Astrophys. J. 171, 209 (1972).
[26]If this idea provides a resolution to the cosmological mass problem, it leaves unresolved any galactic missing-mass difficulty. See Ref. 25 for example.
[27]I. I. Shapiro *et al.*, Phys. Rev. Lett. 26, 27 (1971).
[28]R. Cowsik, lecture at 1973 Summer Meeting of Division of Particles and Fields of the American Physical Society, University of California, Berkeley (unpublished).
[29]Evidence on q_0 is probably only good enough for us to say with assurance that $0<q_0<4$ or so. See Refs. 3, 23, and 24.
[30]R. H. Dicke, Astrophys. J. 152, 1 (1968); G. S. Greenstein, Astron. Lett. 1, 139 (1968); R. E. Morganstern, Phys. Rev. D 4, 278 (1971); 4, 282 (1971).
[31]P. A. M. Dirac, Nature 139, 323 (1937); Proc. R. Soc. A165, 198 (1938).

PHYSICAL REVIEW D VOLUME 10, NUMBER 2 15 JULY 1974

Initial-value problem of general relativity. I. General formulation and physical interpretation*

Niall Ó Murchadha and James W. York, Jr.
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
and Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27514†
(Received 3 April 1974)

The initial-value equations of Einstein's theory of general relativity are formulated as a system of four coupled quasilinear elliptic equations. These equations result from a covariant orthogonal decomposition of symmetric tensors and a generalized technique of conformal deformation of initial data. Mathematical properties and global integrability conditions of the equations are discussed. Physical interpretation of the independent and dependent data is given for both spatially closed and asymptotically flat initial-data sets. In the latter case, the four dependent functions constitute long-range scalar and vector potentials which determine the total mass and total linear and angular momenta of an isolated system. The definitions of linear and angular momenta suggest a unique extension to asymptotically flat three-spaces of the group of translations and rotations of flat three-space. In turn, the "almost symmetries" thus defined lead to Gaussian theorems expressing the equality of certain surface and volume integrals for total linear and angular momenta. An interpretation of the scalar and vector potentials for closed three-spaces is also given. In the Appendix we treat the special case of conformally flat initial data.

I. INTRODUCTION

The initial-value problem of general relativity is the problem of constructing a complete set of Cauchy data on a spacelike hypersurface for Einstein's equations.[1] These data are subject to constraints or initial conditions. One must separate the freely specifiable or independent quantities from the dependent ones, which are determined in terms of the independent data as solutions of the initial-value equations. In this paper we present the initial-value problem as a system of four coupled quasilinear elliptic equations.[2] The independent data describe the "wave" degrees of freedom of the gravitational field and the freely specifiable parts of any matter or other field sources that may be present. The dependent data are generalized potentials. The long-range behavior in an asymptotically flat space of these potentials determines the mass and linear and angular momenta of the gravitational field.

There are four key ideas which underlie the present approach: (1) The Cauchy data refer only to the instantaneous physical state of a gravitational field, not to quantities which describe the velocities and accelerations of observers relative to a given spacelike slice, which are irrelevant to the initial-value problem.[3] Therefore, we shall use as initial data the spatial metric g_{ab} and the "momentum" $\pi^{ab}=g^{1/2}(Kg^{ab}-K^{ab})$, K_{ab} = extrinsic curvature, which do not depend on how the spacetime coordinate system is to be continued away from the initial surface. These quantities form Cauchy data for the gravitational field. However, the initial-value constraints can be written completely in terms of g_{ab} and π^{ab}, and show that these objects are not freely specifiable. (2) For a given three-geometry, represented by g_{ab}, the "momentum" π^{ab} can be orthogonally and covariantly decomposed into a transverse-traceless part, a traceless part determined by a three-vector, and a trace part proportional to the metric. This decomposition plays a key role in identifying the independent and dependent variables. (3) The scalar determining the trace part above may be regarded as a freely specifiable function that serves to label or identify spacelike surfaces. This is a useful realization of the idea of employing extrinsic or momentumlike functions as "time" variables.[4] The particular scalar we have used is denoted by $\tau = \frac{2}{3} g^{-1/2} \pi$. This measures the rate of contraction of local three-volume elements with respect to local proper time and may therefore be called the "Hubble function" of the slice. It has proven convenient to stipulate τ = constant (spatially constant) in previous work, but, as was pointed out,[5] this is not at all necessary. (4) The independent data are invariant with respect to conformal transformations $g_{ab} \rightarrow \bar{g}_{ab} = \phi^4 g_{ab}$, $\phi(x) > 0$. The decomposition of π^{ab} maintains all of its important features under such a deformation of structure. Moreover, $\phi(x)$ and the vector part of π^{ab} are the dependent data which act as generalized long-range potentials for the gravitational field.

In Sec. II the momentum constraints are analyzed. In Sec. III the behavior of initial data under conformal transformations is presented, including transformation of the stress-energy tensor $T^{\mu\nu}$. Section IV assembles the previous results and presents the general initial-value equations as a coupled, quasilinear elliptic system. Boundary conditions for isolated systems and physical interpretation of the initial data for asymptotically flat spacetimes is given in Sec. V. The definition of linear and angular momentum in terms of the dependent variables leads naturally to a unique extension of the standard flat-space symmetries (i.e., translations and rotations) to asymptotically flat spaces. This is discussed in Sec. VI. The final section, Sec. VII, describes the interpretation of initial data for spatially closed universes.

Certain simplifications occur in the special case of conformally flat metrics on the initial spacelike hypersurface. In this case, the method of producing part of the independent data by global decomposition of arbitrary symmetric tensors (Secs. II and III) may be replaced by a simpler, partially local, technique that is discussed in the Appendix.

In the present paper, we have emphasized the general formulation and physical interpretation of the initial-value problem. In subsequent works[6] we shall treat important technical aspects of the initial-value equations ("linearization stability") and formulate and interpret the initial-value problem when the sources of the gravitational field are specific massless fields (spins 0, $\frac{1}{2}$, and 1).[7]

II. MOMENTUM CONSTRAINTS

In terms of g_{ab} and $p^{ab} = g^{-1/2} \pi^{ab}$, the initial-value equations have the form

$$\nabla_b p^{ab} = 8\pi T^a_* = -8\pi S^a \ , \tag{2.1}$$

$$p_{ab} p^{ab} - \tfrac{1}{2} p^2 - R = -16\pi T^*_* \ . \tag{2.2}$$

The external sources are described by the local three-vector current $S^a = -T^a_* = -T^\mu_\nu u^\nu B^a_\mu$ and the local energy density $T^*_* = T^\mu_\nu u^\nu u_\mu$, where T^μ_ν is the stress-energy tensor, u^μ is the timelike unit normal of the initial surface, and B^a_ν is an integrable projection operator ($B^a_\nu u^\nu = 0$).

