightarrow$ flat space), and the singularity at $R = 0$ is attractive (i.e., timelike geodesics can reach it). If we define the "axis" as where the norm of the Killing vector $\partial/\partial\varphi$ vanishes, then $C < 0$ will also guarantee that $R = 0$ is the axis, and space-time is inextendable beyond $R = 0$. Suppose we now cut away the region of space-time bounded by a timelike geodesic world tube, say, $R = R_s(T)$, which strikes the axial singularity at some finite T (Fig. 3). We like to know if we can smoothly join onto the exterior static space-time across $R = R_s(T)$ a collapsing dust solution. First, we know from the form of metric (14) that the dust generating the interior solution must be irrotational. Second, since the interior solution must be singular somewhere (Raychaudhuri's theorem[3]), we expect the interior singularity to match onto our exterior singularity.

The most general cylindrical metric in comoving irrotational geodesic normal coordinates has the form:

$$ds^2 = -dt^2 + e^{2\alpha} d\rho^2 + e^{2\beta} dz^2 + e^{2\gamma} d\psi^2, \tag{15}$$

$U^\mu = \delta^\mu_0$, α, β, γ are functions of ρ, t alone,

allowed coordinate freedom: $\rho \to \rho'(\rho)$.

The field equations for metric (15) are worked out in the Appendix B [Eqs. (B2)]. The coupled nonlinear equations are not soluble in general. So one has to appeal to approximation techniques. Noting that all the field equations (B2), with the exception of the source equation ($G^0{}_0 = \mu$), should also hold in the vacuum region, we expect than an interior solution that goes over to metric (14) when $\mu \to 0$ would be very similar to metric (14) expressed in radial geodesic coordinates.

The radial geodesics for metric (14) are explicitly integrated in a power series form in Appendix A, away from the singularity $R = 0$. Expressing everything in terms of the proper time t and another label ρ for the geodesics, we can put metric (14) into the form (15), with the metric coefficents expanded as power series in ${}_0t(\rho) - t$, where $t = {}_0t(\rho)$ is the singularity $R = 0$. Assuming a similar power series expansion for the interior dust solution, but now with undetermined coefficients, we put them

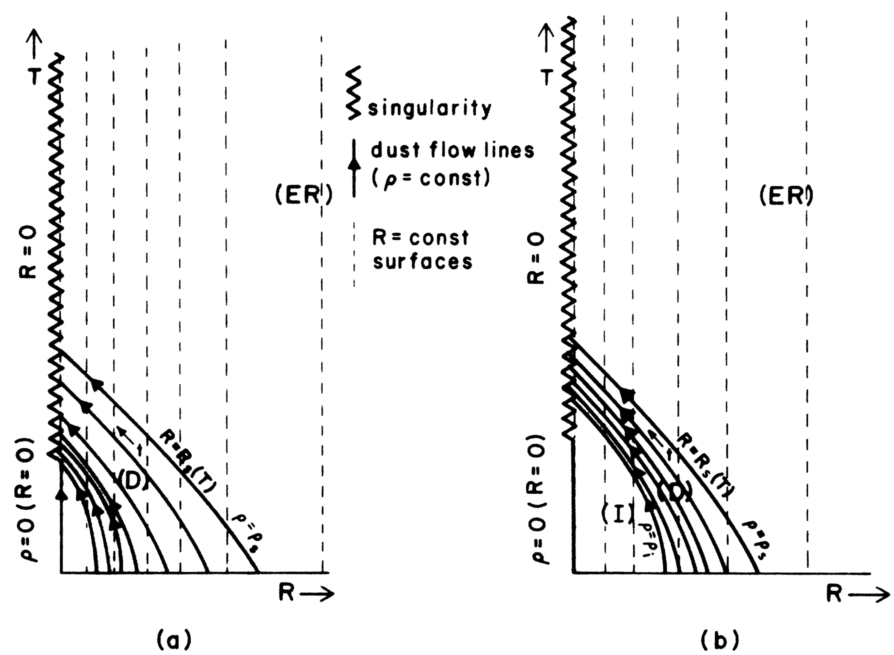

FIG. 3. (a) Space-time diagram of a dust cylinder collapsing into a naked singularity ($R = 0$) from regular initial conditions. (D) is the interior dust region. (ER) is the static exterior field (static Einstein-Rosen solution). R is the Einstein-Rosen coordinate. (b) A similar diagram for a shell of cylindrical dust. Space-time, however, is not flat in the vacuum region inside (I), but is dynamical [metric (19)] and even singular. The double solid line $R = 0$ is the regular world line of the "axis" in the vacuum region.

