PHYSICAL REVIEW D VOLUME 10, NUMBER 12 15 DECEMBER 1974

Gauge invariance and mass

Richard A. Brandt*
Department of Physics, New York University, New York, New York 10003

Ng Wing-Chiu[†‡]
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794
(Received 19 August 1974)

The invariance of a theory involving a vector field $A_\mu(x)$ under local gauge transformations $A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$, etc., for all c-number functions $\Lambda(x)$ in some gauge group $\mathcal{G}$, does not imply that the theory contains a zero-mass gauge particle. It is shown that what *is* relevant to the existence of zero-mass excitations is not the *existence* of $\mathcal{G}$ but the *presence* in $\mathcal{G}$ of the simple gauge functions $\Lambda(x) = R(x) \equiv r \cdot x$, $r_\mu =$ constants, under which $A_\mu(x) \to A_\mu(x) + r_\mu$. If $R(x) \in \mathcal{G}$, then the *transverse* gauge particle propagator has a singularity at zero mass. This result and similar results for the other proper vertex functions are deduced by both structural and functional methods. In conventional Lorentz-gauge four-dimensional QED, $R \in \mathcal{G}$ and so the *physical* photon can be interpreted as a Goldstone boson arising from the spontaneous breakdown of the R-transformation invariance. In two-dimensional massless QED (Schwinger model), $R \notin \mathcal{G}$ and so there the photon can be (and is) massive. The point is further illustrated in other two-dimensional soluble models and four-dimensional perturbative models.

I. INTRODUCTION

At a formal level, there is a strong connection between the existence of a local gauge symmetry in a field theory and the existence of massless gauge particles in the theory. For example, the canonical mass term $M^2 : A_\mu A^\mu :$ is certainly not invariant with respect to the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)\,, \tag{1.1}$$

where $\Lambda(x)$ is a c-number function. But at a more precise level, because of typically quantum effects such as renormalization and vacuum polarization, there is no simple connection between mass parameters in a Lagrangian and the physical mass spectrum of the theory. In fact, as first pointed out by Schwinger,[1] a zero-mass pole in the transverse part

$$D(q^2) = \frac{1}{q^2 + q^2 \Pi(q^2)} \tag{1.2}$$

of the gauge particle propagator can be avoided if the invariant function $\Pi(q^2)$ in the vacuum-polarization tensor $\Pi_{\mu\nu}(q)$ (gauge particle self-energy tensor) has the singular behavior

$$\Pi(q^2) \sim -M^2/q^2 \tag{1.3}$$

for $q^2 \sim 0$, a behavior which is quite consistent with the usual gauge invariance which says nothing about $\Pi(q^2)$. Schwinger showed further that (1.3) is precisely what happens in two-dimensional massless quantum electrodynamics (QED). Such possibilities destroy any simple relation between gauge invariance and mass.

In this paper we shall describe and illustrate a new formulation of this point. We will show that what is relevant to the existence of zero-mass excitations is not the *existence* of a gauge symmetry group $\mathcal{G}$ but the *presence* in $\mathcal{G}$ of the simple gauge functions

$$R(x) \equiv r_\mu x^\mu \tag{1.4}$$

for constant r_μ, so that the corresponding field transformation

$$A_\mu(x) \to A_\mu(x) + r_\mu \tag{1.5}$$

is a (necessarily spontaneously broken) symmetry transformation. If R is in $\mathcal{G}$, then $q^2\Pi(q^2) \to 0$ for $q^2 \to 0$, so that $D(q^2)$ has a singularity at $q^2 = 0$. We shall deduce this result and similar results for the other proper functions by both structural and functional methods.[2]

This conclusion is nicely illustrated in QED. In the conventional (Gupta-Bleuler) four-dimensional perturbative formulation, $R \in \mathcal{G}$ and so the photon here is massless. In the exactly soluble two-dimensional formulation, $R \notin \mathcal{G}$ and so the photon can be (and is) massive. The point will be further illustrated in other two-dimensional soluble models and four-dimensional perturbative models.

The connection between the R transformation (1.5), or its scalar counterpart

$$\phi(x) \to \phi(x) + r\,, \quad r = \text{const} \tag{1.6}$$

and zero-mass excitations is of course well known from the early work on the Goldstone theorem. For example, the free massless scalar particle is the Goldstone boson corresponding to the spontaneous breakdown of (1.6) in the free-field theory ($\Box\phi = 0$).[3] The symmetry has also been used to formulate pion low-energy theorems,[4] and we have previously employed it as a means of obtaining exact Bjorken scaling in theories with reducible scale invariance.[5] What is new here is the observation that the presence or absence of R in $\mathcal{G}$ can be used as a simple criterion for the existence of the dynamical suppression of the zero-mass excitation naively expected in gauge-invariant theories, and hence as a tool to study the infrared structure of gauge theories.[6] The usefulness of this tool is that it is usually much easier to determine the symmetry group of a theory than the mass spectrum.

In a sense our work realizes the often expressed hope of describing the physical photon as a Goldstone particle. It is well known that (in the Lorentz gauge) the two unphysical photon modes are interpretable as the Goldstone bosons arising from the spontaneous breakdown of symmetry under the usual gauge transformations (1.1). Our conclusion is that the *physical* photon modes are interpretable as the Goldstone bosons arising from the spontaneous breakdown of symmetry under the R transformations (1.5). That is, the presence of (spontaneourly broken) R symmetry in ordinary QED guarantees the presence of a zero-mass excitation in the physical (transverse) part of the photon propagator.

In order to determine the symmetry group of a theory, it is important to study transformations on the fundamental independent fields in the theory. These are the fields which determine the Hilbert space and the observables of the theory. A symmetry transformation must transform these fields in such a way that the observables remain invariant. Thus, for example, it is *not* sufficient to find transformations on the fields in a Lagrangian $\mathcal{L}$ which leave $\mathcal{L}$ invariant if the fields mix. As a simple example, consider a free massive vector field A_μ and a free massless scalar field ϕ:

$$(\Box + m^2)A_\mu = 0\,, \quad \Box\phi = 0\,. \tag{1.7}$$

Invariance transformations are

$$\phi(x) \to \phi(x) + r\cdot x\,, \quad A_\mu(x) \to A_\mu(x)\,. \tag{1.8}$$

If one now defines the new vector field

$$V_\mu \equiv A_\mu + \partial_\mu \phi\,, \tag{1.9}$$

one has the field equation

$$(\Box + m^2)V_\mu = m^2 \partial_\mu \phi\,, \tag{1.10}$$

and (1.8) become

$$\phi(x) \to \phi(x) + r\cdot x\,, \quad V_\mu(x) \to V_\mu(x) + r_\mu\,; \tag{1.11}$$

and this seems to contradict our assertions because the transverse part of the V_μ propagator is just $(q^2 - m^2)^{-1}$, which is nonsingular at $q^2 = 0$. The point is that the Lagrangian expressed in terms of V_μ and ϕ contains V_μ-ϕ-mixing mass terms and so nothing can be concluded from (1.11). The fundamental (commuting) fields which determine the Hilbert space are A_μ and ϕ (the Hilbert space is here the simple tensor product of the Fock spaces of A_μ and ϕ), and only R transformations on them are relevant for our considerations.