For any g_{ab}, the momentum tensor p^{ab} can be orthogonally and covariantly decomposed into the form[8]

$$p^{ab} = S^{ab} + (LW)^{ab} + \tfrac{1}{2} \tau g^{ab} \ , \tag{2.3}$$

where $\tau = \frac{2}{3} p$, $p = g_{ab} p^{ab}$. Here, S^{ab} is the transverse-traceless part of p^{ab} ($S^{ab} = p^{ab}_{\mathrm{TT}}$), and the longitudinal or vector part of p^{ab} is given by

$$(LW)^{ab} = \nabla^a W^b + \nabla^b W^a - \tfrac{2}{3} g^{ab} \nabla_c W^c \ . \tag{2.4}$$

Thus,

$$\pi^{ab} = \sigma^{ab} + \mu^{ab} + \tfrac{1}{2} g^{1/2} \tau g^{ab} \ ,$$

where $\sigma^{ab} = g^{1/2} S^{ab}$ and $\mu^{ab} = g^{1/2} (LW)^{ab}$. The momentum constraint (2.1) may now be written as three equations determining W^a, regarding τ and S^a as given:

$$\nabla_b (LW)^{ab} \equiv (\Delta_L W)^a = -\tfrac{1}{2} \nabla^a \tau - 8\pi S^a \ . \tag{2.5}$$

The vector "Laplacian" Δ_L is given by[8]

$$(\Delta_L W)^a = \Delta W^a + \tfrac{1}{3} \nabla^a (\nabla_b W^b) + R^a_b W^b \ , \tag{2.6}$$

where $\Delta = g^{ab} \nabla_a \nabla_b$ is the standard Laplacian.

From Eq. (2.5) we see that the transverse-traceless part of p^{ab} (i.e., S^{ab}) is not constrained by Eq. (2.1). The trace-free part of p^{ab} is essentially the shear of the field $u^\mu(x)$ generating the initial surface. Since $p^{ab} - \frac{1}{3} p g^{ab} = S^{ab} + (LW)^{ab}$ we see from (2.5) that the constrained part of the shear, $(LW)^{ab}$, has as its local sources both currents S^a and nonuniform contraction $\nabla^a \tau \neq 0$. Given τ and S^a, (2.5) is solved for W^b. Then p^{ab} is constructed by adding any transverse-traceless tensor S^{ab} to $[(LW)^{ab} + \frac{1}{2} \tau g^{ab}]$.

Transverse-traceless tensors may be constructed by a straightforward procedure.[5] Given any symmetric tensor T^{ab}, define its trace-free part $\Psi^{ab} = T^{ab} - \frac{1}{3} T g^{ab}$. Then one can put $S^{ab} = \Psi^{ab}_{\mathrm{TT}}$, where

$$\Psi^{ab}_{\mathrm{TT}} = \Psi^{ab} - (LV)^{ab} \tag{2.7}$$

and

$$\nabla_b (LV)^{ab} \equiv (\Delta_L V)^a = \nabla_b \Psi^{ab} \ . \tag{2.8}$$

Notice that the operator Δ_L is used twice: once to construct the longitudinal part of the solution of (2.1), and again to construct S^{ab}. These two procedures may be regarded as being independent because S^{ab} is not constrained by (2.1).

Although (2.5) and (2.8) both employ Δ_L, there is an important difference between them on closed manifolds. A global integrability condition can arise for (2.5) that does not affect (2.8). Before deducing this condition, let us note a few properties of Δ_L. It is a linear, second-order vector operator that has been shown to be both strongly elliptic and self-adjoint.[8] In closed manifolds, the only nonvanishing vectors it can map to zero are conformal Killing vectors, i.e., special symmetry vectors C^a satisfying $\mathcal{L}_C g_{ab} = \frac{2}{3} g_{ab} \nabla_d C^d$ or $(LC)^{ab} = 0$. This means that for an equation of the form $(\Delta_L Z)^a = \lambda^a$, we can solve for Z^a if and only if λ^a is globally orthogonal to C^a. Applying this to (2.5) gives

$$\tfrac{1}{2}\int g^{1/2} C^a \nabla_a \tau d^3x + 8\pi \int g^{1/2} C^a S_a d^3x = 0 \ . \quad (2.9)$$

The same fact applied to (2.8) gives

$$\int g^{1/2} C_a \nabla_b \Psi^{ab} d^3x = -\tfrac{1}{2}\int g^{1/2} (LC)_{ab} \Psi^{ab} d^3x = 0. \quad (2.10)$$

However, (2.10) is *identically satisfied* for all trace-free Ψ^{ab}'s since $(LC)_{ab} = 0$. Thus the existence of a conformal Killing vector leads to no restrictions in the construction of Ψ^{ab}_{TT}; a unique $(LV)^{ab}$ always exists.

Equation (2.9) constitutes, however, a global integrability condition on the permissible choices of τ and S^a on closed manifolds with conformal Killing vectors. Let us look a little more closely at the vacuum case, where we have $\nabla_b p^{ab} = 0$. It is not difficult to see that there must arise, in the case that C^a exists, such a restriction on the trace of p^{ab}. Multiplying $\nabla_b p^{ab}$ by an arbitrary ξ^a and integrating gives

$$0 = \int g^{1/2} \xi_a \nabla_b p^{ab} d^3x = -\tfrac{1}{2}\int g^{1/2} p^{ab} (K\xi)_{ab} d^3x \ , \quad (2.11)$$

since there is no boundary term in a closed manifold. We have put

$$(K\xi)_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \ . \quad (2.12)$$

For an arbitrary ξ^a, (2.11) simply expresses the orthogonality of divergence-free tensors and arbitrary "Killing forms" $(K\xi)_{ab}$. However, if we have a conformal Killing vector C^a [i.e., $(LC)^{ab} = 0$], since

$$(LC)^{ab} = (KC)^{ab} - \tfrac{2}{3} g^{ab} \nabla_d C^d = 0 \ ,$$

the orthogonality condition says that

$$\int g^{1/2} p \nabla_a C^a d^3x = -\int g^{1/2} C^a \nabla_a p \, d^3x = 0 \ . \quad (2.13)$$

Thus, on a closed manifold with conformal symmetries, the trace of a transverse tensor is not an arbitrary scalar function. A similar result holds when $S^a \neq 0$. This shows why there can be global integrability conditions on the functions on the right-hand side of (2.5) but not in (2.8). It is simply a consequence of Einstein's equations.

