[13]Notice that the fields A_μ and A'_μ have the same vacuums.

[14]This is no problem in practice. We are only interested in $q=0$, where δ functions cannot occur.

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Higher-order ϵ terms in Reggeon field theory*

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We calculate the $O(\epsilon^2)$ terms of the Wilson expansion of the critical exponents in the Reggeon field theory with a bare linear trajectory and a triple-Regge interaction. We find that the $O(\epsilon^2)$ and $O(\epsilon)$ terms are comparable at $\epsilon = 2$, and we obtain $\sigma_{\rm tot}(s)\underset{s\to\infty}{\simeq}(\ln s)^{0.38}$. We also show that the Gell-Mann–Low function $\beta(g)$ expanded to finite order in both ϵ and g carries no information about the existence of the Gell-Mann–Low zero at finite ϵ.

I. INTRODUCTION

The technique of using the renormalization group[1] and the Wilson ϵ expansion[2] to derive scaling properties of proper vertices in Reggeon field theory[3] was introduced by Migdal, Polyakov, and Ter-Martirosyan,[4] and by Abarbanel and Bronzan.[5,6] In their work the behavior of the proper vertices in the infrared limit $j\approx 1$ and $t\approx 0$ was examined, and a number of conclusions were reached. The most important of these was a prediction that in a theory with a linear unrenormalized Pomeron trajectory and a triple-Pomeron coupling, the asymptotic behavior of the elastic amplitude is

$$T(s,t)=s(\ln s)^{-\gamma}F(t(\ln s)^z)\ , \tag{1}$$

with $\gamma<0$. This behavior arises from the coincidence at $j=1$ and $t=0$ of an infinite number of branch points. The scaling exponent γ specifies the logarithmic rise of the total cross section,

$$\sigma_{\rm tot}\sim(\ln s)^{-\gamma}\ , \tag{2}$$

and the exponent z specifies the trajectories of Pomeron cuts and pole for small t,

$$\alpha(t)=1+{\rm const}\times(t)^{1/z}\ . \tag{3}$$

The exponents γ and z can be determined in an ϵ expansion, where $\epsilon=4-D$ is the difference between the natural scaling dimension (=4) and the number of transverse dimensions D; we want answers for $\epsilon=2$. Although ϵ is large, it was shown[4-6] that to order ϵ, $-\gamma=\frac{1}{12}\epsilon=\frac{1}{6}$, $z=1+\frac{1}{24}\epsilon=\frac{13}{12}$. If ϵ were always accompanied by a factor like $\frac{1}{12}$, a few terms in the ϵ expansion would give good results for γ and z. We have determined that[7]

$$\begin{aligned}-\gamma&=\tfrac{1}{12}\epsilon+\left(\tfrac{257}{12}\ln\tfrac{4}{3}+\tfrac{37}{24}\right)\left(\tfrac{1}{12}\epsilon\right)^2+O(\epsilon^3)\ ,\\ z&=1+\tfrac{1}{24}\epsilon+\left(\tfrac{155}{24}\ln\tfrac{4}{3}+\tfrac{79}{48}\right)\left(\tfrac{1}{12}\epsilon\right)^2+O(\epsilon^3)\ .\end{aligned} \tag{4}$$

Since the coefficients of the $(\frac{1}{12}\epsilon)^2$ terms are about 7.7 and 3.5, respectively, the $O(\epsilon^2)$ terms are larger than the $O(\epsilon)$ terms at $\epsilon=2$. It would therefore seem that the ϵ expansion is a questionable means of calculating γ and z at $\epsilon = 2$. Our results agree with those obtained independently by Baker.[8]

In Sec. II we review the Reggeon field theory and obtain the basic formulas from which γ and z can be calculated. In Sec. III we enumerate the required perturbation-theory graphs and obtain Eq. (4). Integrals are evaluated in the Appendix.

II. REGGEON FIELD THEORY AND THE RENORMALIZATION GROUP

We begin our discussion with a review of the Reggeon field theory.[6] We define a free Lagrangian

$$\mathcal{L}_0 = \tfrac{1}{2} i \psi^\dagger \frac{\overleftrightarrow{\partial}}{\partial t} \psi - \alpha_0' \nabla\psi^\dagger \cdot \nabla\psi - \Delta_0 \psi^\dagger \psi \ . \qquad (5)$$

Here $\psi = \psi(\vec{x}, t)$ is the unrenormalized Reggeon field, written as a function of $\vec{x}$, a D-dimensional space vector conjugate to the D-dimensional transverse momentum vector $\vec{k}$, and t, a variable conjugate to $E \equiv 1 - j$. The equation of motion corresponding to $\mathcal{L}_0$ yields

$$E = \alpha_0' \vec{k}^2 + \Delta_0 \ . \qquad (6)$$

Defining $\Delta_0 = 1 - \alpha_0$ as a bare "energy gap" then leads to the linear unrenormalized trajectory $(t = -\vec{k}^2)$

$$j = \alpha_0 + \alpha_0' t \ . \qquad (7)$$

We choose $\Delta_0 = 0$, corresponding to $\alpha_0 = 1$ for the Pomeron. No mass counterterm is required to keep $\alpha = 1$ in the presence of interactions within the ϵ expansion.

The interaction we choose is the triple-Pomeron coupling with nonzero bare coupling ir_0. The factor i is dictated by signature factors of the even-signature Pomeron.[3] It is sufficient to retain only the triple-Pomeron coupling because it induces higher couplings or proper vertices. According to Kogut and Wilson,[2] the scaling behavior (and exponents) is independent of the bare couplings we retain in the theory. In general, scaling is the same in our one-coupling theory (with a triple-Pomeron coupling) as it would be in a theory with other bare couplings in addition.

We write our full Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 - \tfrac{1}{2} i r_0 (\psi^\dagger \psi^2 + \text{H.c.}) \ . \qquad (8)$$

As in Ref. 6, we define dimensions for our theory by

$$[x] = k^{-1}, \quad [t] = E^{-1} \ , \qquad (9)$$

and

$$\left[\int d^D x \, dt \, \mathcal{L} \right] = 1 \ . \qquad (10)$$

We find

$$[\psi] = k^{D/2} \ ,$$
$$[\alpha_0'] = E k^{-2} \ ,$$
$$[\Delta_0] = E \ ,$$
and
$$[r_0] = E k^{-D/2} \ . \qquad (11)$$

The Green's functions for n incoming and m outgoing Reggeons are defined as

$$G^{(n,m)}(\vec{x}_i, t_i; \vec{y}_j, \tau_j) = \prod_{i=1}^{n} \prod_{j=1}^{m} \langle 0 | T \psi^\dagger(\vec{y}_j, \tau_j) \psi(\vec{x}_i, t_i) | 0 \rangle \ . \qquad (12)$$

The Fourier transform of the Green's function is defined by

$$\delta(\textstyle\sum E) \delta^D(\textstyle\sum k) G^{(n,m)}(E_i, \vec{k}_i) = \int \prod_{i=1}^{n} d^D x_i \, dt_i e^{i(E_i t_i - \vec{k}_i \cdot \vec{x}_i)} \prod_{j=1}^{m} d^D y_j \, d\tau_j \, e^{-i(E_j \tau_j - \vec{k}_j \cdot \vec{y}_j)} G^{(n,m)}(\vec{x}_i, t_i ; \vec{y}_j, \tau_j) \ . \qquad (13)$$

The δ functions conserve overall energy and momentum in the Green's functions. The Feynman rules for $G^{(n,m)}(E_i, \vec{k}_i)$ are the same as those listed in Ref. 6. They are:

(1) Draw all topologically distinct graphs with arrows indicating the direction of propagation.