back into Eq. (B2), and find the following solution near the singularity:

$$e^{\alpha}=({}_0t')[1-(P+Q)mR_0{}^{-P-1}({}_0t-t)^{Q/P}+\tfrac{3}{2}(P+Q)^2m^2R_0{}^{-2P-2}({}_0t-t)^{2Q/P}+\cdots]$$
$$+P(\ln R_0)'({}_0t-t)+P(mR_0{}^{-1})'R_0{}^{-P}({}_0t-t)^{1+Q/P}+\cdots, \quad (16a)$$
$$e^{\beta}=R_0{}^{C}({}_0t-t)^{C/P}[1-CmR_0{}^{-P-1}({}_0t-t)^{Q/P}+O(({}_0t-t)^{2Q/P})+\cdots], \quad (16b)$$
$$e^{\gamma}=R_0{}^{(1-C)}({}_0t-t)^{(1-C)/P}[1-(1-C)mR_0{}^{-P-1}({}_0t-t)^{Q/P}+O(({}_0t-t)^{2Q/P})+\cdots], \quad (16c)$$

where $R_0(\rho)$, $m(\rho)$, ${}_0t(\rho)$ are arbitrary integration functions of ρ, and C is the same const as in metric (14). The prime indicates $\partial/\partial\rho$ and $P\equiv 1-2C+2C^2>1$, $Q\equiv P-1>0$, and all higher-order terms depend on m. From the Bianchi identity we also have:

$$\mu=\mu_0(\rho)e^{-\alpha-\beta-\gamma}, \quad (16d)$$

where μ_0 is arbitrary integration function. After a dreary calculation for $G^0{}_0$ up to $O(({}_0t-t)^{-1/P})$, we find

$$\mu_0(\rho)=\frac{2Q(P+Q)}{P}R_0{}^{-P}m'(\rho). \quad (17)$$

Thus $\mu=0\Longleftrightarrow\mu_0=0\Longleftrightarrow m=\text{const}$, say, M, and solutions (16) automatically reduce to metric (14). (See Appendix A.) The situation is therefore completely analogous to the spherical case and we know we are on the right track. m is our cylindrical analog of the effective gravitational mass (per unit length):

$$m(\rho)=\frac{P}{2Q(P+Q)}\int_0^{\rho}R_0{}^{P}(\rho)\mu_0(\rho)d\rho. \quad (18)$$

The intriguing thing, however, is that when we go to the external field the total mass M is proportional to E^{-2}, and, contrary to expectations along Newtonian lines, is not determined by the constant C. This will be discussed again later.

Let us look at the boundary conditions at the axis a little more carefully. Suppose we have just a shell of dust, so that in Eq. (18) $m(\rho\le\rho_i)=0$ for some inner coordinate radius ρ_i. In this region we obtain an *exact* solution to the EFE's by setting $m=0$ in Eqs. (16):

$$e^{\alpha}=P(\ln R_0)'({}_0t-t)+{}_0t',$$
$$e^{\beta}=R_0{}^{C}({}_0t-t)^{C/P}, \quad (19)$$
$$e^{\gamma}=R_0{}^{(1-C)}({}_0t-t)^{(1-C)/P}.$$

If we now choose initial conditions $R_0(0)=0$, then we are guaranteed that $\rho=0$ is the axis ($e^{\gamma}=0$). Moreover, computing the curvature tensor components (Appendix C), we see that $\rho=0$ is in fact regular (curvature tensor does not depend on R_0 at all). So we have perfectly regular initial conditions on the axis in this case. Metric (19), however, is not flat, and is in fact dynamical and singular at $t={}_0t(\rho)$ (Appendix C), unlike the spherical case.