For asymptotically flat spaces, the above argument does *not* lead to global integrability conditions. This is because the neglected boundary term in (2.11) is nonvanishing even when we have an exact conformal symmetry C^a, because $C^a \neq 0$ at spacelike infinity. We shall point out in Sec. V that it is this very surface integral which defines the total linear and angular momenta of a gravitational field.[9] Therefore, global integrability conditions may be said to arise in closed worlds with exact conformal symmetries because the total momentum of a closed world must vanish. This is analogous to the fact that the total mass and electric charge of a closed world must also vanish. We shall see in Sec. III that equivalent global integrability conditions arise when the "Hamiltonian constraint" (2.2) is taken into account.

III. CONFORMAL TRANSFORMATIONS

In order to incorporate the Hamiltonian constraint (2.2), we introduce a strictly positive scalar function $\phi(x)$, which will be determined by (2.2) as we shall see in Sec. IV. This function will be treated as a "conformal factor." We subject the metric to a conformal transformation $g_{ab} \to \phi^4 g_{ab}$ which leaves the "conformal metric" $\tilde{g}_{ab} \equiv g^{-1/3} g_{ab}$ invariant. We wish to see how $\phi(x)$ may be taken into account in the analysis of the momentum constraints. In order to do this, we assume that

$$\bar{p}^{ab} = \bar{S}^{ab} + (\bar{L}W)^{ab} + \tfrac{1}{2}\tau \bar{g}^{ab} \quad (3.1)$$

satisfies (2.1) for a metric $\bar{g}_{ab} = \phi^4 g_{ab}$, where we regard g_{ab} as given and $\phi(x)$ is arbitrary (so far). The longitudinal part of $\bar{p}^{ab}$ is

$$(\bar{L}W)^{ab} \equiv \bar{\nabla}^a W^b + \bar{\nabla}^b W^a - \tfrac{2}{3}\bar{g}^{ab} \bar{\nabla}_c W^c \ . \quad (3.2)$$

The transverse-traceless part of $\bar{p}^{ab}$ is $\bar{S}^{ab}$. We construct $\bar{S}^{ab}$ independently of the other parts of $\bar{p}^{ab}$ according to the following prescription: For g_{ab} given, construct S^{ab} as described in Sec. II. Note that if we define

$$\bar{S}^{ab} \equiv \phi^{-10} S^{ab} \ , \quad (3.3)$$

then $\bar{S}^{ab}$ is transverse-traceless with respect to $\bar{g}_{ab}$ since S^{ab} is transverse-traceless with respect to g_{ab}. That is,

$$\bar{\nabla}_b \bar{S}^{ab} \equiv \phi^{-10} \nabla_b S^{ab} = 0 \ , \quad (3.4)$$

$$\bar{S} \equiv \bar{g}_{ab}\bar{S}^{ab} = \phi^{-6}S = 0. \tag{3.5}$$

Just as a conformal metric $\tilde{g}_{ab}$ is defined by $\tilde{g}_{ab} = g^{-1/3}g_{ab}$, so we may likewise define an explicitly conformally invariant form of S^{ab} by $\tilde{\sigma}^{ab} \equiv g^{1/3}\sigma^{ab} = g^{5/6}S^{ab}$. Then (3.4) and (3.5) can be written in a form totally independent of ϕ, i.e., $\tilde{\nabla}_b\tilde{\sigma}^{ab} = 0$ and $\tilde{g}_{ab}\tilde{\sigma}^{ab} = 0$. Here $\tilde{\nabla}_a$ refers to covariant differentiation with respect to the "conformal connection" $\tilde{\Gamma}^a_{bc}$, defined in terms of $\tilde{g}_{ab}$ and $\tilde{g}^{ab}(\tilde{g}_{ab}\tilde{g}^{bc} = \delta^c_a)$ just as Γ^a_{bc} is defined in terms of g_{ab}. These results hold for any $\phi(x) > 0$ and follow from the definition of covariant differentiation, with $\bar{g}_{ab} = \phi^4 g_{ab}$ and

$$\bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + 2\phi^{-1}(\delta^a_b \nabla_c\phi + \delta^a_c\nabla_b\phi - g_{bc}\nabla^a\phi) . \tag{3.6}$$

From (3.2) and (3.6) we find for any $\phi(x)$

$$(\bar{L}W)^{ab} = \phi^{-4}(LW)^{ab} , \tag{3.7}$$

$$\tfrac{1}{2}\tau\bar{g}^{ab} = \tfrac{1}{2}\phi^{-4}\tau g^{ab} . \tag{3.8}$$

Notice that we have not transformed τ and W^a. The stress-energy current is chosen to transform according to

$$S^a \to \bar{S}^a = \phi^{-10}S^a . \tag{3.9}$$

The reasoning behind the transformation (3.9) is explained in Sec. V.

Since $\bar{p}^{ab}$ satisfies by assumption

$$\bar{\nabla}_b\bar{p}^{ab} = -8\pi\bar{S}^a \tag{3.10}$$

and because $\bar{S}^{ab}$ is not constrained by (3.10), we may write the momentum constraint (2.1) in terms of unbarred variables. First we write

$$\begin{aligned}\bar{\nabla}_b\bar{p}^{ab} &= \bar{\nabla}_b(\bar{L}W)^{ab} + \tfrac{1}{2}\bar{\nabla}^a\tau \\ &= \bar{\nabla}_b(\bar{L}W)^{ab} + \tfrac{1}{2}\bar{g}^{ab}\partial_b\tau \\ &= -8\pi\bar{S}^a = -8\pi\phi^{-10}S^a .\end{aligned} \tag{3.11}$$

Multiplying through by ϕ^{10} and using (3.7), we obtain

$$\phi^6[\nabla_b(LW)^{ab} + 6(LW)^{ab}\nabla_b\ln\phi] = -\tfrac{1}{2}\phi^6 g^{ab}\partial_b\tau - 8\pi S^a \tag{3.12}$$

or

$$\nabla_b[\phi^6(LW)^{ab}] = -\tfrac{1}{2}\phi^6\nabla^a\tau - 8\pi S^a . \tag{3.13}$$