The plan of the paper is as follows. After reviewing the Lorentz-gauge formalism in Sec. II, the Ward-Takahashi identities are described in Sec. III, where new derivations, which can be simply extended to R invariance, are given. The connection between $\Pi(q^2)$, $D(q^2)$, and the electric charge operator are then noted in Sec. IV. The zero-momentum theorems arising from R invariance, including $D^{-1}(0) = 0$, are derived in Sec. V. In Sec. VI our results are illustrated in some two-dimensional soluble models: the Schwinger model and a new simple model which we define and solve. The same is done in Sec. VII for four-dimensional perturbative models. The final Sec. VIII contains a short discussion of the possible usefulness of our results.

II. FORMALISM

We shall present our analysis in the framework of ordinary QED. The (renormalized) field equations are the Dirac equation

$$(i\partial\!\!\!/ - m)\psi(x) = -eh(x) \tag{2.1}$$

and the Maxwell equation

$$\partial^\nu F_{\mu\nu}(x) = ej_\mu(x)\,, \tag{2.2}$$

where the electromagnetic field strength is

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu\,, \tag{2.3}$$

in terms of the field potential A_μ, the spinor source is

$$h(x) = :A\!\!\!/(x)\psi(x):\,, \tag{2.4}$$

and the vector current is

$$j_\mu(x) = :\bar\psi(x)\gamma_\mu\psi(x):\,. \tag{2.5}$$

We do not now specify the normal products in (2.4) and (2.5) further except that (2.4) is assumed to be covariant and (2.5) invariant under the group $\mathcal{G}$ of gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x)\,, \tag{2.6a}$$

$$\psi(x) \to e^{ie\Lambda(x)}\psi(x)\,, \tag{2.6b}$$

$$\bar\psi(x) \to e^{-ie\Lambda(x)}\bar\psi(x)\,, \tag{2.6c}$$

so that $\mathcal{G}$ is a symmetry group of the field equations. For now we assume only that $\mathcal{G}$ contains all square-integrable infinitely differentiable functions $\partial_\mu \Lambda$. The subgroup corresponding to $\Lambda(x) = \text{const}$ has $j_\mu(x)$ as its Noether current, the conservation of which follows also from the structure of (2.2) and (2.3):

$$\partial^\mu j_\mu(x) = 0\,. \tag{2.7}$$

The theory will be quantized in the Lorentz gauge,

$$\partial^\mu A_\mu = 0 \ \text{ between physical states}\,, \tag{2.8}$$

in the manifestly Lorentz-covariant Gupta-Bleuler manner with

$$\langle 0 | A_\mu(x) | 0 \rangle = 0 \tag{2.9}$$

and

$$\langle 0 | j_\mu(x) | 0 \rangle = 0\,, \tag{2.10}$$

where $\langle 0 |$ is the vacuum state. Then $\mathcal{G}$ is further restricted by

$$\Box \Lambda(x) = 0\,. \tag{2.11}$$

We define the propagators[7]

$$G(p) = i\left(\frac{1}{2\pi}\right)^4 \int d^4x\, e^{ip\cdot x} \langle 0 | T[\psi(x)\bar{\psi}(0)] | 0 \rangle\,, \tag{2.12}$$

$$D_{\mu\nu}(q) = -\left(\frac{1}{2\pi}\right)^4 \int d^4x\, e^{iq\cdot x} \langle 0 | T[A_\mu(x) A_\nu(0)] | 0 \rangle \tag{2.13}$$

$$\equiv D(q^2) g_{\mu\nu} + E(q^2) q_\mu q_\nu\,; \tag{2.14}$$

we define the proper photon self-energy function $\Pi_{\mu\nu}$ by

$$e\left(\frac{1}{2\pi}\right)^4 \int d^4x\, e^{iq\cdot x} \langle 0 | T[j_\mu(x) A_\nu(0)] | 0 \rangle \equiv \Pi_{\mu\kappa}(q) D_\nu^\kappa(q)\,, \tag{2.15}$$

and the proper vertex part Γ_μ by

$$-\left(\frac{1}{2\pi}\right)^4 \int d^4x\, d^4y\, e^{iq\cdot x + ip\cdot y} \langle 0 | T[A_\mu(x)\psi(y)\bar{\psi}(0)] | 0 \rangle \equiv G(p)\Gamma^\nu(p,q) D_{\nu\mu}(q) G(p+q)\,. \tag{2.16}$$

The Maxwell equations (2.2), (2.3), and (2.8) immediately give

$$q^2 D_{\mu\nu}(q) = g_{\mu\nu} + \Pi_{\mu\kappa}(q) D_\nu^\kappa(q) \tag{2.17}$$

and

$$q^2 G(p)\Gamma^\nu(p,q) D_{\nu\mu}(q) G(p+q) = \left(\frac{1}{2\pi}\right)^4 \int d^4x\, d^4y\, e^{iq\cdot x + ip\cdot y} \langle 0 | T[j_\mu(x)\psi(y)\bar{\psi}(0)] | 0 \rangle\,. \tag{2.18}$$

The inhomogeneous term in (2.17) has been chosen by convention since equal-time commutation relations among the A_μ have not been assumed.[8,9] Iteration of (2.17), etc., leads to renormalized perturbative solutions to these equations in which $\Pi_{\mu\nu}$ and Γ_μ have their usual interpretation in terms of Feynman diagrams.[8,9]

III. GAUGE INVARIANCE

The content of the invariance of the theory under the gauge transformations (2.6) is contained in the Ward-Takahashi[10] (WT) identities obeyed by the Green's and proper functions. The simplest derivation[11] of these identities is based on the current conservation condition (2.7) and the equal-time commutation relations

$$[j_0(x), A_\nu(0)]\,\delta(x_0) = 0\,, \tag{3.1}$$

$$[j_0(x), \psi(0)]\,\delta(x_0) = \psi(0)\delta^4(x)\,, \tag{3.2}$$

which express the charge contents of the fields. Applied to (2.15), (2.7) and (3.1) give[12]

$$q^\mu \Pi_{\mu\nu}(q) = 0\,, \tag{3.3}$$

so that

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) \tag{3.4}$$

and

$$q^\nu \Pi_{\mu\nu}(q) = 0\,. \tag{3.5}$$

Thus, from (2.17),

$$q^\mu q^2 D_{\mu\nu}(q) = q_\nu\,, \tag{3.6}$$

so that, using (2.7) and (3.2), (2.18) gives

$$q_\nu \Gamma^\nu(p,q) = G^{-1}(p) - G^{-1}(p+q)\,. \tag{3.7}$$

Equations (3.3) and (3.7) are the simplest WT identities. All of the identities can be similarly derived.[10] We state only one more. The proper photon-photon amplitude $T_{\alpha_1\alpha_2\alpha_3\alpha_4}(q_1,q_2,q_3,q_4)$ $(\Sigma q_i = 0)$ satisfies

$$q_i^{\alpha_i} T_{\alpha_1\alpha_2\alpha_3\alpha_4}(q_1,q_2,q_3,q_4) = 0\,, \quad i = 1, \ldots, 4\,. \tag{3.8}$$