In the more general case in which μ remains nonzero all the way up to $\rho=0$, if we again assume $R_0(0)=0$ then in order to make metric (16) and the density μ finite at $R_0=0$ we need to impose regularity conditions on m, e.g.,[9]

$$m'(\rho)R_0{}^{-P-1}\underset{\rho\to 0}{\sim}O(1)$$

and

$$m(\rho)R_0{}^{-P-2}R_0{}'\underset{\rho\to 0}{\sim}O(1).$$

Such conditions will automatically make $\rho=0$ the axis ($e^{\gamma}=0$). The space-time pictures of both possibilities are sketched in Fig. 3.

If one traces the reason why space-time cannot be made flat (or, at least, regular) in the interior vacuum $m=0$, one finds that the crucial reason is that the constant C, which determines whether space-time is flat or not [Appendix C shows that metric (19) is flat if and only if $C=0$, 1, or $\frac{1}{2}$], cannot vary. Could this particular phenomenon be due to our restriction that the external field be static? Let us look at some other example *with* radiation.

Rao[17] has discovered that if we choose an energy-momentum tensor of the form

$$T_{\mu\nu}=\sigma k_{\mu}k_{\nu},$$
$$k_{\mu}k^{\mu}=0 \quad (20)$$
$$k_{\mu;\nu}k^{\nu}=0,$$

then we can obtain explicitly a class of Einstein-Rosen solutions if k_{μ} is chosen to lie in the radial direction. $T_{\mu\nu}$ represents high-frequency outgoing (incoming) radiation (photons, neutrinos, or gravitons) depending on whether k_{μ} is outgoing (incoming). The general solution is:

ψ any solution to

$$\Box_C{}^2\psi\equiv\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right)-\frac{\partial^2}{\partial T^2}\right]\psi=0,$$

and (i) if we choose $k^0=k^R$ (outgoing), this implies

$$k_0=-k_R=k(U),$$
$$\sigma=f(U)/R,$$

where k, f are arbitrary functions of the retarded coordinate $U\equiv T-R$, and

$$\chi_{,R}=R(\psi_{,R}{}^2+\psi_{,T}{}^2+fk^2/R)\,,\qquad \chi_{,T}=2R(\psi_{,R}\psi_{,T}-fk^2/R)\,, \tag{21}$$

and (ii) if we choose $k^0=-k^R$ (incoming), this implies

$$k_0=k_R=k(V)\,,\qquad \sigma=f(V)/R\,,$$

where k, f are arbitrary function of the advanced coordinate $V\equiv T+R$.

$$\chi_{,R}=R(\psi_{,R}{}^2+\psi_{,T}{}^2+fk^2/R)\,,\qquad \chi_{,T}=2R(\psi_{,R}\psi_{,T}+fk^2/R)\,. \tag{22}$$

Take, for example, the incoming case, with f nonzero only for a finite range of V. This would represent a "shell" of incoming high frequency photons converging onto the axis ($\sigma\to\infty$ at $R=0$). From Eqs. (22) we see that the σ term contributes nothing to Ψ, and only contributes a constant to χ if we integrate from inside the shell to the exterior of the shell. So if we have for Ψ a static solution $\Psi=C\ln R$ plus other wave solutions, then we also have for χ a static solution $\chi=C^2\ln R+\int_{v_1}^{v_2}dv\,fk^2+\ln E$ plus other wave solutions in the vacuum region outside of the shell. We are therefore convinced that the constant E in χ is related to mass since it is affected by matter or radiation whereas C remains unchanged. This phenomenon is independent of whether we have gravitational waves or not. This peculiarity of the constant C has been discussed by various authors[18] before, with similar conclusions. In fact, Stachel[18] has proposed that it might be the cylindrical analog of the Newman-Penrose constants.[19] (Compare, however, the C energy of Thorne.[20])