We may denote the operator on the left-hand side of (3.13) by $(\tilde{\Delta}_L W)^a$. For any $\phi(x) > 0$, $\tilde{\Delta}_L$ is strongly elliptic and self-adjoint, as can be readily verified. Its only "harmonic" functions on closed manifolds are, again, conformal Killing vectors of the given g_{ab}. This leads in vacuum to a global integrability condition on τ:

$$\int g^{1/2}\phi^6 C^a\nabla_a\tau\, d^3x = 0 , \tag{3.14}$$

if C^a is a conformal Killing vector. Moreover, for any $\phi(x)$, (3.14) is precisely equivalent to the global integrability condition found in Sec. II. This follows from the fact that if $\bar{g}_{ab} = \phi^4 g_{ab}$, then $\bar{g}^{1/2} = \phi^6 g^{1/2}$, $\bar{\nabla}_a\tau = \nabla_a\tau = \partial_a\tau$, and the (contravariant) conformal Killing vectors C^a of $\bar{g}_{ab}$ and g_{ab} are identical [cf. (3.7)]. Similar conclusions follow when $S^a \neq 0$. In the case of asymptotically flat initial data, no such global integrability condition arises.

In summary, suppose we are given g_{ab}, τ, and S^a. We construct S^{ab} as described in Sec. II. For any $\phi(x) > 0$, we solve

$$(\tilde{\Delta}_L W)^a = -\tfrac{1}{2}\phi^6\nabla^a\tau - 8\pi S^a \tag{3.15}$$

or, equivalently,

$$(\Delta_L W)^a + 6(LW)^{ab}\nabla_b\ln\phi = -\tfrac{1}{2}\nabla^a\tau - 8\pi\phi^{-6}S^a \tag{3.16}$$

for W^a. Then

$$\bar{p}^{ab} = \phi^{-10}\bar{S}^{ab} + \phi^{-4}[(LW)^{ab} + \tfrac{1}{2}g^{ab}\tau] \tag{3.17}$$

satisfies the momentum constraint and $\bar{S}^{ab} = \phi^{-10}S^{ab}$, $(\bar{L}W)^{ab} = \phi^{-4}(LW)^{ab}$, and $\tfrac{1}{2}\bar{g}^{ab}\tau = \tfrac{1}{2}\phi^{-4}g^{ab}\tau$ are, for any $\phi(x) > 0$, mutually orthogonal in the global inner product formed with respect to the $\bar{g}_{ab}$ metric.

IV. INITIAL-VALUE EQUATIONS

The conformal factor ϕ is determined in the following manner. In terms of the barred variables defined above, which are assumed to satisfy the constraints, (2.2) becomes

$$(\bar{g}_{ac}\bar{g}_{bd} - \tfrac{1}{2}\bar{g}_{ab}\bar{g}_{cd})\bar{p}^{ab}\bar{p}^{cd} - \bar{R} = -16\pi\,\bar{T}^*_* . \tag{4.1}$$

The scalar curvatures of $\bar{g}_{ab}$ and g_{ab} are related by

$$\bar{R} = R\phi^{-4} - 8\phi^{-5}\Delta\phi , \tag{4.2}$$

where $\Delta\phi \equiv \nabla^2\phi = g^{ab}\nabla_a\nabla_b\phi$ is the ordinary scalar Laplacian. The local proper energy density T^*_* is transformed according to

$$\bar{T}^*_* = \phi^{-8}T^*_* , \tag{4.3}$$

as will be discussed in Sec. V. Substituting (4.2), (4.3), and (3.17) into (4.1) gives the Hamiltonian constraint as a quasilinear elliptic equation determining ϕ:

$$\begin{aligned}-8\Delta\phi = -R\phi + M_{\mathrm{TT}}\phi^{-7} + 2M_{\mathrm{TL}}\phi^{-1} \\ + (M_{\mathrm{L}} - \tfrac{3}{8}\tau^2)\phi^5 + 16\pi T^*_*\phi^{-3} ,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}M_{\mathrm{TT}} &= g_{ac}g_{bd}S^{ab}S^{cd} , \\ M_{\mathrm{TL}} &= g_{ac}g_{bd}S^{ab}(LW)^{cd} , \\ M_{\mathrm{L}} &= g_{ac}g_{bd}(LW)^{ab}(LW)^{cd} .\end{aligned} \tag{4.5}$$

The combined operations of decomposition of π^{ab} and conformal mapping of g_{ab} and π^{ab} may be regarded as defining a certain mapping $\mathcal{G}$, whose properties are useful in discussing the initial-value equations. To define this mapping, let $\{g_{ij}\}\times\{\pi^{ij}\}$ be the collection of all Riemannian metrics and symmetric tensors. Let C be the subset of this collection that satisfies the constraints (2.1) and (2.2). Define $E=\{\tilde{g}_{ab}\}\times\{\tilde{\sigma}^{ab}\}\times\{\tau\}$. Then $\mathcal{G}$ is defined as the mapping $\mathcal{G}$: $C\to E$ corresponding to our reformulation of the Einstein constraints (2.1) and (2.2) as a quasilinear coupled elliptic system (3.13) and (4.4). One of the central issues in the initial-value problem is to determine the true space of independent data $D\subset E$, with D defined as the range of $\mathcal{G}$. The method of approach followed to resolve this question is to determine those elements of E that lead to unique solutions of (3.13) and (4.4). For these elements, (3.13) and (4.4) define a mapping $\mathcal{K}$: $D\to C$ and $\mathcal{K}=\mathcal{G}^{-1}$. This problem is treated in the following paper.[6]

The initial-value equations have now been written as four quasilinear coupled elliptic equations for the four unknown functions W^a and ϕ. If one chooses $\tau=$ constant and $S^a=0$, (3.13) implies $(LW)^{ab}=0$. Equation (4.4) is then simplified by the fact that $M_{\rm L}=M_{\rm TL}=0$. This case has been analyzed previously.[10] It was found that for "almost every" choice of independent data (g_{ab}, S^{ab}, T^*_*, and $\tau=$ constant), a solution ϕ exists. Whenever it exists, it is unique, except in the special case $T^*_*=S^{ab}=\tau=0$. In this case, (4.4) becomes linear and homogeneous in ϕ and any constant multiple of a given solution is also a solution. However, this special case of nonuniqueness only holds for *closed* vacuum manifolds. On asymptotically flat spaces, the boundary condition $\phi=1$ at infinity assures a unique solution.