(2) Integrate $d^D q \, dE_q$ around each loop.

(3) Supply a factor for each vertex: $r_0/(2\pi)^{(D+1)/2}$.

(4) Each Reggeon propagator is the retarded "nonrelativistic" expression

$$G_0^{(1,1)}(E, \vec{k}) = i/(E - \alpha_0' \vec{k}^2 - \Delta_0 + i\epsilon) \ . \qquad (14)$$

(5) Supply a factor $\frac{1}{2}$ for closed loops with Reggeon loops having momenta in the same direction.

The unrenormalized connected proper vertex functions $\Gamma^{(n,m)}$ are now defined by taking off the external legs of the connected part of $G^{(n,m)}$. We write

$$\Gamma^{(n,m)}(E_i, \vec{k}_i) = \prod_{l=1}^{n+m} [G^{(1,1)}(E_l, \vec{k}_l)]^{-1} G_c^{(n,m)}(E_i, \vec{k}_i) \ . \qquad (15)$$

The vertex functions $\Gamma^{(n,m)}$ also depend on the unrenormalized parameters α_0' and r_0. We shall use dimensional regularization to define the integrals, as in Ref. 6. The renormalized proper vertex functions $\Gamma_R(E_i, \vec{k}_i, \alpha', r, E_N)$ depend on the renormalized slope α', the renormalized coupling r, and a normalization point E_N. $E=-E_N$ is chosen as a point at which to define the coupling r and the slope α' through conditions on the appropriate vertex functions Γ_R. Normalization is imposed away from the perturbative singularities of the calculus, i.e., $E_N>0$. Hence r and α' are functions of E_N. A variation of E_N involves a finite renormalization and thus a change in r, in α', and in Γ_R. The connection between Γ_R and Γ is

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, \alpha', r, E_N) = Z^{(n+m)/2}\Gamma^{(n,m)}(E_i, \vec{k}_i, \alpha_0', r_0, \Delta_0) \;. \quad (16)$$

The wave-function renormalization Z is a function of α_0', r_0, and E_N.

The normalization conditions on Γ_R are, then,

$$\Gamma_R^{(1,1)}(E, \vec{k}^2)\Big|_{\substack{E=0\\ \vec{k}^2=0}} = 0 \;, \quad (17)$$

$$\frac{\partial}{\partial E}\, i\Gamma_R^{(1,1)}(E, \vec{k}^2)\Big|_{\substack{E=-E_N\\ \vec{k}^2=0}} = 1 \;, \quad (18)$$

$$\frac{\partial}{\partial \vec{k}^2}\, i\Gamma_R^{(1,1)}(E, \vec{k}^2)\Big|_{\substack{E=-E_N\\ \vec{k}^2=0}} = -\alpha'(E_N) \;, \quad (19)$$

$$\Gamma_R^{(1,2)}(E_i, \vec{k}_i)\Big|_{\substack{E_1=-E_N=2E_{2,3}\\ \vec{k}_i\cdot\vec{k}_j=0}} = \frac{r(E_N)}{(2\pi)^{(D+1)/2}} \;. \quad (20)$$

In the weak-coupling limit, $r\to r_0$, $\alpha'\to\alpha_0'$.

It is convenient to define dimensionless couplings $g_0(E_N)$ and $g(E_N)$ by

$$g_0(E_N) = \frac{r_0}{(\alpha_0')^{D/4}} E_N^{(D/4)-1} \;,$$
$$g(E_N) = \frac{r}{(\alpha')^{D/4}} E_N^{(D/4)-1} \;. \quad (21)$$

We should note at this point that we could have multiplied these definitions by an arbitrary function of D. This freedom will play a role in some of our considerations later.[7]

The renormalization-group equation for $\Gamma_R^{(n,m)}$ is obtained by noting that $\Gamma^{(n,m)}$ does not depend upon E_N, so that its derivative with respect to E_N is zero. Using Eq. (16) and the chain rule,

$$\left[E_N\frac{\partial}{\partial E_N} + \beta(g)\frac{\partial}{\partial g} + \zeta(\alpha', g)\frac{\partial}{\partial \alpha'} - \frac{(n+m)}{2}\gamma(g)\right] \times \Gamma_R^{(n,m)}(E_i, \vec{k}_i, g, \alpha', E_N) = 0 \;. \quad (22)$$

We have substituted in our dimensionless coupling g. Here, the coefficients in Eq. (22) are

$$\gamma(g) = E_N \frac{\partial}{\partial E_N} \ln Z(\alpha_0', r_0, E_N)\Big|_{\alpha_0', r_0 \text{ fixed}} \;, \quad (23)$$

$$\zeta(\alpha', g) = E_N \frac{\partial}{\partial E_N} \alpha'(E_N)\Big|_{\alpha_0', r_0 \text{ fixed}} \;, \quad (24)$$

$$\beta(g) = E_N \frac{\partial}{\partial E_N} g(E_N)\Big|_{\alpha_0', r_0 \text{ fixed}} \;. \quad (25)$$

As in Ref. 6 we now use the dimensional-analysis representation for $\Gamma_R^{(n,m)}$ which is defined by the statement that

$$[\Gamma_R^{(n,m)}] = Ek^{D[1-(n+m)/2]} \;. \quad (26)$$

It is

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, g, \alpha', E_N) = E_N\left(\frac{E_N}{\alpha'}\right)^{(2-n-m)D/4}\psi_{nm}\left(\frac{E_i}{E_N}, \frac{\alpha'}{E_N}\vec{k}_i\cdot\vec{k}_j, g\right) \;. \quad (27)$$

Using this we obtain the equation[1]

$$\left\{\xi\frac{\partial}{\partial\xi} - \beta(g)\frac{\partial}{\partial g} + [\alpha' - \zeta(\alpha', g)]\frac{\partial}{\partial\alpha'} + [\tfrac{1}{2}(n+m)\gamma(g) - 1]\right\}\Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, g, \alpha', E_N) = 0 \;. \quad (28)$$

Here, $\xi = e^t$ is a scaling parameter whose value we are at liberty to choose. It has been introduced in place of the explicit E_N dependence through the dimensional-analysis representation. [This t is not that in Eq. (9).]