With the forms (2.14) and (3.4), (2.17) gives

$$q^2 D(q^2) = 1 - q^2\Pi(q^2) D(q^2)\,, \tag{3.9}$$

which is just (1.2), and

$$q^2E(q^2)=\Pi(q^2)D(q^2). \tag{3.10}$$

We now present an alternative derivation of (3.3) which can be used to deduce the consequences of R invariance as well. Under (2.6a), the photon propagator (2.13) undergoes the transformation

$$D_{\mu\nu}(q)\to D'_{\mu\nu}(q)\equiv D_{\mu\nu}(q)+q_\mu\hat{\Lambda}(q)\Lambda_\nu(0), \tag{3.11}$$

where

$$\hat{\Lambda}(q)\equiv i\left(\frac{1}{2\pi}\right)^4\int d^4x\, e^{iq\cdot x}\Lambda(x) \tag{3.12}$$

and

$$\Lambda_\nu(0)\equiv\partial_\nu\Lambda(x)|_{x=0}. \tag{3.13}$$

In the theory constructed from the transformed fields

$$A'_\mu=A_\mu+\partial_\mu\Lambda,\quad j'_\mu=j_\mu, \tag{3.14}$$

etc., we define the polarization tensor $\Pi'_{\mu\nu}(q)$ as in (2.15). Equations (2.10) and (3.14) then give the identity[13]

$$\Pi'_{\mu\kappa}(q)D'^{\kappa}_{\nu}(q)=\Pi_{\mu\kappa}(q)D^{\kappa}_{\nu}(q). \tag{3.15}$$

Similarly, consideration of $\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle$ gives the identity

$$\Pi_{\mu\nu}+\Pi_{\mu\kappa}D^{\kappa\lambda}\Pi_{\lambda\nu}=\Pi'_{\mu\nu}+\Pi'_{\mu\kappa}D'^{\kappa\lambda}\Pi'_{\lambda\nu}, \tag{3.16}$$

in which substitution of (3.15) gives

$$(g^\lambda_\mu+\Pi_{\mu\kappa}D^{\kappa\lambda})\Pi_{\lambda\nu}=(g^\lambda_\mu+\Pi_{\mu\kappa}D^{\kappa\lambda})\Pi'_{\lambda\nu}. \tag{3.17}$$

This implies that

$$\Pi_{\lambda\nu}(q)=\Pi'_{\lambda\nu}(q) \tag{3.18}$$

except perhaps at isolated points.[14] Equations (3.11), (3.15), and (3.18) thus imply the WT identity (3.3).

The WT identity (3.8) can be immediately derived in the same manner. It is, however, more difficult to derive (3.7) with this method. In the remainder of this section, we will deduce all of the WT identities in a concise and simple manner using functional techniques. The same method will afterwards be used to study R invariance.

The generating functional for Green's functions in quantum electrodynamics is defined by the functional integral

$$W[J_\mu,\xi,\bar{\xi}]=\int[dA][d\psi][d\bar{\psi}]\exp\left\{i\int d^4x\left[\mathcal{L}(x)+\frac{1}{2\alpha}(\partial_\mu A^\mu)^2-J_\mu(x)A^\mu(x)-\xi(x)\psi(x)-\bar{\xi}(x)\bar{\psi}(x)\right]\right\}, \tag{3.19}$$

where $\mathcal{L}(x)$ is the usual Dirac-Maxwell Lagrangian, and the usual gauge-fixing and source terms have been included. The gauge transformations (2.6) leave $\mathcal{L}(x)$ invariant, and so

$$\begin{aligned}W[J_\mu,\xi,\bar{\xi}]=\int[dA][d\psi][d\bar{\psi}]\exp\Big\{i\int d^4x\Big[&\mathcal{L}(x)+\frac{1}{2\alpha}(\partial_\mu A^\mu)^2+\frac{1}{\alpha}\partial_\mu A^\mu(x)\Box\Lambda(x)-J_\mu(x)A^\mu(x)\\&-J_\mu(x)\partial^\mu\Lambda(x)-e^{ie\Lambda(x)}\xi(x)\psi(x)-e^{-ie\Lambda(x)}\bar{\xi}(x)\bar{\psi}(x)\Big]\Big\},\end{aligned} \tag{3.20}$$

which implies that

$$0=\left[\frac{i}{\alpha}\Box\partial_\mu\frac{\delta}{\delta J_\mu(x)}-\partial_\mu J^\mu(x)-e\xi(x)\frac{\delta}{\delta\xi(x)}+e\bar{\xi}(x)\frac{\delta}{\delta\bar{\xi}(x)}\right]W[J_\mu,\xi,\bar{\xi}]. \tag{3.21}$$

We make the usual Legendre transformation to obtain the generating functional $\Gamma[\mathcal{A}_\mu,\Psi,\bar{\Psi}]$ of proper vertices:

$$\Gamma[\mathcal{A}_\mu,\Psi,\bar{\Psi}]=Z[J_\mu,\xi,\bar{\xi}]-\int d^4x[J_\mu(x)\mathcal{A}^\mu(x)+\xi(x)\Psi(x)+\bar{\xi}(x)\bar{\Psi}(x)],\quad iZ=\ln W, \tag{3.22}$$

where

$$\begin{aligned}&\mathcal{A}_\mu=\frac{\delta Z}{\delta J_\mu},\quad J_\mu=-\frac{\delta\Gamma}{\delta\mathcal{A}_\mu},\\&\Psi=\frac{\delta Z}{\delta\xi},\quad \xi=-\frac{\delta\Gamma}{\delta\Psi},\\&\bar{\Psi}=\frac{\delta Z}{\delta\bar{\xi}},\quad \bar{\xi}=-\frac{\delta\Gamma}{\delta\bar{\Psi}}.\end{aligned} \tag{3.23}$$

Then we obtain from (3.21)

$$\begin{aligned}0=&\frac{i}{\alpha}\Box\partial_\mu\mathcal{A}^\mu(x)+\partial_\mu\frac{\delta\Gamma}{\delta\mathcal{A}_\mu(x)}\\&+e\frac{\delta\Gamma}{\delta\Psi(x)}\Psi(x)-e\frac{\delta\Gamma}{\delta\bar{\Psi}(x)}\bar{\Psi}(x),\end{aligned} \tag{3.24}$$

which is the general form of the WT identities.

None of the above three methods of derivation is totally satisfactory from a mathematical point of view. It is, however, possible in QED to rigorously deduce all of the WT identities and prove their equivalence to the gauge invariance of the field equations by consistently using the renormalized Schwinger-Dyson equations.[8] In the following section we will rely on the above more formal methods, believing they can be substantiated by the more precise methods used in QED.

IV. THE PHOTON PROPAGATOR AT ZERO MASS

According to Eq. (1.2), the behavior of the invariant photon polarization amplitude $\Pi(q^2)$ for $q^2 \to 0$ determines the behavior of the invariant transverse photon propagator for $q^2 \to 0$. If $\Pi(0) = -1$, $D(q^2)$ is more singular than $1/q^2$. For $\Pi(0) = \text{const} \neq -1$, $D(q^2)$ has a simple pole at $q^2 = 0$. This happens in conventional perturbative QED where $\Pi(0) = 0$ is maintained so that $D(q^2) \to 1/q^2$. If $\Pi(0) = \infty$ but $q^2\Pi(q^2) \to 0$, then $D(q^2) \sim [q^2\Pi(q^2)]^{-1}$ is singular for $q^2 \to 0$, but less so than is a simple pole. A zero-mass "excitation" can then be said to exist, although the extent to which this can be identified with a zero-mass "particle" is not immediately clear. If $q^2\Pi(q^2) \to M^2 = \text{const}$, then $D(q^2) \to -1/M^2$, and so there is no contributing zero-mass excitation in the spectrum. This is the Schwinger mechanism (1.3). If (1.3) is valid also for q^2 near M^2, then $D(q^2) \sim (q^2 - M^2)^{-1}$ for $q^2 \sim M^2$ and so there is a mass M particle in the spectrum. Finally, if $q^2\Pi(q^2) \to \infty$ for $q^2 \to 0$, $D(q^2) \to 0$ and again there is no zero-mass excitation.