In the most general case, (3.13) and (4.4) are coupled and we have not yet achieved a complete theory of the existence and uniqueness of solutions for all assignments of $\tau(x)$. However, in the following paper,[6] we analyze the case in which $\tau=$ constant $+\,\delta\tau(x)$ and $S^a=\delta S^a(x)$. We find that for small $\delta\tau(x)$ and $\delta S^a(x)$, a solution for W^a and ϕ always exists and is unique.

V. PHYSICAL INTERPRETATION AND BOUNDARY CONDITIONS FOR ISOLATED SYSTEMS

The conformally invariant, freely specified quantities $\tilde{g}_{ab}$, $\tilde{\sigma}^{ab}$, and τ constitute the independent data. We pointed out in the Introduction that τ may be regarded as an essentially kinematical variable serving to label spacelike hypersurfaces, either directly or implicitly. It has been argued elsewhere that $\tilde{g}_{ab}$ and $\tilde{\sigma}^{ab}$ characterize the "wave-like" or "pure spin-two" part of the gravitational field.[11] They form dynamical "coordinates" and conjugate momenta in an unconstrained Hamiltonian formalism for gravity, similar in spirit to the quantities denoted $g^{\rm TT}_{ij}$ and $\pi^{ij}_{\rm TT}$ in the work of Arnowitt, Deser, and Misner.[12] The canonical framework provides, as is well known, a convenient framework for the definition of gravitation radiation in terms of excitations of the dynamical variables. As has been previously shown,[11] the excitation of $\tilde{g}_{ab}$, its deviation from (conformal) flatness, is determined by computing the three-dimensional conformal curvature tensor $\tilde{\beta}^{ab}$.

In the case of nonvacuum problems, data characterizing currents and energy density must also be prescribed on the initial hypersurface. In this paper, we have regarded these quantities as possible vector and scalar point functions on the initial manifold, rather than, for example, building them up from other fields or from "fluid" models. We have found it convenient to prescribe only those features of S^a and T^*_* that are consistent with an *a priori* knowledge of the initial metric only up to a conformal factor.[13] To justify our transformations $\overline{T}^*_*=\phi^{-8}T^*_*$ and $\overline{S}^a=\phi^{-10}S^a$, we may appeal to dimensional considerations. In keeping with the desire that the decompositions of g_{ab} and π^{ab} be consistent with a canonical interpretation, we regard the unit of action to be fixed, as well as the velocity of light, $c=1$. In this case, inertial mass scales as reciprocal length and the dimensions of T^*_* are (length)$^{-4}$, which implies $T^*_*\to\phi^{-8}T^*_*$. Similarly, if S^a refers to the components of $\vec{S}$ in an arbitrary coordinate basis, then the physical components $S^{\hat{a}}$ (referred to a local orthonormal triad), scale as $\overline{S}^{\hat{a}}=\phi^{-8}S^{\hat{a}}$. This implies $\overline{S}^a=\phi^{-10}S^a$. Of course, arguments such as this one can only lead to an over-all power of ϕ and do not tell us whether the quantities should pick up inhomogeneous terms involving derivatives of ϕ. The simplest choice is the one we have used. Moreover, as is discussed in detail in a subsequent paper, these transformations are uniquely derived by considering the sources to arise from massless fields, which have well-known "good" conformal properties. In this case of massive fields, the problem has a different character and will be treated elsewhere. If the stress-energy tensor is phenomenological, then the local rest mass of the source is not fundamental and the use of the simplest transformation is indicated.

It is of considerable interest that the so-called dominant energy condition is preserved under our transformations.[14] This condition is of great importance in proving a number of the singularity theorems and is obeyed by all known forms of matter. It may be stated as follows: Let u^μ be the four-velocity of a local observer at rest with respect to the initial manifold, i.e., u^μ is the unit normal of the surface. The local four-momentum per unit proper three-volume T^μ can be written as $T^{\mu\nu}u_\nu$. The "dominant energy condition" asserts that T^μ is nonspacelike. Therefore,

$$T_\mu T^\mu = S^a S_a - (T^*_*)^2 \leq 0 \tag{5.1}$$

or

$$\frac{S_a S^a}{(T^*_*)^2} \leq 1 \ . \tag{5.2}$$

Our conformal transformations assert that $S^a \to \phi^{-10}S^a$, $S_a \to \phi^{-6}S_a$, $T^*_* \to \phi^{-8}T^*_*$. Therefore, (5.2) may be incorporated into the independent initial data and it is guaranteed that the resulting stress-energy tensor $T^{\mu\nu}$ will satisfy the dominant energy condition on the initial manifold.

The physical interpretation of the conformal factor ϕ goes back to Brill's work on gravitational waves at a moment-of-time symmetry.[15] Brill showed in that case that the asymptotically $O(r^{-1})$ part of ϕ contains the total gravitational mass of the system if the initial data are assumed to be asymptotically Schwarzschildian, i.e., asymptotically conformally flat and spherically symmetric. Only with this requirement can the total mass be physically well defined.

In recent work, this idea has been generalized by Geroch[16] and ourselves.[17] We have shown that the total energy E is given by

$$16\pi E = -8\oint_\infty \vec{\nabla}\phi \cdot d\vec{S} \ , \tag{5.3}$$

where it is assumed that the arbitrarily specified "base" metric g_{ab} asymptotically approaches a flat metric f_{ab} faster than $O(r^{-1})$. This assumption is justified by the requirement that the total energy E be finite. In that case, it was shown that $g_{ab} \to f_{ab} + O(r^{-(3/2+\epsilon)})$, $\epsilon > 0$. Applying Gauss's theorem to (5.3), we can express the total energy in terms of a specific volume integral by using (4.4):

$$\begin{aligned} 16\pi E &= -8\oint_\infty \vec{\nabla}\phi \cdot d\vec{S} \\ &= -8\int_v \sqrt{g}\ \Delta\phi\, d^3x \\ &= \int_v \sqrt{g}\ d^3x\,[16\pi T^*_*\phi^{-3} + M_{TT}\phi^{-7} + 2M_{TL}\phi^{-1} \\ &\qquad + (M_L - \tfrac{3}{8}\tau^2)\phi^5 - R\phi] \ . \end{aligned} \tag{5.4}$$

The finiteness of E also leads to the requirement that $T^*_* \to O(r^{-(3+\epsilon)})$, $\sigma^{ab} \to O(r^{-(3/2+\epsilon)})$.