The solution of Eq. (28) is then

$$\Gamma_R^{(n,m)}(\xi E_i, \vec{k}_i, g, \alpha', E_N) = \Gamma_R^{(n,m)}(E_i, \vec{k}_i, \bar{g}(-t), \bar{\alpha}'(-t), E_N) \times \exp\int_{-t}^{0} dt'[1 - \tfrac{1}{2}(n+m)\gamma(\bar{g}(t'))] \;, \quad (29)$$

where

$$\frac{d\bar{g}(t)}{dt} = -\beta(\bar{g}(t)) \;, \quad (30)$$

$$\frac{d\bar{\alpha}'(t)}{dt} = \bar{\alpha}'(t) - \zeta(\bar{\alpha}'(t), \bar{g}(t)) \;, \quad (31)$$

and

$$\bar{g}(0) = g \;, \quad (32)$$

$$\bar{\alpha}'(0) = \alpha' \;. \quad (33)$$

Scaling expressions for $\Gamma_R^{(n,m)}$ are obtained by examining the solution of Eq. (29) as $\xi\to 0$ or $t\to-\infty$. In this limit, $\bar{g}(-t)$ goes to g_1, the Gell-Mann–Low zero, where $\beta(g_1)=0$ and $\beta'(g_1)>0$. We find[6]

$$\Gamma_R^{(n,m)}(E_i, \vec{k}_i, g, \alpha', E_N) \sim C_\gamma E_N \left(\frac{E_N}{C_{\alpha'}\alpha'}\right)^{(2-m-n)D/4} \left(-\frac{E}{E_N}\right)^{1+z(g_1)(2-m-n)D/4-\gamma(g_1)(m+n)/2} \times \phi_{n,m}\left(\frac{E_i}{E}, \left(-\frac{E}{E_N}\right)^{-z(g_1)} \frac{\vec{k}_i\cdot\vec{k}_j}{E_N} C_{\alpha'}\alpha'\right) . \tag{34}$$

Here C_γ and $C_{\alpha'}$ are constants, E is any linear combination of the E_i's, and

$$z(g_1) = 1 - \left.\frac{\zeta(\alpha', g)}{\alpha'}\right|_{g=g_1} , \tag{35}$$

$z(g_1)$ and $\gamma(g_1)$ are the exponents z and γ in Eq. (4).

The key to the ϵ-expansion calculation of z and γ is that g_1^2 is $O(\epsilon)$. Therefore the perturbation expansion of β, γ, and ζ becomes the ϵ expansion. To obtain Eq. (4) we must calculate $\Gamma^{(1,1)}$ and $\Gamma^{(1,2)}$ to order g_0^4 and g_0^5, respectively. Using Eqs. (16)–(25) the renormalization can be carried out to get ζ/α' and γ to order g^4, and β to order g^5. From these expressions g_1^2, z, and γ are obtained to $O(\epsilon^2)$.

III. THE SCALING EXPONENTS TO ORDER ϵ^2

We begin by calculating $\Gamma^{(1,1)}$ to order g_0^4 and $\Gamma^{(1,2)}$ to order g_0^5.

The $O(g_0^4)$ contributions to $\Gamma^{(1,1)}$ are illustrated in Fig. 1. Using the Feynman rules of Sec. II and integrating over E_1 and E_2 by Cauchy's theorem, we obtain

$$\Gamma_a^{(1,1)}(-E_N, \vec{k}^2) = \frac{-ir_0^4}{2(2\pi)^{2D}} \int d^Dk_1 d^Dk_2 [E_N + \alpha_0'(\vec{k}-\vec{k}_1)^2 + \alpha_0'\vec{k}_1^2]^{-2} \times [E_N + \alpha_0'(\vec{k}-\vec{k}_1)^2 + \alpha_0'\vec{k}_2^2 + \alpha_0'(\vec{k}_1-\vec{k}_2)^2]^{-1} , \tag{36}$$

$$\Gamma_b^{(1,1)}(-E_N, \vec{k}^2) = \frac{-ir_0^4}{(2\pi)^{2D}} \int d^Dk_1 d^Dk_2 [E_N + \alpha_0'\vec{k}_1^2 + \alpha_0'(\vec{k}-\vec{k}_1)^2]^{-1} [E_N + \alpha_0'\vec{k}_2^2 + \alpha_0'(\vec{k}-\vec{k}_2)^2]^{-1} \times [E_N + \alpha_0'(\vec{k}-\vec{k}_2)^2 + \alpha_0'\vec{k}_1^2 + \alpha_0'(\vec{k}_2-\vec{k}_1)^2]^{-1} . \tag{37}$$

In the Appendix we evaluate these integrals and their derivatives with respect to $\vec{k}^2$, at $\vec{k}^2=0$. Of course, since we are calculating to order ϵ^2 we do not have to calculate the integrals exactly; we only need the terms proportional to $1/\epsilon^2$ and $1/\epsilon$. We find

$$-i\Gamma_a^{(1,1)}(-E_N, \vec{0}) = \frac{-r_0^4 E_N}{2(8\pi\alpha_0')^4}\left\{\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left[\tfrac{5}{2} - \gamma_{\rm EM} + \ln\left(\frac{16\pi\alpha_0'}{\sqrt{3}\,E_N}\right)\right]\right\} + O(\epsilon^0) , \tag{38}$$

$$-\frac{\partial}{\partial\vec{k}^2} i\Gamma_a^{(1,1)}(-E_N, \vec{k})\bigg|_{\vec{k}^2=0} = \frac{-3\alpha_0' r_0^4}{8(8\pi\alpha_0')^4}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\tfrac{31}{36} - \gamma_{\rm EM} + \ln\frac{16\pi\alpha_0'}{\sqrt{3}\,E_N}\right)\right] + O(\epsilon^0) , \tag{39}$$

$$-i\Gamma_b^{(1,1)}(-E_N, \vec{0}) = \frac{4r_0^4 E_N}{(8\pi\alpha_0')^4}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\tfrac{3}{2} - \gamma_{\rm EM} + \ln\frac{4\pi\alpha_0'\sqrt{3}}{E_N}\right)\right] + O(\epsilon^0) , \tag{40}$$

$$-\frac{\partial}{\partial\vec{k}^2} i\Gamma_b^{(1,1)}(-E_N, \vec{k})\bigg|_{\vec{k}^2=0} = \frac{2\alpha_0' r_0^4}{(8\pi\alpha_0')^4}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\tfrac{7}{12} - \gamma_{\rm EM} + \ln\frac{6\pi\alpha_0'}{E_N}\right)\right] + O(\epsilon^0) . \tag{41}$$

$\gamma_{\rm EM}$ is the Euler-Mascheroni constant.

The $O(g_0^5)$ contributions to $\Gamma^{(1,2)}$ are illustrated in Fig. 2. After integrating over the two-loop energies, all diagrams can be expressed in terms of five integrals. These are

$$J_1(a,b,c) = \int d^Dk_1 d^Dk_2 [aE_N + 2\alpha_0'\vec{k}_1^2]^{-1}[bE_N + 2\alpha_0'\vec{k}_2^2]^{-1}[cE_N + \alpha_0'\vec{k}_1^2 + \alpha_0'\vec{k}_2^2 + \alpha_0'(\vec{k}_1-\vec{k}_2)^2]^{-1} , \tag{42}$$

$$J_2(a,b,c) = \int d^Dk_1 d^Dk_2 [aE_N + 2\alpha_0'\vec{k}_1^2]^{-1}[bE_N + 2\alpha_0'\vec{k}_2^2]^{-1}[cE_N + \alpha_0'\vec{k}_1^2 + \alpha_0'\vec{k}_2^2 + \alpha_0'(\vec{k}_1-\vec{k}_2)^2]^{-2} = -\frac{1}{E_N}\frac{\partial J_1(a,b,c)}{\partial c} , \tag{43}$$

$$J_3(a,b) = \int d^Dk_1 d^Dk_2 [aE_N + 2\alpha_0'\vec{k}_1^2]^{-1}[bE_N + \alpha_0'\vec{k}_1^2 + \alpha_0'\vec{k}_2^2 + \alpha_0'(\vec{k}_1-\vec{k}_2)^2]^{-1} , \tag{44}$$

$$J_4(a, b) = \int d^Dk_1 d^Dk_2 [aE_N + 2\alpha_0' \vec{k}_1^2]^{-2} [bE_N + \alpha_0' \vec{k}_1^2 + \alpha_0' \vec{k}_2^2 + \alpha_0' (\vec{k}_1 - \vec{k}_2)^2]^{-1}$$

$$= -\frac{1}{E_N} \frac{\partial J_3(a, b)}{\partial a} , \tag{45}$$

$$J_5(a, b, c) = \int d^Dk_1 d^Dk_2 [aE_N + 2\alpha_0' \vec{k}_1^2]^{-2} [bE_N + 2\alpha_0' \vec{k}_1^2]^{-1} [cE_N + \alpha_0' \vec{k}_1^2 + \alpha_0' \vec{k}_2^2 + \alpha_0' (\vec{k}_1 - \vec{k}_2)^2]^{-1}$$

$$= -\frac{1}{E_N^2} \left[\frac{J_3(a, c) - J_3(b, c)}{(b-a)^2} + \frac{1}{b-a} \frac{\partial J_3(a, c)}{\partial a} \right] . \tag{46}$$