How well does the collapse picture of our special model with dust represent more general cylindrical situations (e.g., with pressure and gravitational radiation)? At present we can only say a couple of speculative remarks. Since the form of the singularity in the interior solution (16) is *not* of the "caustic" type,[21] we have high hopes that it will be stable against pressure perturbations.[22] Thorne[23] has shown that a general vacuum cylindrical metric, with waves or not, does not possess event horizons. So the question concerning models with gravitational radiation is whether naked singularities can again arise from collapse with *regular initial conditions*. Preliminary studies of dust models seem to indicate that this is indeed possible, and the situation is similar to our solution. Models with both pressure and gravitational radiation remain to be investigated.

V. CONCLUSIONS

We have compared and contrasted three collapsing-dust models with high symmetry. We see that in the spherical case, the existence of trapped surfaces in the immediate vicinity of the singularity is associated with the space-like nature of the singularity. We fail to construct a plane-symmetric model that collapses into the naked singularity of the external static field because the singularity is repulsive. This together with the charged spherical dust model demonstrate that in many situations repulsive gravity (or negative gravitational mass) is associated with timelike singularity. On the other hand, we succeed in explicitly constructing a cylindrical dust model, which, starting with regular initial conditions, collapses into the naked singularity of an external static field. Although the assumptions of cylindrical symmetry and no radiation make this model a little too artificial, its existence casts doubts, to a certain extent, on the hypothesis of "cosmic censorship".[1]

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APPENDIX A: RADIAL TIMELIKE GEODESICS OF THE STATIC EINSTEIN-ROSEN METRIC

$$ds^2=-e^{2\chi-2\psi}(dT^2-dR^2)+e^{2\psi}dz^2+R^2e^{-2\psi}d\varphi^2\,,\qquad \psi=C\ln R\,,\quad \chi=C^2\ln R+\ln E\,,\quad C<0\,. \tag{A1}$$

The radial geodesic equations are ($\varphi=$ const, z = const):

$$(\dot T^2-\dot R^2)e^{2\chi-2\psi}=1\,, \tag{A2}$$

where the dot indicates $\partial/\partial t$ with t the proper time, and

$$(\dot T R^{2C^2-2C})^\cdot=0\,. \tag{A3}$$

The solution is (for incoming geodesics)

$$\begin{aligned}&\dot T=k_0R^{2C-2C^2}=k_0R^{-Q}\,,\\ &Q\equiv 2C^2-2C>0\,,\\ &R=R_0({}_0t-t)^{1/P}+R_1({}_0t-t)\\ &\quad+R_2({}_0t-t)^{1+Q/P}+O(({}_0t-t)^{1+2Q/P})\,,\\ &P\equiv Q+1>1\,,\end{aligned} \tag{A4}$$

where

$$R_0{}^P=k_0P\,,$$

$$R_1=-\frac{P^2}{2E^2(P+Q)}R_0^{-P},$$

$$R_2=\frac{R_1^2}{2R_0}\frac{(3P^2-5PQ+Q^2)}{(P+2Q)},$$

etc. and $t={}_0t \Leftrightarrow R=0$, the axial singularity. If we now let R_0 (i.e., k_0) and ${}_0t$ depend on a certain label ρ, we can express metric (A1) in (t,ρ,z,φ) coordinates in the form

$$ds^2=-dt^2+e^{2\alpha_e}d\rho^2+e^{2\beta_e}dz^2+e^{2\gamma_e}d\varphi^2,$$

$$e^{\alpha_e}=\frac{(R^P)'}{R_0^P}=({}_0t')\left[1+\frac{R_1}{R_0}(P+Q)({}_0t-t)^{Q/P}+\left(\frac{R_2}{R_0}+\frac{Q}{2}\frac{R_1^2}{R_0^2}\right)(P+2Q)({}_0t-t)^{2Q/P}+O(({}_0t-t)^{3Q/P})+\cdots\right]$$

$$+P(\ln R_0)'({}_0t-t)+PQR_1R_0'R_0^{-2}({}_0t-t)^{1+Q/P}+O(({}_0t-t)^{1+2Q/P})+\cdots, \quad \text{(A5a)}$$