It is thus clear that gauge invariance is consistent with any behavior of $D(q^2)$ for $q^2 \to 0$. The physical photon can therefore not be thought of as a Goldstone boson of gauge invariance. Unless

$$D(q^2) \underset{q^2 \to 0}{\sim} 1/q^2 + O(q^2),$$

however, it follows from Eq. (2.10) that $E(q^2)$ is singular at $q^2 = 0$, and so at least the unphysical photons can be considered as Goldstone bosons of conventional gauge invariance.

Let us now note how the above considerations are related to the electric charge operator

$$Q \equiv e \int d^3x j_0(x). \tag{4.1}$$

Here e is the electric coupling constant defined, for example, by the normalization condition[15]

$$\bar{u}(p)\Gamma_\mu(p,q)u(p+q) = e\bar{u}(p)\gamma_\mu u(p+q), \quad q^2 = 0 \tag{4.2}$$

on the proper vertex function. In conventional QED, the charge serves both as a conserved quantum number ($\dot{Q} = 0$) and as a measure of the coupling strength of the photon to charged particles:

$$\langle p|Q|p'\rangle = e\delta(\vec{p} - \vec{p}\,')p_0 \quad \text{(conventional QED)}. \tag{4.3}$$

This dual role played by (4.1) can, however, break down. To see this, note that (2.14), (2.18), and (4.1) give in general

$$\langle p|Q|p'\rangle = e\delta(\vec{p} - \vec{p}\,')p_0(1-\delta), \tag{4.4}$$

where

$$1 - \delta \equiv \lim_{q^2 \to 0} q^2 D(q^2) \tag{4.5}$$

or

$$\delta = \lim_{q^2 \to 0} q^2\Pi(q^2)D(q^2). \tag{4.6}$$

In conventional QED $\delta = 0$; but if $\delta = 1$, then Q vanishes between electron states and one says that the electron charge is completely screened by vacuum polarization. By (3.9), if $\delta = 1$ then $D(q^2)$ is less singular than $1/q^2$ for $q^2 \to 0$. The conventional connection between the charge and the coupling constant is therefore lost if the transverse photon propagator has no zero-mass pole. This is actually expected by Gauss's theorem (charge $\neq 0$ $\Leftrightarrow 1/r$ potential $\Leftrightarrow$ mass-zero particle). We repeat that conventional gauge invariance precludes none of this.[16]

V. R TRANSFORMATIONS

We have seen in Sec. IV that (3.3) is the only consequence of gauge invariance on $\Pi_{\mu\nu}$ when the gauge group $\mathcal{G}$ consists only of smooth square-integrable functions $\Lambda(x)$. We will now show that more information is obtainable if $\mathcal{G}$ contains as well certain smooth functions which do not decrease for $x \to \infty$. Specifically, we consider the gauge function $\Lambda(x) = R(x) = r \cdot x$ in Eq. (1.4), so that $\partial_\mu \Lambda(x) = r_\mu$ and A_μ transforms as in (1.5). The Fourier-transformed field transforms as

$$\hat{A}_\mu(q) \to \hat{A}_\mu(q) + r_\mu \delta^4(q). \tag{5.1}$$

Because this $\Lambda(x)$ does not decrease at infinity, the Fourier transform exists only as a distribution. Note that the transformed field $A'_\mu(x) = A_\mu(x) + r_\mu$ continues to transform correctly under translations, and so leads to a translationally invariant theory with the same vacuum (zero energy-momentum eigenstate of the generator P_μ of translations, i.e., of the momentum operator).

Under (1.5), the photon propagator transforms as

$$D_{\mu\nu}(q) \to D'_{\mu\nu}(q) = D_{\mu\nu}(q) - r_\mu r_\nu \delta^4(q). \tag{5.2}$$

The assumed invariance of (2.15) again leads to the identity (3.15), where now $\Pi'_{\mu\kappa}$ is the polarization tensor appropriate to the field A'_μ. We again obtain (3.17), which tells us that $\Pi_{\mu\nu}$ and $\Pi'_{\mu\nu}$ can only differ at $q = 0$ by a distribution with support at $q = 0$. The conserved structures, however, forbid the existence of such a difference and so $\Pi_{\mu\nu}$

$=\Pi'_{\mu\nu}$ near $q=0$. Equation (3.15) thus implies that

$$\Pi_{\mu\nu}(q)\delta^4(q)=0\,, \tag{5.3}$$

or

$$\Pi_{\mu\nu}(0)=0\,. \tag{5.4}$$

From (3.4), this means that

$$\Pi(q^2)<1/q^2 \text{ for } q^2\to 0 \tag{5.5}$$

and hence that the transverse photon propagator (1.2) has a singularity at $q^2=0$:

$$D(0)=\infty\,. \tag{5.6}$$

We see that the presence of R in $\mathcal{G}$ implies the existence of a zero-mass excitation and so provides more information than gauge invariance alone. This result is of the Goldstone-theorem type (R symmetry is of course spontaneously broken), although there is here no conserved local current associated with R invariance. As already noted in the Introduction, it is the *physical* photon which here plays the role of the Goldstone boson.

There are a couple of subtle points in the above which we have glossed over which we would now like to return to. Using

$$\left(\frac{1}{2\pi}\right)^4\int d^4x\, e^{iq\cdot x}\partial_\mu(r\cdot x)=r_\mu\delta^4(q)$$
$$=-q_\mu(r\cdot\partial_q)\delta^4(q)\,, \tag{5.7}$$

(5.2) can also be written as

$$D'_{\mu\nu}(q)=D_{\mu\nu}(q)+q_\mu(r\cdot\partial_q)\delta^4(q)r_\nu\,. \tag{5.8}$$

In view of (3.3), the identity (3.15) now gives no further information. The point is that the identity (5.7) is only correct when applied to testing functions. The amplitudes $\Pi_{\mu\nu}(q)$ and $D_{\mu\nu}(q)$ are, however, not testing functions and so must be separately evaluated before multiplication in (3.15). That is, the associativity law $A(BC)=(AB)C$ fails for distributions. With this interpretation, (5.3), and not (3.3), is the correct conclusion. A related source of confusion is based on the observation that

$$(q_\mu q_\nu-q^2g_{\mu\nu})\delta^4(q)=0\,, \tag{5.9}$$

which, in view of (3.4), again suggests that (5.3) is trivially satisfied for any $\Pi(q^2)$. The resolution is again the fact that the distributions $\Pi_{\mu\nu}$ and $D_{\mu\nu}$ must be separately evaluated before multiplication. For example, if $\Pi(q^2)$ is $(q^2-i\epsilon)^{-1}$, (5.3) is not satisfied since

$$\left(\frac{1}{q^2}q^2\right)\delta^4(q)=\delta^4(q)\neq 0\,, \tag{5.10}$$

although the wrong interpretation would give

$$\frac{1}{q^2}[q^2\delta^4(q)]=0\,, \tag{5.11}$$

suggesting that (5.3) is satisfied. Equation (5.10) is the correct conclusion. A more picturesque way of phrasing these points is that, although (5.7) looks longitudinal, it really is not because it has support at $q=0$ where no direction in q space can be defined.