Define

$$R = \frac{r_0^5 (2\pi)^2}{(2\pi)^{5(D+1)/2}} . \tag{47}$$

Then at the normalization point $\vec{k}_i = 0$, $E_2 = E_3 = \frac{1}{2}E_1 = -\frac{1}{2}E_N$,

$$\Gamma_a^{(1,2)} = \frac{R}{E_N} [J_4(\tfrac{1}{2}, \tfrac{1}{2}) - J_4(1, 1)] , \tag{48}$$

$$\Gamma_b^{(1,2)} = \tfrac{1}{2} R\, J_5(1, \tfrac{1}{2}, 1) , \tag{49}$$

$$\Gamma_c^{(1,2)} = \tfrac{1}{2} R\, J_5(\tfrac{1}{2}, 1, \tfrac{1}{2}) , \tag{50}$$

$$\Gamma_d^{(1,2)} = \frac{2R}{E_N} [J_1(1, \tfrac{1}{2}, \tfrac{1}{2}) - J_1(1, 1, 1)] , \tag{51}$$

$$\Gamma_e^{(1,2)} = \frac{2R}{E_N} [J_1(1, \tfrac{1}{2}, 1) - J_1(1, 1, 1)] , \tag{52}$$

$$\Gamma_f^{(1,2)} = \frac{2R}{E_N} [J_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) - J_1(1, \tfrac{1}{2}, 1)] , \tag{53}$$

$$\Gamma_g^{(1,2)} = \frac{2R}{E_N} [J_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) - J_1(1, \tfrac{1}{2}, \tfrac{1}{2})] , \tag{54}$$

$$\Gamma_h^{(1,2)} = \frac{2R}{E_N} [J_1(\tfrac{1}{2}, \tfrac{1}{2}, 1) - J_1(1, \tfrac{1}{2}, 1)] , \tag{55}$$

$$\Gamma_i^{(1,2)} = -\frac{4R}{E_N^2} [J_3(\tfrac{1}{2}, 1) - J_3(1, 1) - J_3(\tfrac{1}{2}, \tfrac{1}{2}) + J_3(1, \tfrac{1}{2})] , \tag{56}$$

$$\Gamma_j^{(1,2)} = R[J_2(1, \tfrac{1}{2}, 1) + J_2(\tfrac{1}{2}, \tfrac{1}{2}, 1) + J_2(\tfrac{1}{2}, 1, 1)] , \tag{57}$$

$$\Gamma_k^{(1,2)} = \frac{2R}{E_N} [J_1(1, \tfrac{1}{2}, \tfrac{1}{2}) - J_1(1, \tfrac{1}{2}, 1)] . \tag{58}$$

The complete vertex function $\Gamma^{(1,2)}$ in fifth order is twice the sum of the above contributions, with the exception of $\Gamma_j^{(1,2)}$, which is counted once. It is

$$\Gamma^{(1,2)}|_{\text{fifth order}} = \frac{r_0}{(2\pi)^{(D+1)/2}} \frac{g_0^4}{(8\pi)^4} \left[\frac{20}{\epsilon^2} + \frac{1}{\epsilon} (26 - 20\gamma_{\text{EM}} + 20 \ln\pi - 6\ln 3 + 52 \ln 2) + O(\epsilon^0) \right] . \tag{59}$$

The above expressions must be augmented by the lower-order terms calculated in Ref. 6. The order r_0^2 terms in $\Gamma^{(1,1)}$ are

$$-i\Gamma^{(1,1)}(-E_N, \vec{k}^2) = E_N + \alpha_0' \vec{k}^2 + \frac{r_0^2}{2(2\pi)^D} \left(\frac{\pi}{2\alpha_0'} \right)^{D/2} \Gamma(1 - \tfrac{1}{2}D)(E_N + \tfrac{1}{2}\alpha_0' \vec{k}^2)^{(D/2)-1} . \tag{60}$$

The order r_0 and r_0^3 terms in $\Gamma^{(1,2)}$ are

$$\Gamma^{(1,2)}|_{\text{norm. pt.}} = \frac{r_0}{(2\pi)^{(D+1)/2}} \left[1 + \frac{r_0^2}{(2\pi)^{(D+1)}} \frac{8\pi}{E_N} \left(\frac{\pi}{2\alpha_0'} \right)^{(D/2)} \Gamma(1 - \tfrac{1}{2}D) E_N^{(D/2)-1} (1 - 2^{1-D/2}) \right] . \tag{61}$$

To be consistent with other expressions, we should expand the right-hand side of Eq. (60) and the bracket on the right-hand side of Eq. (61) in powers of ϵ, and retain the terms of $O(1/\epsilon)$ and $O(\epsilon^0)$ only.

We are now ready to renormalize and calculate the scaling exponents. From Eqs. (16) and (18),

$$Z^{-1}(r_0, \alpha_0', E_N) = -\frac{\partial}{\partial E_N} i\Gamma^{(1,1)}(-E_N, \vec{k}^2 = 0) . \tag{62}$$

From Eqs. (38), (40), and (60)

$$Z^{-1}(g_0, \alpha_0', E_N) = 1 + a_2 g_0^2/\epsilon + a_4 g_0^4/\epsilon^2$$
$$= 1 - \frac{g_0^2}{(8\pi)^2} \left[\frac{1}{\epsilon} + (\tfrac{3}{2}\ln 2 + \tfrac{1}{2}\ln\pi - \tfrac{1}{2}\gamma_{\text{EM}}) \right] + \frac{g_0^4}{(8\pi)^4} \left[\frac{7}{2\epsilon^2} + \frac{1}{\epsilon} (\tfrac{5}{4} + 6\ln 2 + \tfrac{9}{4}\ln 3 + \tfrac{7}{2}\ln\pi - \tfrac{7}{2}\gamma_{\text{EM}}) \right] . \tag{63}$$

The constants a_2 and a_4 will appear later, and can be read off Eq. (63).