$$e^{\beta_e}=R^C=R_0^C({}_0t-t)^{C/P}\left[1+C\frac{R_1}{R_0}({}_0t-t)^{Q/P}+C\left(\frac{R_2}{R_0}-\frac{(1-C)}{2}\frac{R_1^2}{R_0^2}\right)({}_0t-t)^{2Q/P}+\cdots\right], \quad \text{(A5b)}$$

$$e^{\gamma_e}=R^{(1-C)}=R_0^{(1-C)}({}_0t-t)^{(1-C)/P}\left[1+(1-C)\frac{R_1}{R_0}({}_0t-t)^{Q/P}+(1-C)\left(\frac{R_2}{R_0}-\frac{C}{2}\frac{R_1^2}{R_0^2}\right)({}_0t-t)^{2Q/P}+\cdots\right], \quad \text{(A5c)}$$

where the prime indicates $\partial/\partial\rho$.

APPENDIX B: EFE'S FOR METRIC (15) AND THE SOLUTION NEAR THE SINGULARITY

$$ds^2=-dt^2+e^{2\alpha}d\rho^2+e^{2\beta}dz^2+e^{2\gamma}d\varphi^2,\quad u^\mu=\delta^\mu_0, \quad \text{(B1)}$$

where α,β,γ are functions of ρ, t and the allowed coordinate freedom is $\rho\to\rho'(\rho)$. The EFE's with dust source $G_{\mu\nu}=-\mu u_\mu u_\nu$ reduce to the following essential equations [where the dot indicates $\partial/\partial t$ and the prime indicates $\partial/\partial\rho$]:

$$G^1{}_0=e^{-2\alpha}[\dot\beta'+\dot\beta\beta'+\dot\gamma'+\dot\gamma\gamma'-\dot\alpha(\beta'+\gamma')]=0, \quad \text{(B2a)}$$

$$G^1{}_1=(\ddot\beta+\ddot\gamma+\dot\beta^2+\dot\gamma^2+\dot\beta\dot\gamma)-e^{-2\alpha}\beta'\gamma'=0, \quad \text{(B2b)}$$

$$G^2{}_2=(\ddot\alpha+\dot\alpha^2+\ddot\gamma+\dot\gamma^2+\dot\alpha\dot\gamma)-e^{-2\alpha}(\gamma''+\gamma'^2-\alpha'\gamma')=0, \quad \text{(B2c)}$$

$$G^3{}_3=(\ddot\alpha+\dot\alpha^2+\dot\beta^2+\ddot\beta+\dot\alpha\dot\beta)-e^{-2\alpha}(\beta''+\beta'^2-\beta'\alpha')=0, \quad \text{(B2d)}$$

$$G^0{}_0=(\dot\alpha\dot\beta+\dot\alpha\dot\gamma+\dot\beta\dot\gamma)-e^{-2\alpha}(\beta''+\beta'^2+\gamma''+\gamma'^2+\beta'\gamma'-\beta'\alpha'-\gamma'\alpha')=\mu. \quad \text{(B2e)}$$

A more useful combination of $G^1{}_1$, $G^2{}_2$, $G^3{}_3$, and $G^0{}_0$ is given by

$$\tfrac{1}{2}(G^2{}_2+G^3{}_3-G^0{}_0-G^1{}_1)=\ddot\alpha+\dot\alpha^2-\dot\beta\dot\gamma+e^{-2\alpha}\beta'\gamma'=-\tfrac{1}{2}\mu. \quad \text{(B3)}$$