Similar arguments give

$$T_{\alpha_1\alpha_2\alpha_3\alpha_4}(q_1,q_2,q_3,q_4)|_{q_i=0}=0\,,\quad i=1-4 \tag{5.12}$$

and the similar vanishing of the n-photon proper amplitude when any external momentum vanishes. These low-energy theorems are among the consequences of R invariance, i.e., of the presence of R in $\mathcal{G}$. They are similar to the low-energy theorems previously deduced for the assumed-to-exist S matrix from R invariance in scalar field theories.[4] Our results are, however, valid off the mass shell and this is of extreme importance in massless theories where the existence of an S matrix is in question.

The further consequences of R invariance are most easily deduced by functional methods. The Dirac-Maxwell Lagrangian, together with the gauge-fixing term, is invariant under the R transformation (1.5) and the associated phase transformations

$$\begin{aligned}\psi&\to\psi e^{ier\cdot x}\,,\\ \bar\psi&\to\bar\psi e^{-ier\cdot x}\,.\end{aligned} \tag{5.13}$$

The derivation of the WT identity proceeds in the same way as before, except that the constant R transformation necessitates the integrated form

$$0=\int d^4x\left[-\frac{\delta\Gamma}{\delta\mathcal{A}_\mu(x)}-ex_\mu\frac{\delta\Gamma}{\delta\Psi(x)}\Psi(x)+ex_\mu\frac{\delta\Gamma}{\delta\bar\Psi}\bar\Psi(x)\right] \tag{5.14}$$

as the counterpart of (3.24). These relations are the consequences of R invariance.

From (5.14) we obtain by functional differentiation

$$0=\int d^4x\left[-\frac{\delta^3\Gamma}{\delta\mathcal{A}_\mu(x)\delta\Psi(y)\delta\bar\Psi(z)}\right]-ey^\mu\frac{\delta^2\Gamma}{\delta\Psi(y)\delta\bar\Psi(z)}$$
$$+ez^\mu\frac{\delta^2\Gamma}{\delta\Psi(y)\delta\bar\Psi(z)}\,, \tag{5.15}$$

which implies for the proper vertices in momentum space

$$\Gamma_\mu(0,p)=-\frac{\partial}{\partial p^\mu}G^{-1}(p)\,. \tag{5.16}$$

Equations (5.16) would follow from (3.7) if

$$\lim_{q\to 0}q_\lambda\frac{\partial}{\partial q_\mu}\Gamma^\lambda(q,p)=0\,. \tag{5.17}$$

R invariance thus implies the validity of (5.17).

From (5.14), we can perform functional differentiation with respect to $\mathcal{A}_\alpha$ any number of times, and we get

$$0=\int d^4y_1\cdots d^4y_m d^4x\,\frac{\delta^{(n+1)}\Gamma}{\delta\mathcal{A}_\mu(x)\delta\mathcal{A}_{\alpha_1}(y_1)\cdots\delta\mathcal{A}_{\alpha_n}(y_n)}\,,\qquad m<n\,. \quad (5.18)$$

This statement is translated in momentum space to be

$$0=\Gamma^{\mu\alpha_1\cdots\alpha_n}(q,p_1=0,\ldots,p_m=0;\,p_{m+1},\ldots,p_n)\,. \quad (5.19)$$

In particular, the inverse photon propagator $(D^{-1})_{\mu\nu}(q)$ vanishes as $q\to 0$. This was our previous conclusion (5.6).

In conclusion, let us note that one can try to generalize our analysis by considering the presence in $\mathcal{G}$ of

$$R_n(x)=r_{\alpha_1\cdots\alpha_n}x^{\alpha_1}\cdots x^{\alpha_n}\,. \quad (5.20)$$

This would lead naively, e.g., to the vanishing of $n-1$ derivatives of the photon amplitudes at zero four-momentum, even if $\Box R_n(x)=0$. This naive conclusion is actually not necessarily correct because of the loss of translational invariance caused by an R_n transformation. We omit the details in this paper and will study only the $R_1=R$ transformation.

VI. TWO-DIMENSIONAL SOLUBLE MODELS

Two-dimensional massless QED (Schwinger model[1]) is exactly soluble. It is gauge-invariant but has no physical zero-mass excitation. It therefore follows from our results in the previous section that R is not in the Schwinger gauge group $\mathcal{G}_S$. We will show this explicitly below. For clarity, we will first define and solve a different but related two-dimensional model which illustrates the same points in a simpler way.

As field equations, we take

$$\partial^\nu F_{\mu\nu}+m^2A_\mu=ek_\mu\,, \quad (6.1)$$

$$\partial\!\!\!/\chi=0\,, \quad (6.2)$$

where

$$F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu \quad (6.3)$$

and

$$k_\mu=:\bar\chi\gamma_\mu\chi: \quad (6.4)$$

is the free-field Wick product. The spinor χ satisfies the massless free Dirac equation (6.2) and we can append the free-field anticommutation relations

$$\{\chi(x),\bar\chi(0)\}=-i\gamma\cdot x\,\delta(x^2)\,. \quad (6.5)$$

The conserved current (6.4) is therefore well defined and is taken as the source of the massive vector field A_μ. (The field equations do not follow from a Lagrangian.) We will further impose the divergence condition

$$\partial^\mu A_\mu=0\,, \quad (6.6)$$

so that (6.1) becomes

$$(\Box+m^2)A_\mu=ek_\mu\,, \quad (6.7)$$

although this is not necessary. Equations (6.1)–(6.6) now define our model.

We will use the Dirac matrices[17–19]

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \quad (6.8)$$

and also

$$\gamma^5=\gamma^0\gamma^1=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad \epsilon_{\mu\nu}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \quad (6.9)$$

which are related by

$$\gamma_\mu\gamma_5=\epsilon_{\mu\nu}\gamma^\nu\,. \quad (6.10)$$

Some useful relations are

$$\epsilon_{\mu\nu}\epsilon^{\nu\kappa}=g_\mu^{\ \kappa}\,, \quad (6.11)$$

$$\mathrm{tr}\gamma_\mu\gamma_\nu=2g_{\mu\nu}\,,\quad \mathrm{tr}\gamma_5\gamma_\mu\gamma_\nu=2\epsilon_{\mu\nu}\,. \quad (6.12)$$

Before solving the model, we will show that the field equations are invariant under the local gauge transformations

$$A_\mu(x)\to A_\mu(x)+\partial_\mu\Lambda(x)\,, \quad (6.13)$$

$$\chi(x)\to\exp\{+(i\pi m^2/e)[\Lambda(x)-\gamma_5\lambda(x)]\}\chi(x)\,, \quad (6.14)$$

where $\Lambda(x)$ is any c-number function satisfying

$$\Box\Lambda(x)=0 \quad (6.15)$$

and such that

$$\lambda(x)\equiv\int_{-\infty}^{x_1}dx_1'\,\partial_0\Lambda(x_0,x_1') \quad (6.16)$$

exists. Such functions constitute the gauge group $\mathcal{G}$ of the model. The key property of (6.16) is

$$\partial_\mu\lambda(x)=\epsilon_{\mu\nu}\partial^\nu\Lambda(x)\,. \quad (6.17)$$