Equations (16) and (19) show that

$$\frac{\alpha'(E_N)}{Z} = -\frac{\partial}{\partial \vec{k}^2}\, i\Gamma^{(1,1)}(-E_N, \vec{k}^2)\bigg|_{\vec{k}^2=0} . \tag{64}$$

We find

$$\begin{aligned}\frac{\alpha'(E_N)}{\alpha_0' Z} &= 1 + c_2\frac{g_0^{\,2}}{\epsilon} + c_4\frac{g_0^{\,4}}{\epsilon^2}\\ &= 1 - \frac{g_0^{\,2}}{(8\pi)^2}\left[\frac{1}{2\epsilon} + \tfrac{1}{4}(3\ln 2 + \ln\pi - \gamma_{\rm EM})\right]\\ &\quad + \frac{g_0^{\,4}}{(8\pi)^4}\left[\frac{13}{8\epsilon^2} + \frac{1}{\epsilon}\left(\tfrac{27}{32} + \tfrac{1}{2}\ln 2 + \tfrac{35}{16}\ln 3 + \tfrac{13}{8}\ln\pi - \tfrac{13}{8}\gamma_{\rm EM}\right)\right] .\end{aligned} \tag{65}$$

Finally, we obtain the renormalized coupling by evaluating

$$\frac{g(E_N)}{(2\pi)^{(D+1)/2}} = \frac{E_N^{\,(D/4)-1}}{[\alpha'(E_N)]^{D/4}}\,\Gamma^{(1,2)}\bigg|_{\rm norm.\ pt.} . \tag{66}$$

Using Eq. (63) for Z^{-1} and Eq. (65) for $\alpha'(E_N)$ we obtain, after some algebra,

$$g(E_N) = g_0\left(1 + \frac{w g_0^{\,2}}{\epsilon} + \frac{w_4 g_0^{\,4}}{\epsilon^2}\right) , \tag{67}$$

which inverts to give

$$g_0 = g - (w/\epsilon)g^3 + (3w^2 - w_4)g^5/\epsilon^2 . \tag{68}$$

Here,

$$w = \frac{1}{(8\pi)^2}\left[-3 + \epsilon\left(-\tfrac{15}{8} - \tfrac{5}{2}\ln 2 - \tfrac{3}{2}\ln\pi + \tfrac{3}{2}\gamma_{\rm EM}\right)\right] , \tag{69}$$

$$w_4 = \frac{1}{(8\pi)^4}\left[\tfrac{27}{2} + \epsilon\left(\tfrac{697}{32} + \tfrac{329}{8}\ln 2 - \tfrac{149}{16}\ln 3 + \tfrac{27}{2}\ln\pi - \tfrac{27}{2}\gamma_{\rm EM}\right)\right] . \tag{70}$$

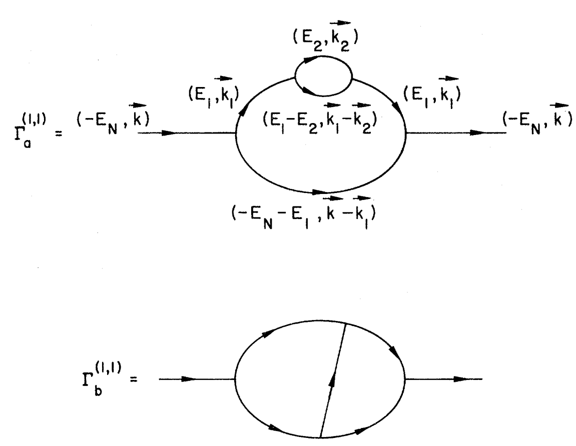

FIG. 1. The contributions to the unrenormalized self-energy $\Gamma^{(1,1)}$ in $O(g_0^{\,4})$.

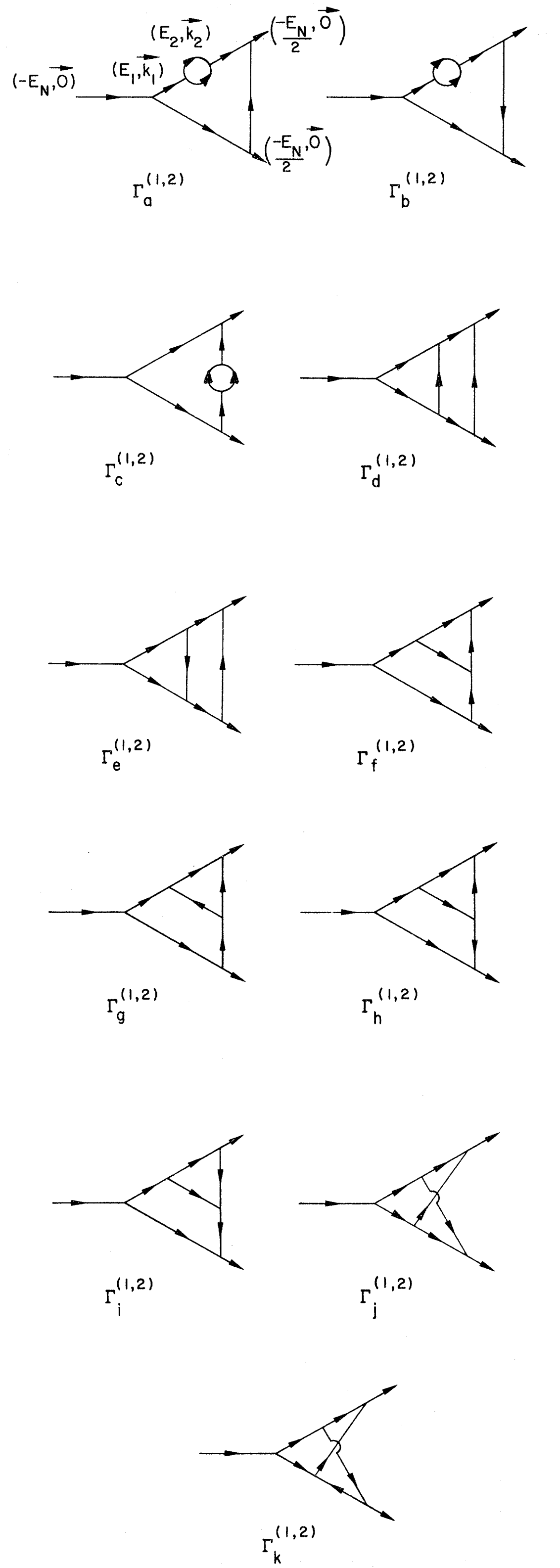

FIG. 2. The $O(g_0^{\,5})$ contributions to the unrenormalized vertex $\Gamma^{(1,2)}$.

The E_N dependence of all these quantities is hidden in g_0. We use the fact that

$$E_N \frac{\partial}{\partial E_N} g_0^{\,p} = -\tfrac{1}{4} p \epsilon g_0^{\,p} \ .$$

We obtain, using Eq. (66) and (67),

$$\beta(g) = E_N \frac{\partial g(E_N)}{\partial E_N}\bigg|_{\text{fixed } r_0, \alpha_0'}$$

$$= -\tfrac{1}{4}\epsilon g_0 \left(1 + 3w\frac{g_0^{\,2}}{\epsilon} + 5\frac{w_4 g_0^{\,4}}{\epsilon^2}\right) \tag{71}$$

$$= -\tfrac{1}{4}\epsilon g \left[1 + 2w\frac{g^2}{\epsilon} + (4w_4 - 6w^2)\frac{g^4}{\epsilon^2}\right] \tag{72}$$

$$= -\tfrac{1}{4}\epsilon g + \frac{g^3}{(8\pi)^2}\left[\tfrac{3}{2} + \epsilon\left(\tfrac{15}{16} + \tfrac{5}{4}\ln 2 + \tfrac{3}{4}\ln\pi - \tfrac{3}{4}\gamma_{\text{EM}}\right)\right]$$

$$- \frac{g^5}{(8\pi)^4}\left(\tfrac{157}{32} + \tfrac{149}{16}\ln\tfrac{4}{3}\right) . \tag{73}$$