Also from the Bianchi identity we have

$$\mu=\mu_0(\rho)e^{-\alpha-\beta-\gamma}, \quad \text{(B4)}$$

where $\mu_0(\rho)$ is an integration function. Let us now try a power-series solution near the singularity of the form

$$e^{\alpha}=e^{\alpha_0}[1+\alpha_1({}_0t-t)^{Q/P}+\alpha_3({}_0t-t)^{2Q/P}+\cdots+O(({}_0t-t)^{nQ/P})+\cdots]+\alpha_2({}_0t-t)+\alpha_4({}_0t-t)^{1+Q/P}+\cdots+O(({}_0t-t)^{1+nQ/P}),$$

n integers (B5)

$$\beta=\beta_0+\frac{C}{P}\ln({}_0t-t)+\beta_1({}_0t-t)^{Q/P}+\beta_2({}_0t-t)^{2Q/P}+\cdots,$$

$$\gamma=\gamma_0+\frac{1-C}{P}\ln({}_0t-t)+\gamma_1({}_0t-t)^{Q/P}+\gamma_2({}_0t-t)^{2Q/P}+\cdots.$$

where ${}_0t$, $\alpha_0,\alpha_1,\alpha_2,\ldots,\beta_0,\beta_1,\ldots,\gamma_0,\gamma_1,\ldots,C$ are undetermined arbitrary functions of ρ. Plugging these into Eqs. (B2) we find that the equations are identically satisfied provided the coefficients satisfy the constraints:

(i) $C=\text{const}$;

(ii) $\beta_n/\gamma_n=\dfrac{C}{1-C}\Leftrightarrow\beta_0=C\ln R_0$;

$$\gamma_0=(1-C)\ln R_0,\quad \beta_1=CR_1R_0^{-1};\quad \gamma_1=(1-C)R_1R_0^{-1};\quad R_1=-m(\rho)R_0^{-P}; \quad \text{(B6)}$$

where $R_0(\rho), m(\rho)$ can be arbitrary functions of ρ; higher-order coefficients are determined in terms of R_0, R_1 in exactly the same way as Eq. (A5);

(iii) $e^{\alpha_0} = {}_0t'$;

$\alpha_1 = -(P+Q)mR_0^{-P-1}$,

$\alpha_2 = P(\ln R_0)'$;

$\alpha_3 = \frac{3}{2}(P+Q)^2 m^2 R_0^{-2P-2}$,

$\alpha_4 = P(mR_0^{-1})' R_0^{-P}$, etc.

Thus the interior is identical to the exterior solution (A5) except for the fact that $R_1 R_0^P$ is no more a constant, but is proportional to $m(\rho)$, an arbitrary function. Using (B3) we find that the nonconstancy of m precisely leads to nonzero mass density

$$\mu_0(\rho) = \frac{2Q(P+Q)}{P} m'(\rho) R_0^{-P} \tag{B7}$$

and $m(\rho)$ is the cylindrical analog of total gravitating mass (per unit length) inside the ρth shell.

In general the series (B5) with coefficients (B6) should converge in the same way as the vacuum metric (A5). We expect the solution to be unique (as determined by the external static field) because it contains correctly the two arbitrary degrees of freedom $[m(\rho)$, the initial mass distribution, and $[R_0(\rho)]^P$, the initial "kinetic energy" distribution$]$ expected for a general irrotational nonradiative cylindrical dust solution. (${}_0t$ is of course nonessential because it can be changed by rescaling ρ). A specification of the functions m and R_0 on an initial partial Cauchy surface uniquely determines the future (or past) development of the solution.

APPENDIX C: CURVATURE TENSOR FOR THE METRIC (19)

We have

$$e^{\alpha} = P(\ln R_0)'({}_0t - t) + {}_0t',$$
$$e^{\beta} = R_0^{C}({}_0t - t)^{C/P},$$
$$e^{\gamma} = R_0^{(1-C)}({}_0t - t)^{(1-C)/P}. \tag{C1}$$

It is convenient to introduce orthonormal frames and use the Cartan formalism to compute the curvature tensor in this case. Let ω^i (Greek indices; $i, j, k = 0, \ldots, 3$) be the tetrad

$$\omega^0 = dt, \quad \omega' = e^{\alpha} d\rho, \quad \omega^2 = e^{\beta} dz, \quad \omega^3 = e^{\gamma} d\varphi,$$

$$ds^2 = \sum_{ij} \eta_{ij} \omega^i \otimes \omega^j, \quad \eta_{ij} \equiv \mathrm{diag}(-1, 1, 1, 1). \tag{C2}$$