The vector W^a is, like ϕ, basically a long-range potential whose dominant asymptotic term is $O(r^{-1})$. From (3.13), we require that $\tau \to O(r^{-(2+\epsilon)})$ and $S^a \to O(r^{-(3+\epsilon)})$. Not surprisingly, in our approach the asymptotically dominant terms of W^a determine the total linear and angular momentum of the gravitational field.[18] The total linear and angular momentum of a gravitational field can be defined as an integral at spacelike infinity over the gravitational momentum as[9]

$$P^{\hat{a}} = \oint_\infty \xi^{\hat{a}}_i \pi^{ij} d^2S_j, \quad \hat{a} = 1, \ldots, 6 \tag{5.5}$$

where $\xi^{\hat{a}}_i$ are the three translational and the three rotational Killing vectors at spacelike infinity. Now we know that

$$\pi^{ij} = \sigma^{ij} + g^{1/2}(LW)^{ij} + \tfrac{1}{2}g^{1/2}g^{ij}\tau \ ,$$

and if we assume that the independent variables (σ^{ij}, τ) are chosen to fall off fast enough, then (5.5) reduces to

$$P^{\hat{a}} = \oint_\infty g^{1/2}\xi^{\hat{a}}_i (LW)^{ij} d^2S_j \ . \tag{5.6}$$

This can be seen from that fact that an elementary canonical analysis[19] of the action principle of general relativity shows that the generator of spatial displacements is the expression[20]

$$-2\nabla_b[\phi^6(LW)^{ab}] \equiv 16\pi p^a \ , \tag{5.7}$$

whose value is determined by (3.13). We wish to pass via Gauss's theorem to a surface integral for total momentum. In order to do this in a physically and mathematically well-defined manner, one needs to take the inner product of (5.7) with a vector field ξ^a defined everywhere on the manifold. This vector field must have special properties in order that the final result express the equality of a surface integral and a volume integral over "sources." The source integral should not involve W^a. Multiplying by ξ_a and integrating, we find

$$\begin{aligned} 16\pi P_\xi &= 16\pi \int p^a \xi_a g^{1/2} d^3x \\ &= -2\int g^{1/2}\xi_a \nabla_b[\phi^6(LW)^{ab}]\, d^3x \ . \end{aligned} \tag{5.8}$$

From Gauss's theorem we find

$$\begin{aligned} 16\pi P_\xi &= -2\oint_\infty \phi^6\xi_a (LW)^{ab} g^{1/2} d^2S_b \\ &\quad + \int g^{1/2}\phi^6(L\xi)_{ab}(LW)^{ab} d^3x \ . \end{aligned} \tag{5.9}$$

We see that if ξ^a is an exact Killing vector, $(K\xi)^{ab} = 0$, or an exact conformal Killing vector, $(L\xi)^{ab} = 0$, then the second term on the right-hand side of (5.9) vanishes. Hence, using (3.13) and (5.9), in the case of ξ^a being an exact symmetry, we have the desired relation between total momentum and its sources. If ξ^a corresponds to a translational

Killing vector, then P_ξ represents a component of the total linear momentum. If ξ^a represents a rotational Killing vector, the P_ξ represents a component of total angular momentum. Of course, it is well known that in the presence of exact symmetries, exact "Gaussian" conservation laws can be established. The advantage of the present formalism is that *exact symmetries are not needed to establish the "surface integral equals volume integral over sources" result.* In other words, this formulation suggests a unique generalization of the entire flat three-space group of exact translations and rotations to asymptotically flat spaces corresponding to the gravitational fields of isolated systems possessing finite total energy and momentum.

VI. UNIQUE EXTENSION OF FLAT-SPACE SYMMETRIES TO ASYMPTOTICALLY FLAT SPACES; LINEAR AND ANGULAR MOMENTA

To see how this unique extension of the orthogonal group comes about, consider the second term on the right-hand side of (5.9) and note that its value, as well as the value of all the other integral expressions entering this analysis, may be computed in either the base metric g_{ab} or the final metric $\bar{g}_{ab} = \phi^4 g_{ab}$, with no change in value. Thus,

$$\int \phi^6 g^{1/2} (LW)_{ab} (L\xi)^{ab} d^3x = \int (\bar{g})^{1/2} \bar{g}_{ac} \bar{g}_{bd} (\bar{L}W)^{ab} (\bar{L}\xi)^{cd} d^3x \ . \quad (6.1)$$

We may again apply Gauss's theorem to obtain

$$\int (\bar{g})^{1/2} \bar{g}_{ac} \bar{g}_{bd} (\bar{L}W)^{ab} (\bar{L}\xi)^{cd} d^3x = 2 \oint_\infty \bar{g}_{ac} W^c (\bar{L}\xi)^{ab} (\bar{g})^{1/2} d\bar{S}_b - 2 \int (\bar{g})^{1/2} \bar{g}_{ac} W^c \bar{\nabla}_b (\bar{L}\xi)^{ab} d^3x \ . \quad (6.2)$$

We want both integrals on the right-hand side of (6.2) to vanish. Thus, we first require that ξ^a must satisfy everywhere on the manifold the equations

$$\bar{\nabla}_b (\bar{L}\xi)^{ab} \equiv (\bar{\Delta}_L \xi)^a = 0 \ . \quad (6.3)$$

These equations have only the solution $\xi^a = 0$ if as boundary condition we demand ξ^a goes to zero at infinity. Thus, on a topologically Euclidean manifold, $(\bar{\Delta}_L W)^a = 0 \Rightarrow W^a = 0$ since W^a vanishes at infinity. However, the natural boundary condition on ξ^a is that it becomes at infinity a flat-space translational or rotational Killing vector (expressed in any coordinate system that is being employed). In this case, ξ^a does not vanish at infinity and (6.3) can be solved uniquely with the stated boundary condition and asymptotic falloffs listed earlier in this section.