The invariance of the free Dirac equation (6.2) under (6.14) follows immediately from (6.17) and (6.10). We note next that under (6.14) the current (6.4) transforms as

$$k_\mu(x)\to k_\mu(x)+\frac{m^2}{e}\partial_\mu\Lambda(x)\,. \quad (6.18)$$

This follows from the explicit form

$$G(\xi)=\frac{\gamma\cdot\xi}{2\pi i\xi^2} \tag{6.19}$$

of the free Green's function, the trace expressions (6.12), and the relation

$$\xi_\mu\xi_\nu-\epsilon_{\mu\kappa}\xi^\kappa\epsilon_{\nu\lambda}\xi^\lambda=\xi^2 g_{\mu\nu}\,. \tag{6.20}$$

The invariance of (6.1) is then an immediate consequence of (6.13) and (6.18).

At this point, everything is in conformity with our expectations. We have a massive vector meson and a local gauge-invariance group $\mathcal{G}$ given by (6.13) and (6.14), but clearly $R\notin\mathcal{G}$ since (6.16) does not exist for $\Lambda(x)=r\cdot x$, $\partial_0\Lambda(x)=r_0\neq 0$. Actually, the analysis is not yet complete for two reasons: (i) The mass spectrum has not yet been determined and so a massless excitation is not yet definitely excluded, and (ii) A_μ and χ are not necessarily independent fields (no appropriate commutation relation has been specified), and so it is not yet clear that the transformations (6.13) and (6.14) can be consistently implemented by symmetry transformations on fundamental independent fields. We will now affirmatively resolve these points by explicitly solving the model.

An operator solution to (6.1)–(6.6) is

$$A_\mu=a\epsilon_{\mu\nu}\partial^\nu\sigma+\frac{e}{m^2}k_\mu\,, \tag{6.21}$$

where $\sigma(x)$ is a mass-m scalar free field,

$$(\Box+m^2)\sigma=0\,, \tag{6.22}$$

and a is an arbitrary constant. That (6.21) solves (6.6) and (6.7) is an immediate consequence of $\partial^\mu k_\mu=0$ and $\Box k_\mu=0$. The mass spectrum is now clear. The complete Hilbert space is the direct product $\mathcal{H}\equiv\mathcal{H}_\sigma\otimes\mathcal{H}_\chi$ of the free-field Fock spaces, and the particle spectrum consists of a free mass-m scalar and a free massless spinor. The gauge group of the model is also clear. It is the one specified above since the transformations (6.13) and (6.14) are induced by the transformations

$$\sigma(x)\to\sigma(x)\,, \tag{6.23}$$

$$\chi(x)\to\exp\{+(i\pi m^2/e)[\Lambda(x)-\gamma_5\lambda(x)]\}\,\chi(x) \tag{6.24}$$

on the fundamental independent fields, as follows from (6.18). The picture is thus confirmed. We have local gauge invariance, no massless photon excitation, and R not in the gauge group, consistently with Sec. V.

The operator field equations for two-dimensional QED have the forms (2.1) ($m=0$) and (2.2) with[19]

$$h(x)=\tfrac{1}{2}\gamma^\mu\lim_{\xi\to 0}\,[A_\mu(x+\xi)\psi(x)+\psi(x)A_\mu(x-\xi)] \tag{6.25}$$

and

$$j_\mu(x)=\lim_{\xi\to 0}\,\{\bar\psi(x+\xi)\gamma^\mu\psi(x)-G_\mu(\xi)[1-ie\xi^\nu A_\nu(x)]\}\,, \tag{6.26}$$

where

$$G_\mu(\xi)=\langle 0|\bar\psi(\xi)\gamma^\mu\psi(0)|0\rangle\,. \tag{6.27}$$

The second term in (6.26) arises from the usual exponential line integral needed for gauge invariance.[1,8,9] Although with (6.25)–(6.27) the field equations are R-invariant, the gauge group $\mathcal{G}_S$ of the theory cannot be determined until the model is solved in terms of fundamental independent fields. Since the photon becomes massive one expects $R\notin\mathcal{G}_S$. To confirm this, we recall that an operator solution can be obtained by the following steps[19]:

(1) Let η be a free massless scalar field[20] quantized with a negative metric, and on the (indefinite metric) Hilbert space ($m_\sigma{}^2\equiv m^2=e^2/\pi$)

$$\mathcal{H}_1\equiv\mathcal{H}_\sigma\otimes\mathcal{H}_\eta\otimes\mathcal{H}_\chi \tag{6.28}$$

define the operators[21]

$$\psi=e^{i\sqrt{\pi}\gamma_5(\eta+\sigma)}\chi\,, \tag{6.29}$$

$$A_\mu=\frac{\sqrt{\pi}}{e}\epsilon_{\mu\nu}\partial^\nu(\eta+\sigma)\,,\quad \partial^\mu A_\mu=0\,. \tag{6.30}$$

Equation (6.29) then satisfies the massless Dirac equation, but (6.26) gives

$$j_\mu=k_\mu-\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu(\sigma+\eta)\,, \tag{6.31}$$

whereas

$$\Box A_\mu=-\frac{e}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\sigma=e(j_\mu-l_\mu)\,, \tag{6.32}$$

where

$$l_\mu\equiv k_\mu-\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\eta\,. \tag{6.33}$$

(2) Consider the non-negative metric subspace $\mathcal{H}_P$ of $\mathcal{H}_1$ consisting of vectors annihilated by $l_\mu^{(-)}$ ($\Box l_\mu=0$):

$$l_\mu^{(-)}|P\rangle=0 \text{ for } |P\rangle\in\mathcal{H}_P\,. \tag{6.34}$$

The Maxwell equations are satisfied on $\mathcal{H}_P$, where one has

$$j_\mu=-\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\sigma\,, \tag{6.35}$$

so that $Q=0$ and $(\Box+m^2)A_\mu=0$, and furthermore the electron has now disappeared from the spectrum.[19,22]

(3) Finally proceed to the positive-normed Hilbert space $\mathcal{H}\equiv\mathcal{H}_P/\mathcal{H}_0$ obtained from $\mathcal{H}_P$ by dividing out the zero-normed states. Here the theo-

ry is equivalent to that of the massive free scalar field σ on $\mathcal{H}_\sigma$.[19] Thus the electron disappears and the photon acquires mass m by the mechanism (1.3).