Using the structure equations[24]

$$d\omega^i = -\sum_k \omega^i{}_k \wedge \omega^k \tag{C3}$$

and

$$\tfrac{1}{2}\sum_{k,l} R^i{}_{jkl}\, \omega^k \wedge \omega^l = d\omega^i{}_j + \sum_k \omega^i{}_k \wedge \omega^k{}_j, \tag{C4}$$

we find

$$\omega^1{}_0 = \omega^0{}_1 = \mathring{\alpha}\, \omega^1,$$
$$\omega^2{}_0 = \omega^0{}_2 = \mathring{\beta}\, \omega^2,$$
$$\omega^2{}_1 = -\omega^1{}_2 = \beta' e^{-\alpha} \omega^2, \tag{C5}$$
$$\omega^3{}_0 = \omega^0{}_3 = \mathring{\gamma}\, \omega^3;$$
$$\omega^3{}_1 = -\omega^1{}_3 = \gamma' e^{-\alpha} \omega^3,$$

others zero, which implies

$$\tfrac{1}{2}\sum_{jk} R^1{}_{0jk}\, \omega^j \wedge \omega^k = 0,$$

$$\tfrac{1}{2}\sum_{jk} R^2{}_{0jk}\, \omega^j \wedge \omega^k = \tfrac{1}{2}\sum_{jk} R^2{}_{1jk}\, \omega^j \wedge \omega^k = \frac{C}{P}\left(\frac{C}{P} - 1\right)({}_0t - t)^{-2}(\omega^0 \wedge \omega^2 + \omega^1 \wedge \omega^2),$$

$$\tfrac{1}{2}\sum_{jk} R^3{}_{0jk}\, \omega^j \wedge \omega^k = \tfrac{1}{2}\sum_{jk} R^3{}_{1jk}\, \omega^j \wedge \omega^k = \frac{C(1-C)(1-2C)}{P^2}({}_0t - t)^{-2}(\omega^0 \wedge \omega^3 + \omega^1 \wedge \omega^3),$$

so that the nonvanishing components of $R^i{}_{jkl}$ are

$$R^2{}_{002} = R^2{}_{012} = R^2{}_{112} = \frac{C}{P}\left(\frac{C}{P} - 1\right)({}_0t - t)^{-2},$$
$$R^3{}_{003} = R^3{}_{013} = R^3{}_{113} = \frac{C(1-C)(1-2C)}{P^2}({}_0t - t)^{-2}, \tag{C6}$$

and those obtained by permutation of the indices. Thus the curvature tensor is:

(a) independent of R_0;

(b) singular at $t = {}_0t(\rho)$ except when $C = 0$, 1, or $\frac{1}{2}$, in which case space-time is flat. $C = 0, 1$ corresponds to Minkowski space in metric (14), whereas $C = \frac{1}{2}$ reduces metric (14) to the plane-symmetric Taub[5] solution.

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†Present address: Department of Physics, The University of Utah, Salt Lake City, Utah 84112.

[1]For a detailed discussion of the concepts of black hole, naked singularity and related concepts, see S. Hawking, *Black Holes*, 1972 Les Houches Lectures, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, N.Y., 1973). In the plane- and cylindrical-symmetric cases, since space-time is not asymptotically flat, we simply define a singularity to be "naked" if it is visible from all values of the variable A in the Taub metric (13) or R in the static Einstein-Rosen metric (14).

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[8]Throughout this paper we adopt the convention:

$$8\pi G = c = 1,$$

$$\mathrm{sgn}\, g_{\mu\nu} = (-1, +1, +1, +1),$$

$$\mu, \nu = (0, \ldots, 3),$$

$$\text{Einstein equations: } G_{\mu\nu} = -T_{\mu\nu}.$$

[9]Here "$\underset{r\to 0}{\simeq} O(1)$" means that the limit is finite or $\to 0$ as some *integer* power of r as $r \to 0$.

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