The proof of this assertion has its simplest form under the certain conditions which are analogous to standard potential-theory assumptions. Suppose we consider a finite system which has only been radiating gravitationally for a finite time. Since, as is well known, gravitational radiation cannot reach spatial infinity in this case, we assume that the conformal curvature $\tilde{\beta}^{ab}$ and $\tilde{\sigma}^{ab}$, as well as the "slicing variable" τ, have compact support. Outside a finite region, the only parts of the gravitational field that can be felt in this case are its long-range potentials ϕ and W^a. Hence, outside a finite domain the metric is conformally flat and possesses in this region exact conformal symmetries denoted by $\xi^a_{(0)}$, including those corresponding to Killing symmetries (translations and rotations) at infinity, where $\phi = 1$. Now put $\xi^a = \eta^a + \xi^a_{(0)}$ and substitute into (6.3) to obtain

$$(\bar{\Delta}_L \eta)^a = -(\bar{\Delta}_L \xi_{(0)})^a \ . \quad (6.4)$$

We wish to find η^a such that η^a goes to *zero* at infinity. Since the right-hand side of (6.4) is actually zero outside a bounded domain, we have a strongly elliptic linear system (similar to Poisson's equation) which always possesses, for each choice of $\xi^a_{(0)}$, a unique solution for η^a. Hence our ξ^a exists uniquely.

There are several further points of considerable interest. First, as expected, if $\xi^a_{(0)}$ is an exact symmetry everywhere on the entire manifold, then we find $\eta^a = 0$ and $\xi^a = \xi^a_{(0)}$. Second, we see that asymptotically $\eta^a \to 0 + O(r^{-1})$. Clearly the value of the $O(r^{-1})$ part will not influence the surface integral for P_ξ. However, the exact unique form of η^a is needed when P_ξ is equated to a volume integral over sources (see below). Finally we note that on *closed* manifolds, $(\bar{\Delta}_L \xi)^a = 0$ is completely equivalent to $(\bar{L}\xi)^{ab} = 0$, the condition for an *exact* symmetry, since there are no boundary terms in this case. This is analogous to the well-known fact[21] that on closed (Riemannian) manifolds, Killing's equations $(K\xi)^{ab} = 0$ are equivalent to the second-order equations $\nabla_b (K\xi)^{ab} = 0$.

Returning to (6.2), we see that the surface integral term also vanishes since $W^a \sim O(r^{-1})$ and $(\bar{L}\xi)^{ab} \sim O(r^{-2})$ at the slowest. Thus, if ξ^a is constructed as indicated, the total linear and angular momenta obey the following Gaussian theorem:

$$16\pi P_\xi = -2 \oint_\infty (\bar{g})^{1/2} \xi^a (\bar{L}W)^{bc} \bar{g}_{ab} d\bar{S}_c = 16\pi \int (\bar{g})^{1/2} \bar{S}^a \xi^b \bar{g}_{ab} d^3x + \int (\bar{g})^{1/2} \xi^a \partial_a \tau d^3x \ . \quad (6.5)$$

This expression may be readily rewritten in terms of the freely specified (unbarred) data and ϕ:

$$16\pi P_\xi = -2\oint_\infty \phi^6 \xi_a (LW)^{ab} (g)^{1/2}\, dS_b$$
$$= 16\pi \int (g)^{1/2} S_a \xi^a\, d^3x$$
$$+ \int (g)^{1/2} \phi^6 \xi^a \partial_a \tau\, d^3x\,. \qquad (6.6)$$

We have verified that these expressions transform in the expected manner if the "almost symmetry" frame is subjected to "displacement of origin" and "rotation of axes." Their correct behavior under asymptotically Lorentzian transformations (tilting the slice at infinity) can be inferred from their numerical equality with the surface integrals appearing in correct pseudotensor-type surface integrals. We plan to treat these matters in more detail elsewhere.

As a final observation, we can see why Eqs. (3.13) and (4.4) must in general be coupled in ϕ and W^a. This is simply because in relativistic mechanics the total energy and (linear) three-momentum are coupled by the well-known relation $E^2 = m_0^2 + P^2$. Of course, when $P=0$, $E=m_0$. Therefore, consider an asymptotically flat, nonsingular, topologically Euclidean vacuum solution of Einstein's equations. Moreover, suppose $\tau = 0$, a maximal slicing or "foliation" of the solution. Then we have vanishing total momentum. *In this case, the choice of a maximal foliation defines a global "rest frame" for a collection of singularity-free gravitational "waves."*

VII. CLOSED UNIVERSES

In closed universes, it is meaningless to speak of total mass, linear momentum, angular momentum, or electric charge. The latter quantities are defined by boundary integrals, and a closed world by definition has no boundaries. Yet it is still meaningful to speak of gravitational "waves" when a smoothed-out background can be defined.[22] It is also meaningful to speak of dynamical field "coordinates" and momenta, as exemplified, for example, in the "quantum cosmology" program.[23] We have discussed elsewhere[24] the relation of our variables $\tilde{g}_{ab}$ to the anisotropy variables and of the τ = constant foliations of closed worlds to the Ω-time foliations of Misner[23] and others.

In a closed world, our prescription does not allow an *a priori* specification of the total volume of the universe. This scale of the universe is found from the solutions of the initial value equations by computing

$$\text{volume} = \int \sqrt{\bar{g}}\, d^3x$$
$$= \int \phi^6 \sqrt{g}\, d^3x\,, \qquad (7.1)$$

where $\sqrt{g}$ is arbitrary. In this case, W^a does not define total momentum, but when $W^a \neq 0$ in a closed world, a "preferred" spatial direction is singled out. This arises from nonuniform expansion and/or currents of matter or other fields. Suppose we describe a closed world in a frame comoving with the objects of nonvanishing rest mass. Furthermore, assume that the spatial slices one obtains in this way have an average (i.e., volume) expansion that is at any epoch, independent of position in space. Then the only way W^a can arise is from existence of currents of massless objects such as photons or neutrinos. Such currents cause the anisotropy of the world to vary. Another cause of "dynamic" anisotropy is of purely gravitational origin through $\tilde{\sigma}^{ij}$. This transverse-traceless object determines, locally, a direction in space k^a, where $\tilde{\sigma}^{ab} k_b = 0$. Both $\tilde{\sigma}^{ab}$ and W^a result in "dynamic anisotropy" through

$$\mathcal{L}_u \tilde{g}_{ab} = 2\tilde{g}_{ac}\tilde{g}_{bd}\, g^{-1/2} (\tilde{\sigma}^{cd} + \tilde{\mu}^{cd})\,, \qquad (7.2)$$

where $\tilde{\mu}^{cd} = g^{5/6}(LW)^{cd}$. However, the effects of gravitational and nongravitational anisotropy changes can always be distinguished in the large because $\tilde{\sigma}^{ab}$ and $\tilde{\mu}^{ab}$ are globally orthogonal tensors.