We can now proceed to investigate the gauge-invariance properties of the model. We must determine for which functions $\Lambda(x)$ the theory is invariant under the transformations (2.6). This means by definition that we must determine for which $\Lambda(x)$ the transformations (2.6) can be implemented via (6.29), (6.30), and (6.34), by transformations on the independent fundamental fields σ, χ, and η. Note that although ψ and A_μ depend only on the sum $\eta+\sigma$, the subsidiary condition (6.34) depends on χ and η alone, and so there are three distinct independent fields; i.e., (6.34) is not a condition on ψ and A_μ alone and (6.29) and (6.30) do not determine the theory, and $l_\mu(x)$ must be gauge-invariant:

$$l_\mu(x) \to l_\mu(x). \tag{6.36}$$

The desired transformation laws are

$$\sigma(x) \to \sigma(x), \tag{6.37}$$

$$\eta(x) \to \eta(x) + \frac{e}{\sqrt{\pi}}\lambda(x), \tag{6.38}$$

$$\chi(x) \to e^{ie[\Lambda(x)-\gamma^5\lambda(x)]}\chi(x), \tag{6.39}$$

where $\lambda(x)$ is given by (6.16). That (6.37)–(6.39) induce (2.6) and (6.36) follows as for the previous model. The symmetry group $\mathcal{G}_S$ of the theory is thus determined and we see that $R \notin \mathcal{G}$, as expected on the basis of Sec. V from the absence of a massless photon. Although the electron confinement aspect of the theory is interesting,[22,23] the model is physically even more trivial than the preceding one—it is equivalent to the massive free scalar field on $\mathcal{H}_\sigma$.

There is another way of looking at the absence of a zero-mass physical excitation in the Schwinger model. Because the operator solution (6.29), (6.30) gives noncanonical equal-time commutation relations and does not satisfy the operator Maxwell equation, the formalism of Sec. II is not directly applicable. Because of (6.30), Eq. (2.17) has no solution. A $\Pi_{\mu\nu}$ can only be defined if $g_{\mu\nu}$ in (2.17) is replaced by $(q^2 g_{\mu\nu} - q_\mu q_\nu)P(q^2)$ with P a polynomial, as required by locality. Then $\Pi_{\mu\nu}(0)=0$, but this does *not* imply the presence of a singularity in $D(q^2)=P(q^2)/[1+\Pi(q^2)]$ at $q^2=0$. Equation (1.3) is obtained only if $P(q^2)$ is taken to be $1/q^2$, but this choice is not possible in all gauges.

VII. FOUR-DIMENSIONAL PERTURBATIVE MODELS

We consider first conventional QED. The gauge-invariance group $\mathcal{G}_e$ of the theory is just the gauge-invariance group of the field equations (2.1) and (2.2) since these (renormalized) equations determine the perturbative solution in which ψ and A_μ are independent fields.[8,9] It follows from the explicit form of the finite local field equations given in Ref. 8 that $R \in \mathcal{G}_e$. Therefore, by Sec. V, $D(q^2)$ has a zero-mass excitation—the physical photon.

As our second example, we will construct a model with R invariance but no other gauge invariance. The interaction Lagrangian is

$$\mathcal{L}_I = g\partial_\mu A B^\mu C, \tag{7.1}$$

where A is a massless scalar field, B is a vector field, and C is a scalar field. Because of the derivative coupling of A, the theory is formally invariant to the R transformations

$$A(x) \to A(x) + r, \tag{7.2a}$$

$$B_\mu(x) \to B_\mu(x), \tag{7.2b}$$

$$C(x) \to C(x). \tag{7.2c}$$

When B^μ or C have finite masses, the theory can be renormalized so as to preserve the symmetry under (7.2). All of the expected zero-momentum theorems will hold in each order of perturbation theory. For example,

$$[D^A(0)]^{-1} = \Gamma_\mu(0,p) = T^A(q_1,q_2,q_3,q_4)|_{q_i=0} = 0, \tag{7.3}$$

where $D^A(q^2)$ is the A propagator, $\Gamma_\mu(q,p)$ is the $A(q)$-$B_\mu(p)$-$C(p-q)$ proper vertex, and $T^A(q_i)$ is the proper A four-point function. In fact, the proper A self-energy function $\Pi^A(q^2)$ vanishes linearly for $q^2 \to 0$, and so $D^A(q^2)$ has a simple pole at $q^2=0$.

The formal generator of (7.2) is the conserved current

$$J_\mu \equiv \partial_\mu A - gB_\mu \phi. \tag{7.4}$$

The WT identities arising from $\partial^\mu J_\mu = 0$ give no new information, however, because $\partial^\mu J_\mu = 0$ is just the A field equation

$$\Box A = g\partial^\mu(B_\mu\phi). \tag{7.5}$$

If B^μ and C are also massless, the theorems (7.3) remain valid in each order, but now $\Pi^A(p^2) \sim p^2(\ln p^2)^N$ and so $D^A(p^2)$ loses its pole but remains singular.[24] The vanishing of $\Pi^A(0)$ is a direct consequence of the R invariance of the model, as manifested in the derivative coupling. In this case, since the theory is infrared-free, the zero-momentum behavior can be exactly determined by Symanzik's[25] method.

As our final examples, we consider the class of models recently studied by Cornwall.[26] These models are gauge-invariant, but the gauge particles nevertheless are massive because of the presence of mass terms of the form

$$M^2\left(A_\mu - \frac{1}{\square}\partial_\mu\partial\cdot A\right)^2. \tag{7.6}$$

The expression (7.6) is invariant under (1.1) for $\Lambda(x)$ such that $\square\Lambda(x) \neq 0$, but it is *not* invariant under (1.5). We thus again find consistency with the analysis of Sec. V.

VIII. DISCUSSION

Our study of various models in Secs. VI and VII has confirmed our conclusions deduced in Sec. V. It is the presence of the R transformation (1.5) in the gauge symmetry group $\mathcal{G}$ that implies the presence of a zero-mass excitation. Consequently, if such excitations are to be avoided, $\mathcal{G}$ must not contain R.

Is this observation useful? In finite orders of perturbation theory, the field equations determine the theory and the given fields are the fundamental ones so that the invariance group $\mathcal{G}$ of the theory is the invariance group of the field equations, and it is relatively easy to see if $R \in \mathcal{G}$. It is, however, also easy to study $D(0)$ itself order by order. In some models, the field equations themselves may not be enough to determine whether or not $R \in \mathcal{G}$. It may be necessary to know the exact solution, which of course gives $D(0)$ directly. The models we have studied thus do not indicate that our results are particularly useful. However, in more complicated and hopefully more realistic models, it may be much easier to determine if $R \in \mathcal{G}$ than to calculate $D(0)$. Our analysis can then be used as an effective tool to study the zero-momentum behavior of these theories. This possibility is discussed in detail in another paper.[6]

It is of course possible that $R \in \mathcal{G}$ in each order of perturbation theory but not in the exact sum.[27] For example, a Lagrangian with a term like (7.6) may be equivalent to a Lagrangian without such a term if this latter theory is summed over all orders.[26] Another example is provided by the Schwinger model in which, as we have seen, it is the nonfundamental nature of the original ψ and A_μ fields which is responsible for the lack of R invariance in the exact solutions. We cannot rule out such occurrences in nature, but we note that if they are present, then it seems impossible to perform realistic calculations at the present time. Perturbation theory is the only serious calculation tool presently available.

We note in this connection that the symmetry group of the field equations *cannot* fail to be the symmetry group of the theory because of vacuum breaking. The R symmetry is always a spontaneously broken one, and so some other mechanism must be present in order to ruin the strict correspondence between field equations and symmetries.