We next evaluate γ by differentiating $\ln Z$ with respect to E_N. We obtain

$$\gamma = \tfrac{1}{2} a_2 g_0^{\,2} + (a_4 - \tfrac{1}{2} a_2^{\,2}) g_0^{\,4}/\epsilon \tag{74}$$

$$= \tfrac{1}{2} a_2 g^2 + (a_4 - \tfrac{1}{2} a_2^{\,2} - a_2 w) g^4/\epsilon \tag{75}$$

$$= \left[-\tfrac{1}{2} + \tfrac{1}{4}\epsilon(-3\ln 2 - \ln\pi + \gamma_{\text{EM}})\right]\frac{g^2}{(8\pi)^2}$$

$$+ \left[-\tfrac{5}{2}\ln 2 + \tfrac{9}{4}\ln 3 - \tfrac{5}{8}\right]\frac{g^4}{(8\pi)^4} \ . \tag{76}$$

Finally, we evaluate $\zeta/\alpha'(E_N)$ by differentiating $\ln\alpha'(E_N)$ at fixed α_0', r_0 with respect to E_N. We obtain

$$\frac{\zeta}{\alpha'} = \gamma - \tfrac{1}{2} c_2 g_0^{\,2} + (\tfrac{1}{2} c_2^{\,2} - c_4) g_0^{\,4}/\epsilon \tag{77}$$

$$= \gamma - \tfrac{1}{2} c_2 g^2 + (\tfrac{1}{2} c_2^{\,2} - c_4 + w c_2) g^4/\epsilon \tag{78}$$

$$= \tfrac{1}{2}\left[-\tfrac{1}{2} + \tfrac{1}{4}\epsilon(-3\ln 2 - \ln\pi + \gamma_{\text{EM}})\right]\frac{g^2}{(8\pi)^2}$$

$$+ \left(\tfrac{7}{8}\ln 2 + \tfrac{1}{16}\ln 3 - \tfrac{17}{32}\right)\frac{g^4}{(8\pi)^4} \ . \tag{79}$$

Equations (73), (76), and (79) contain the major results. We have succeeded in evaluating all of the functions appearing in the renormalization-group solution for the Green's functions in perturbation theory. At this point we should step back and notice that these functions have lost all singularities in ϵ, as they must. Second, at this point, the Euler-Mascheroni constant γ_{EM} occurs in each expression. This will eventually cancel out in the final results of the ϵ expansion, and is connected to an invariance of the theory under a rescaling of the dimensionless coupling g by an arbitrary function of D. We shall discuss this point more fully later on.

Proceeding to the final step of the calculation, we now evaluate the zero g_1 of the $\beta(g)$ function in an ϵ expansion. That is, we set

$$\beta(g_1) = 0 \ . \tag{80}$$

This is solved to $O(\epsilon^2)$ by

$$\frac{g_1^{\,2}}{(8\pi)^2} = \tfrac{1}{6}\epsilon + \tfrac{1}{12}\epsilon^2\left[\gamma_{\text{EM}} - \ln\pi\right.$$

$$\left. + \tfrac{1}{144}(356\ln 2 - 298\ln 3 - 23)\right] \ . \tag{81}$$

Inserting this expression into that for γ and ζ/α' leads finally to

$$-\gamma = \tfrac{1}{12}\epsilon + (\tfrac{1}{12}\epsilon)^2(\tfrac{257}{12}\ln\tfrac{4}{3} + \tfrac{37}{24}), \tag{82}$$

$$-\zeta/\alpha' = \tfrac{1}{24}\epsilon + (\tfrac{1}{12}\epsilon)^2(\tfrac{155}{24}\ln\tfrac{4}{3} + \tfrac{79}{48}) \ . \tag{83}$$

These are the final expressions we obtain. We note that the dependence on γ_{EM} has cancelled, and that the final expression for the $O(\epsilon^2)$ terms are relatively compact. Unfortunately, they are also rather large. At $\epsilon = 2$, corresponding to the real world, we obtain

$$-\gamma \cong \tfrac{1}{6} + \tfrac{1}{36}(7.7) = 0.38 \ , \tag{84}$$

$$-\zeta/\alpha' \cong \tfrac{1}{12} + \tfrac{1}{36}(3.5) = 0.18 \ . \tag{85}$$

Thus, the ϵ expansion seems at best a rather slowly convergent series. One might ask the question at this point of whether there might be a sensible alternative procedure to use in obtaining expressions for γ and ζ/α'. A quick look at the expression for $g_1^{\,2}$ in Eq. (81) shows that at $\epsilon = 2$ the $O(\epsilon^2)$ term is negative, and $g_1^{\,2} < 0$. Not only that, but returning to the expression for $\beta(g)$ in Eq. (73) one can imagine setting $\epsilon = 2$ therein and solving for g_1 directly. If one does this, one finds $g_1^{\,2} > 0$. In fact, if one now uses this value of g_1 in the expression for γ in Eq. (76), one obtains $-\gamma \approx \frac{1}{6}$ to within 10%. Unfortunately, this line of reasoning is incorrect. The demonstration involves an invariance of the theory under rescaling of the dimensionless coupling g by an arbitrary function $f(\epsilon)$. Such a rescaling does *not* leave the *finite order-*ϵ perturbative expression for $\beta(g)$ invariant, nor does it leave the resultant ϵ expansion for $g_1^{\,2}$ invariant. However, the ϵ expansions of γ and ζ/α' *are* invariant. To illustrate the point, consider the rescaling

$$g^2 = (8\pi)^{2-\epsilon/2} G^2/\Gamma(1 + \tfrac{1}{2}\epsilon) \tag{86}$$

$$= 8\pi G^2\left[1 + \tfrac{1}{2}\epsilon(\gamma_{\text{EM}} - \ln 8\pi) + O(\epsilon^2)\right] \ . \tag{87}$$

Defining

$$\beta_G(G) = E_N \frac{\partial G}{\partial E_N} \ ,$$

we obtain to $O(\epsilon G^3, G^5)$

$$\beta_G(G) = -\tfrac{1}{4}\epsilon G + \left[\tfrac{3}{2} + \epsilon\left(\tfrac{15}{16} - \ln 2\right)\right]\frac{G^3}{8\pi} - \left(\tfrac{157}{32} + \tfrac{149}{16}\ln\tfrac{4}{3}\right)\frac{G^5}{(8\pi)^2}\,. \quad (88)$$

The equation $\beta_G(G_1) = 0$ is solved to $O(\epsilon^2)$ by

$$\frac{G_1^2}{8\pi} = \frac{\epsilon}{6} + \frac{\epsilon^2}{(12)^3}(788\ln 2 - 298\ln 3 - 23)\,. \quad (89)$$

Now at $\epsilon = 2$, $G_1^2 > 0$, unlike the solution g_1^2, which was negative to $O(\epsilon^2)$. Furthermore, setting $\epsilon = 2$ in Eq. (88) results in 4 complex roots. We see that our ϵ-expanded β functions evaluated at $\epsilon = 2$ provide no insight into the existence or nonexistence of the Gell-Mann–Low zero. Since this discussion revolves around changing the $O(\epsilon g^3)$ coefficient of $\beta(g)$ through transformations like Eq. (87), this ambiguity does not occur in lower order, where β is needed only to $O(\epsilon^0 g^3)$ and where the existence of the zero with $\beta'(g_1) > 0$ is assured. We must *assume* that our $O(\epsilon^2)$ expansion of g_1^2 does not spoil the infrared stability of the theory found in $O(\epsilon)$. We cannot verify stability within the ϵ expansion.