APPENDIX: CONFORMALLY FLAT INITIAL DATA[25]

Suppose the arbitrarily specified "base" metric g_{ab} is flat: $g_{ab} = f_{ab}$. Then the construction of the transverse-traceless part S^{ab} of the momentum p^{ab} can be somewhat simplified since covariant derivatives commute on flat space. Thus, as is well known, the tensor

$$p^{ab} = f^{-1}\epsilon^{amn}\epsilon^{bij}\, \nabla_n \nabla_j A_{mi} \qquad (A1)$$

satisfies $\nabla_b p^{ab} = 0$ on flat space, where A_{mi} is any symmetric tensor and ϵ^{amn} denotes the unit permutation tensor of weight +1. The vanishing divergence of p^{ab} is not a conformally local attribute of p^{ab} unless the trace of p^{ab} also vanishes. This will not be true in general because from (A1) we have

$$p = g_{ab}p^{ab} = \nabla^2 A - \nabla^m \nabla^n A_{mn}\,. \qquad (A2)$$

Hence, (A1) is not useful in conjunction with the conformal mapping that is needed to simultaneously satisfy the Hamiltonian constraint. However, this defect can be easily remedied by defining a symmetric tensor B_{ab} as the trace-free part of A_{ab}:

$$B_{ab} = A_{ab} - \tfrac{1}{3} f_{ab} A\,, \qquad (A3)$$

where $A = f^{ab}A_{ab}$. Now in (A2) we may set $p=0$ to obtain

$$\nabla^2 A = \tfrac{3}{2}\nabla^m \nabla^n B_{mn} . \quad \text{(A4)}$$

Treating the traceless tensor B_{mn} as given, (A4) can be solved for A. Using this result in (A1) will produce a transverse-traceless tensor

$$S^{ab} = f^{-1}\epsilon^{amn}\epsilon^{bij}\nabla_n \nabla_j (B_{mi} + \tfrac{1}{3}Af_{mi}) . \quad \text{(A5)}$$

For any ϕ, we then find that $\bar{S}^{ab} = \phi^{-10}S^{ab}$ is transverse-traceless with respect to any conformally flat metric $\bar{g}_{ab} = \phi^4 f_{ab}$. One may now proceed to satisfy the constraints just as in the general case. In particular, for a maximal ($\tau = 0$) initial data set with no currents ($S^i = 0$) to be constructed on a Euclidean base metric, the complete solution is obtained by solving for the two scalars A and ϕ from (A4) and

$$-8\Delta\phi = M_{TT}\,\phi^{-7} + 16\pi\, T^*_*\,\phi^{-3} , \quad \text{(A6)}$$

which is (4.4) specialized to the present example. Equations (A4) and (A6) are two elliptic equations for scalar "potentials" A and ϕ with ordinary flat-space Laplacians. These equations are *not* coupled: Solve (A4), then (A6). An immediate consequence which has been previously deduced by less transparent methods is that the total mass of such a solution [cf. (5.4)] is manifestly positive.[26]

---

*Work supported in part by National Science Foundation Grant No. GP30799X to Princeton University.

†Present address.

[1]For background and a review of the status of the intial-value problem prior to the past few years, see the article by Y. Choquet-Bruhat, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962).

[2]Recent work leading to the present results and other references are contained in the following articles: J. W. York, Jr., Phys. Rev. Lett. 26, 1656 (1971); J. Math. Phys. 13, 125 (1972); Phys. Rev. Lett. 28, 1082 (1972); J. Math. Phys. 14, 456 (1973); N. Ó Murchadha and J. W. York, Jr., *ibid*. 14, 1551 (1973); Phys. Rev. D (to be published); J. W. York, Jr., Ann. Inst. Henri Poincaré (to be published).

[3]This may be regarded as the basic reason why "thin sandwich" formulations of the initial-value problem have not led to satisfactory results. The sandwich approach involves using as dependent data quantities whose only role is to describe the continuation of the spacetime coordinate system away from the initial surface (e.g., ${}^{(4)}g_{00}$ and ${}^{(4)}g_{0i}$). In our approach these quantities remain arbitrary.

[4]The advantages of "extrinsic time" have also been stressed by K. Kuchař, Phys. Rev. D 4, 955 (1971).

[5]York, Ref. 2, fourth paper.

[6]N. Ó Murchadha and J. W. York, Jr., following paper, Phys. Rev. D 10, 437 (1974).

[7]J. Isenberg, N. Ó Murchadha, and J. W. York, Jr., unpublished work.

[8]York, Ref. 2, fourth and seventh papers.

[9]See also York, Ref. 2, seventh paper.

[10]Ó Murchadha and York, Ref. 2, fifth paper.

[11]York, Ref. 2, first paper.

[12]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Ref. 1).

[13]Similar mappings of the stress-energy tensor, interpreted as generalized transformations of units in a context quite different from the present one, have been discussed by R. H. Dicke, Phys. Rev. 125, 2163 (1962).

[14]This condition is discussed by S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge, England, 1973), Chap. 4.

[15]D. R. Brill, Ann. Phys. (N.Y.) 7, 466 (1959).

[16]R. Geroch, J. Math. Phys. 13, 956 (1972).

[17]Ó Murchadha and York, Ref. 2, sixth paper.

[18]In Ref. 12, Arnowitt, Deser, and Misner were able to define the total linear momentum in terms of a vector. Their vector was not covariantly well defined, because it required in its definition the use of Cartesian-type coordinates and a flat "background" metric everywhere on the three-manifold.

[19]J. W. York, Jr., unpublished work.

[20]Again, this is similar in spirit to the analysis given by Arnowitt, Deser, and Misner (Ref. 12).

[21]K. Yano and S. Bochner, *Curvature and Betti Numbers* (Princeton Univ. Press, Princeton, 1953); R. A. Matzner, J. Math. Phys. 9, 1657 (1968).

[22]See, for example, R. A. Isaacson, Phys. Rev. 166, 1263 (1968); 166, 1272 (1968).

[23]See, for example, C. W. Misner, Phys. Rev. 186, 1319 (1969); 186, 1328 (1969).

[24]York, Ref. 2, third paper.

[25]This topic was treated by D. R. Brill, Nuovo Cimento Suppl. 2, 1 (1964). His treatment was amended and revised by J. W. York, cited as J. W. York, Jr. (1973b) by C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 536.

[26]R. Arnowitt, S. Deser, and C. W. Misner, Ann. Phys. (N.Y.) 11, 116 (1960).