As a final comment, we recall that our analysis only established a *singularity* of $D(q^2)$ at $q^2 = 0$ if $R \in \mathcal{G}$. The question arises as to whether or not this singularity corresponds to a physical massless particle. This is, in general, a difficult problem. If $D(q^2)$ has a pole at $q^2 = 0$, the particle interpretation is clear. In special cases, it may be possible to prove that the singularity is a pole. For example, if a suitable conserved current can be constructed, one may use the method of Ezawa and Swieca.[28] Even if there is no simple pole, it may be possible to construct an S matrix, perhaps using the methods developed by Zwanziger[29] for QED. In any case, the mere presence of a singularity at $q^2 = 0$ indicates the existence of physical massless excitations which are presumably not present experimentally in strong interactions and so must be avoided. The hadronic symmetry group should therefore not contain the R transformation.

ACKNOWLEDGMENT

One of us (R.A.B.) wishes to thank Gianfausto Dell'Antonio for many helpful discussions. Conversations with John Lowenstein, Andre Swieca, and Daniel Zwanziger are also gratefully acknowledged.

---

*Work supported in part by the National Science Foundation under Grant No. GP-36316X.

†Work supported in part by the National Science Foundation under Grant No. GP-32998X.

‡Present address: Department of Physics and Astronomy, University of Maryland, College Park, Md. 20742.

[1] J. Schwinger, Phys. Rev. 128, 2425 (1962).

[2] It would be more precise to say that the R symmetry implies the stated behavior if it is present in the Lorentz-gauge formalism in which we work.

[3] R. F. Streater, Proc. R. Soc. A287, 510 (1967).

[4] G. Kramer and W. F. Palmer, Phys. Rev. 182, 1442 (1969), and references therein.

[5] R. A. Brandt and W. C. Ng, Nuovo Cimento 13A, 1025 (1973).

[6] R. A. Brandt and W. C. Ng, DESY Reports Nos. 74/37 and 74/38 (unpublished).

[7] Our unconventional normalizations are used for later notational simplicity.

[8] R. A. Brandt, Ann. Phys. (N.Y.) 52, 122 (1969).

[9] R. A. Brandt, Fortschr. Phys. 18, 249 (1970).

[10] See Ref. 8 and references therein.

[11] Y. Takahashi, Nuovo Cimento 6, 371 (1957).

[12] It is assumed that $D_{\nu}^{\kappa}(q)$ is invertible.

[13]Notice that the fields A_μ and A'_μ have the same vacuums.

[14]This is no problem in practice. We are only interested in $q=0$, where δ functions cannot occur.

[15]We are here ignoring the possible infrared effects at the electron mass shell.

[16]For a recent attempt to perform calculations in such a framework, see R. L. Stuller, Max-Planck-Institut Report No. PAE-PTh 2 (unpublished).

[17]See Refs. 18 and 19. We will use the conventions and methods of these references.

[18]B. Klaiber, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968) Vol. XA, p. 141.

[19]J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) 68, 172 (1971).

[20]This does not really exist in two dimensions and so a regularization is necessary. See Refs. 18 and 19.

[21]A normal ordering must be understood here. See Ref. 19.

[22]A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D 10, 732 (1974).

[23]For a review, see R. A. Brandt, Lecture at the IX Balaton Symposium, 1974, Berlin report (unpublished).

[24]The renormalization subtractions must now be made at a nonvanishing Euclidean point.

[25]K. Symanzik, Commun. Math. Phys. 23, 49 (1971); 34, 3 (1973).

[26]J. M. Cornwall, Phys. Rev. D 10, 500 (1974).

[27]In the sense of at least an asymptotic expansion.

[28]J. Ezawa and J. A. Swieca, Commun. Math. Phys. 5, 330 (1967).

[29]D. Zwanziger, Phys. Rev. Lett. 19, 934 (1973); New York University Reports Nos. NYU/TR/10/74 and NYU/TR/13/74 (unpublished).

PHYSICAL REVIEW D VOLUME 10, NUMBER 12 15 DECEMBER 1974

Higher-order ϵ terms in Reggeon field theory*

J. B. Bronzan†

Fermi National Accelerator Laboratory, Batavia, Illinois 60510

Jan W. Dash‡

Argonne National Laboratory, Argonne, Illinois 60439

(Received 15 July 1974)

We calculate the $O(\epsilon^2)$ terms of the Wilson expansion of the critical exponents in the Reggeon field theory with a bare linear trajectory and a triple-Regge interaction. We find that the $O(\epsilon^2)$ and $O(\epsilon)$ terms are comparable at $\epsilon = 2$, and we obtain $\sigma_{\text{tot}}(s) \underset{s\to\infty}{\simeq} (\ln s)^{0.38}$. We also show that the Gell-Mann–Low function $\beta(g)$ expanded to finite order in both ϵ and g carries no information about the existence of the Gell-Mann–Low zero at finite ϵ.

I. INTRODUCTION

The technique of using the renormalization group[1] and the Wilson ϵ expansion[2] to derive scaling properties of proper vertices in Reggeon field theory[3] was introduced by Migdal, Polyakov, and Ter-Martirosyan,[4] and by Abarbanel and Bronzan.[5,6] In their work the behavior of the proper vertices in the infrared limit $j \approx 1$ and $t \approx 0$ was examined, and a number of conclusions were reached. The most important of these was a prediction that in a theory with a linear unrenormalized Pomeron trajectory and a triple-Pomeron coupling, the asymptotic behavior of the elastic amplitude is

$$T(s,t) = s(\ln s)^{-\gamma} F(t(\ln s)^z) , \tag{1}$$

with $\gamma < 0$. This behavior arises from the coincidence at $j=1$ and $t=0$ of an infinite number of branch points. The scaling exponent γ specifies the logarithmic rise of the total cross section,

$$\sigma_{\text{tot}} \sim (\ln s)^{-\gamma} , \tag{2}$$

and the exponent z specifies the trajectories of Pomeron cuts and pole for small t,

$$\alpha(t) = 1 + \text{const} \times (t)^{1/z} . \tag{3}$$

The exponents γ and z can be determined in an ϵ expansion, where $\epsilon = 4 - D$ is the difference between the natural scaling dimension ($=4$) and the number of transverse dimensions D; we want answers for $\epsilon = 2$. Although ϵ is large, it was shown[4-6] that to order ϵ, $-\gamma = \frac{1}{12}\epsilon = \frac{1}{6}$, $z = 1 + \frac{1}{24}\epsilon = \frac{13}{12}$. If ϵ were always accompanied by a factor like $\frac{1}{12}$, a few terms in the ϵ expansion would give good results for γ and z. We have determined that[7]

$$\begin{aligned} -\gamma &= \tfrac{1}{12}\epsilon + \left(\tfrac{257}{12}\ln\tfrac{4}{3} + \tfrac{37}{24}\right)\left(\tfrac{1}{12}\epsilon\right)^2 + O(\epsilon^3) , \\ z &= 1 + \tfrac{1}{24}\epsilon + \left(\tfrac{155}{24}\ln\tfrac{4}{3} + \tfrac{79}{48}\right)\left(\tfrac{1}{12}\epsilon\right)^2 + O(\epsilon^3) . \end{aligned} \tag{4}$$

Since the coefficients of the $(\frac{1}{12}\epsilon)^2$ terms are about 7.7 and 3.5, respectively, the $O(\epsilon^2)$ terms are larger than the $O(\epsilon)$ terms at $\epsilon = 2$. It would therefore seem that the ϵ expansion is a questionable