We emphasize that Eqs. (82) and (83) are unchanged by the rescaling procedure. For better or worse, they are the scaling exponents to $O(\epsilon^2)$. In the above example we obtain to $O(\epsilon G^2, G^4)$

$$\gamma = -\frac{1}{2}\frac{G^2}{8\pi} + \left(-\frac{5}{2}\ln 2 + \frac{9}{4}\ln 3 - \frac{5}{8}\right)\frac{G^4}{(8\pi)^2}\,, \quad (90)$$

which becomes Eq. (82) upon insertion of Eq. (89).

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APPENDIX

In this appendix we shall calculate the integrals for the self-energy $\Gamma^{(1,1)}$ in $O(g_0^4)$ and the vertex function $\Gamma^{(1,2)}$ in $O(g_0^5)$ used in the text.

We begin with the integral for $\Gamma_a^{(1,1)}$ in Eq. (36). Introducing the Feynman parameter x, we get

$$\Gamma_a^{(1,1)} = -\frac{i r_0^4}{(2\pi)^{2D}}\int_0^1 x\,dx\int d^Dk_1 d^Dk_2[E_N + 2\alpha_0'\vec{k}_1^2 + 2\alpha_0'\vec{k}_2^2(1-x) + \alpha_0'\vec{k}^2 - 2\alpha_0'\vec{k}\cdot\vec{k}_1 - 2\alpha_0'\vec{k}_1\cdot\vec{k}_2(1-x)]^{-3}\,. \quad (A1)$$

Next, we use the following integral:

$$\int d^Dk_1 d^Dk_2(a\vec{k}_1^2 + b\vec{k}_2^2 + c\vec{k}_1\cdot\vec{k}_2 + d + e\vec{k}\cdot\vec{k}_1 + f\vec{k}\cdot\vec{k}_2)^{-\sigma} = (2\pi)^D\tilde{d}^{D-\sigma}\Gamma(\sigma - D)(4ab - c^2)^{-D/2}/\Gamma(\sigma)\,, \quad (A2)$$

where

$$\tilde{d} = d - \frac{\vec{k}^2}{4ab - c^2}(be^2 + af^2 - cef)\,. \quad (A3)$$

We obtain at $\vec{k}^2 = 0$ and $E = -E_N$

$$-i\Gamma_a^{(1,1)}|_{\vec{k}^2=0} = -\frac{r_0^4\Gamma(3-D)}{2(4\pi\alpha_0')^D}E_N^{D-3}\int_0^1 x\,dx[(1-x)(3+x)]^{-D/2}\,, \quad (A4)$$

$$-\frac{\partial}{\partial\vec{k}^2}i\Gamma_a^{(1,1)}\bigg|_{\vec{k}^2=0} = \frac{\alpha_0' r_0^4\Gamma(4-D)E_N^{D-4}}{2(4\pi\alpha_0')^D}\int_0^1 x\,dx\left(\frac{1+x}{3+x}\right)[(1-x)(3+x)]^{-D/2}\,. \quad (A5)$$

To evaluate the integrals in Eqs. (A4) and (A5) we integrate by parts, using the formula

$$\int_0^1 dx f(x)x^{\epsilon/2-2} = \frac{2}{\epsilon}f'(0) + f'(0) - f(1) - \int_0^1 f''(x)\ln x\,dx + O(\epsilon)\,. \quad (A6)$$

Letting $x \to 1 - x$ in formula (A4) we have

$$f(x) = (1-x)(4-x)^{\epsilon/2-2}\,. \quad (A7)$$

Using the formulas

$$\int_0^1 \frac{\ln x\,dx}{(4-x)^3} = \tfrac{1}{32}\left(\ln\tfrac{3}{4} - \tfrac{1}{3}\right)\,,\qquad \int_0^1 \frac{\ln x\,dx}{(4-x)^4} = \tfrac{1}{192}\left(\ln\tfrac{3}{4} - \tfrac{1}{3} - \tfrac{7}{18}\right) \quad (A8)$$

and expanding everything to $O(\epsilon^0)$, we obtain

$$-i\Gamma_a^{(1,1)}|_{\vec{k}^2=0} = -\frac{r_0^4 E_N}{32(4\pi\alpha_0')^4}\times\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\frac{5}{2} - \gamma_{\rm EM} + \ln\frac{16\pi\alpha_0'}{\sqrt{3}\,E_N}\right)\right]\,, \quad (A9)$$

which is Eq. (38).

To obtain the $\vec{k}^2$ derivative of $\Gamma_a^{(1,1)}$ we again use Eq. (A6) with

$$f(x)=(4-x)^{\epsilon/2-1}-5(4-x)^{\epsilon/2-2}+6(4-x)^{\epsilon/2-3}, \tag{A10}$$

and

$$\int_0^1 \frac{\ln x\, dx}{(4-x)^5}=\tfrac{1}{6144}\left(6\ln\tfrac{3}{4}-\tfrac{191}{27}\right). \tag{A11}$$

This then leads eventually to Eq. (39).

The other self-energy graph in fourth order is $\Gamma_b^{(1,1)}$ [see Eq. (37)]. For this graph we must introduce two Feynman parameters x and y. We obtain, after using Eq. (A2),

$$-i\Gamma_b^{(1,1)}=-\frac{r_0^4\Gamma(3-D)}{(4\pi\alpha_0')^D}\int_0^1 dx\int_0^{1-x}dy[3-2(x+y)-(x-y)^2]^{-D/2}\left[E_N+\frac{\alpha_0'\vec{k}^2(1-x^2-y^2)}{3-2(x+y)-(x-y)^2}\right]^{D-3}. \tag{A12}$$

We set

$$x=\tfrac{1}{2}(u+v),\quad y=\tfrac{1}{2}(u-v). \tag{A13}$$

To evaluate the integral at $\vec{k}^2=0$ we integrate by parts using

$$\int_v^1 du(3-2u-v^2)^{\epsilon/2-2}=\frac{1}{2-\epsilon}[(1-v^2)^{\epsilon/2-1}-(3-2v-v^2)^{\epsilon/2-1}]. \tag{A14}$$

Setting $v=1-w$ and using the formula

$$\int_0^1 f(w)w^{\epsilon/2-1}dw=\frac{2f(0)}{\epsilon}-\int_0^1 f'(w)\ln w\, dw+O(\epsilon), \tag{A15}$$

with

$$f(w)=(2-w)^{\epsilon/2-1}-(4-w)^{\epsilon/2-1}, \tag{A16}$$

then allows us to obtain $\Gamma_b^{(1,1)}$ to $O(\epsilon^0)$ as in Eq. (40).

The integral for $-(\partial/\partial\vec{k}^2)i\Gamma_b^{(1,1)}$ is the same as Eq. (A12) with an additional factor from differentiating. This becomes, after changing variables and integrating by parts,

$$-\frac{\partial}{\partial\vec{k}^2}i\Gamma_b^{(1,1)}\Big|_{\vec{k}^2=0}[\alpha_0' r_0^4 E_N^{D-4}\Gamma(4-D)/(4\pi\alpha_0')^D]^{-1}$$
$$=\int_0^1 dv\int_v^1 du(1-\tfrac{1}{2}u^2-\tfrac{1}{2}v^2)(3-2u-v^2)^{\epsilon/2-3} \tag{A17}$$
$$=\int_0^1\frac{dv(1-v)^{\epsilon/2-1}}{8(1-\frac{1}{2}\epsilon)(1-\frac{1}{4}\epsilon)}[(2-\tfrac{1}{2}\epsilon)(1+v)^{\epsilon/2-1}-(2-\epsilon)(1+v)(3+v)^{\epsilon/2-2}-v(3+v)^{\epsilon/2-1}]$$
$$-\frac{1}{16(1-\frac{1}{2}\epsilon)(1-\frac{1}{4}\epsilon)}\int_0^1 dv\ln\left(\frac{3+v}{1+v}\right). \tag{A18}$$

After some algebra, we obtain Eq. (41) in the text.

We see by Eqs. (42)–(46) that we need only evaluate the integrals J_1 and J_3. For J_1 we use Eq. (A2) after introducing parameters x and y. We obtain

$$J_1(a,b,c)=\left(\frac{\pi}{\alpha_0'}\right)^D\Gamma(3-D)E_N^{D-3}I_1(a,b,c), \tag{A19}$$

where

$$I_1(a,b,c)=\int_0^1 dx\int_0^{1-x}dy\frac{[c+(a-c)x+(b-c)y]^{D-3}}{[3-2(x+y)-(x-y)^2]^{D/2}}. \tag{A20}$$

J_3 is obtained similarly, except that only one Feynman parameter is needed. We get

$$J_3(a,b)=\left(\frac{\pi}{\alpha_0'}\right)^D\Gamma(2-D)E_N^{D-2}I_3(a,b), \tag{A21}$$

where

$$I_3(a,b)=\int_0^1 dx\frac{[a+x(b-a)]^{D-2}}{x^{D/2}(4-x)^{D/2}}. \tag{A22}$$

We now outline the procedure used in evaluating I_1 and I_3. We begin with I_1. We define

$$\alpha=\tfrac{1}{2}(a+b)-c,\quad \beta=\tfrac{1}{2}(a-b), \tag{A23}$$

and change to u, v variables (A13). We get

$$I_1=I_1^+(\alpha,\beta,c)+I_1^-(\alpha,\beta,c), \tag{A24}$$

where

$$I_1^+(\alpha, \beta, c) = \tfrac{1}{2}\int_0^1 dv \int_v^1 du(c+\alpha u+\beta v)^{1-\epsilon} \times (3-2u-v^2)^{\epsilon/2-2} \quad \text{(A25)}$$

and

$$I_1^-(\alpha, \beta, c) = I_1^+(\alpha, -\beta, c)\ . \quad \text{(A26)}$$

Now the singularities in ϵ in Eq. (A25) come from the vanishing of the second term in the integrand at $u=v=1$. Integrating by parts several times yields

$$I_1^+(\alpha, \beta, c) = \frac{1}{4(1-\frac{1}{2}\epsilon)}\int_0^1 dv\left\{h_{\alpha\beta c}{}^{1-\epsilon}w^{\epsilon/2-1} - k_{\alpha\beta c}{}^{1-\epsilon}z^{\epsilon/2-1} + \frac{\alpha}{\epsilon}(1-\epsilon)\left[h_{\alpha\beta c}{}^{-\epsilon}(-1+w^{\epsilon/2}) - k_{\alpha\beta c}{}^{-\epsilon}(-1+z^{\epsilon/2})\right]\right\}\ . \quad \text{(A27)}$$

$h_{\alpha\beta c}$ and $k_{\alpha\beta c}$ are equal to $(c+\alpha+\beta v)$ and $(c+\alpha v+\beta v)$, respectively. w and z are $(1-v^2)$ and $(3-2v-v^2)$, respectively.

We next expand in ϵ and extract the singularities in ϵ by integrating the v integrals by parts, using

$$\int_0^1 dvf(v)(1-v)^{\epsilon/2-1} = \frac{2}{\epsilon}f(1) + \int_0^1 dvf'(v)\ln(1-v) + O(\epsilon)\ . \quad \text{(A28)}$$

After some algebra, we obtain

$$I_1(a, b, c) = \frac{a+b}{8\epsilon} + \left[\tfrac{1}{8}(a+b)\right]\left(\tfrac{1}{2} - \ln 4 + \tfrac{3}{2}\ln 3\right) + \tfrac{1}{4}c\ln\tfrac{4}{3} - \tfrac{1}{8}(a\ln a + b\ln b) + O(\epsilon)\ . \quad \text{(A29)}$$

Using Eq. (A24) we finally get

$$J_1(a, b, c) = -E_N\left(\frac{\pi}{2\alpha_0'}\right)^4\left\{\frac{2(a+b)}{\epsilon^2} + \frac{1}{\epsilon}\left[(a+b)\left(3 - 2\gamma_{\text{EM}} + 2\ln\frac{\alpha_0'}{4\pi E_N} + 3\ln 3\right) + 4c\ln\tfrac{4}{3} - 2(a\ln a + b\ln b)\right]\right\} + O(\epsilon^0)\ . \quad \text{(A30)}$$

Next we turn to the evaluation of the integral $I_3(a, b)$ in Eq. (A22). The singularities in I_3 came from the factor $x^{\epsilon/2-2}$ near $x=0$. We integrate by parts, using Eq. (A6) with

$$f(x) = [a+x(b-a)]^{2-\epsilon}(4-x)^{\epsilon/2-2}\ . \quad \text{(A31)}$$

We also use Eq. (A8). After some algebra we get

$$I_3(a, b) = \frac{1}{4\epsilon}\left(ab - \tfrac{3}{4}a^2\right) + \tfrac{1}{12}b^2 + \tfrac{3}{32}a^2\left(1 + \ln\tfrac{3}{16}a^2\right) + \tfrac{1}{4}ab\left(-1 + \ln 4 - \tfrac{1}{2}\ln 3a^2\right)\ . \quad \text{(A32)}$$

Using Eq. (A21) we then obtain $J_3(a, b)$ as

$$J_3(a, b) = E_N{}^2\left(\frac{\pi}{\alpha_0'}\right)^4\left\{\frac{1}{\epsilon^2}\left(\tfrac{1}{8}ab - \tfrac{3}{32}a^2\right) + \frac{1}{\epsilon}\left[\left(\tfrac{1}{8}ab - \tfrac{3}{32}a^2\right)\left(-\gamma_{\text{EM}} + \ln\frac{\alpha_0'}{\pi E_N}\right) + \tfrac{1}{24}b^2 - \tfrac{3}{32}a^2\left(1 - \tfrac{1}{2}\ln\tfrac{3}{16}a^2\right) + \tfrac{1}{16}ab\left(1 - \ln\tfrac{3}{16}a^2\right)\right]\right\} + O(\epsilon^0)\ . \quad \text{(A33)}$$

This completes our evaluation of the needed integrals J_1 and J_3. J_2, J_4, and J_5 are then obtained by Eqs. (43), (45), and (46).

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