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π-π scattering in the SU_2 σ model

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We explore the freedom of choosing subtraction points in the renormalizable σ model of $\vec{\pi}$ and σ. Phase shifts are computed from Padé approximants in the one-loop approximation. Comparisons are made with a previous calculation and an SU_3 σ-model calculation. The phase of the scalar form factor of the pion is presented.

I. INTRODUCTION

In the preceding paper[1] (referred to as paper II) we described a calculation of phase shifts for the SU_3 σ model in the one-loop approximation. In this paper we do the same for a much simpler model—the SU_2 σ model.[2] Since a similar calculation has been done before by Basdevant and Lee[3] (referred to as BL) we need to justify doing it again. This paper differs from BL in the manner by which finite parts of renormalization counterterms are chosen. We describe a freedom in the renormalization procedure that is not discussed in BL. We advocate adopting a procedure in which perturbation theory is a power series in a physical quantity with a known value—in this case $1/f_\pi^2$—and in which all subtractions are at physically measurable quantities. The procedure in this paper is the direct analog of that in our paper II. Hence, in addition to exploring the renormalization freedom, this paper gives a direct comparison between these two models treated on the same footing and in the light of up-to-date phase-shift data that is substantially different from that used in Ref. 3.

The freedom we refer to can best be described by considering the hypothetical situation in which the σ particle is stable. Then a very natural renormalization procedure suggests itself. Since there are three parameters in the model, three quantities can be chosen to be fixed constants to all orders in perturbation theory—a natural set being m_π, m_σ, and the perturbation expansion parameter, which for us is in $1/f_\pi^2$, for BL in $1/\langle\sigma\rangle^2$ ($\langle\sigma\rangle$ = vacuum expectation value of the σ field). The statement that the expansion parameter has no higher-order corrections is a tautology, but there is a choice involved in what that parameter shall be. Renormalizing at the pion mass is a deep-seated prejudice based on the fact that we know the mass very well. If one knew instead the 10th derivative of a form factor very well, one could make a case of renormalizing at the physical quantity. Now, since the σ is in fact unstable and very wide, even if we were committed to renormalizing at its mass, there are a myriad of possible conditions one could think of to replace the strict mass renormalization condition for the stable σ.

Neither BL nor we are committed to renormalizing at the σ mass, although one of our renormalization prescriptions discussed here is in that spirit. This method (referred to as method II) is to demand that $d\delta_0^0/ds$ be a maximum at the tree value of the σ mass, where δ_0^0 is the $I=0$, $l=0$ $\pi\pi$ phase shift. A second method is given which exactly parallels our paper II (method I), in which the $I=0$, $l=0$ $\pi\pi$ amplitude is renormalized such that there are no second-order corrections at a low-energy on-mass-shell point. A consequence of both these methods is that as $f_\pi \to \infty$, with tree masses fixed, the ratio of (second order)/(tree) for all quantities goes to zero, all dynamically generated states go away, and the scalar resonances approach their zero-width approximations.

For method I, a stronger statement holds in that the perturbation series is a power series in $1/f_\pi{}^2$, as we show.

In addition to giving numerical results for phase shifts calculated from scattering amplitudes, we also calculate the scalar form factor of the pion and extract the $\pi\pi$ phase shift from the unitarity equation for the form factor. Comparing these two determinations of the phase shift gives a handle on the convergence of the approximation.

This paper is written in a way that closely parallels paper II. We hope this has a pedagogical value in that paper II is complicated by many things that are not essential to an understanding of many points. This paper, in effect, serves as a model of a model for us. The numerical calculations in this paper were done by truncating the program used in paper II.

II. TREE APPROXIMATION AND RENORMALIZATION THROUGH ONE-LOOP ORDER

Let us start with the Lagrangian expressed in terms of unrenormalized fields, $\vec{\pi}^0, \sigma^0$:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm sym} + \mathcal{L}_{\rm SB},\\
\mathcal{L}_{\rm sym} &= +\tfrac{1}{2}(\partial_\mu \vec{\pi}^0)^2 + \tfrac{1}{2}(\partial_\mu \sigma^0)^2\\
&\quad - \tfrac{1}{2}(\mu^0)^2[(\vec{\pi}^0)^2 + (\sigma^0)^2]\\
&\quad + \tfrac{1}{4} g^0[(\sigma^0)^2 + (\vec{\pi}^0)^2]^2,\\
\mathcal{L}_{\rm SB} &= -\epsilon^0\sigma^0 .
\end{aligned} \tag{2.1}$$

Introduce a chiral-invariant renormalization constant[4]

$$(\vec{\pi}^0, \sigma^0) = C^{1/2}(\vec{\pi}, \sigma) \tag{2.2}$$

and write all bare quantities $(\mu^0)^2, g^0, \epsilon^0$ in terms of a tree part μ^2, g, ϵ and a second-order part $\delta\mu^2, \delta g, \delta\epsilon$:

$$\begin{aligned}
(\mu^0)^2 &= \mu^2 + \delta\mu^2,\\
g^0 &= g + \delta g,\\
\epsilon^0 &= \epsilon + \delta\epsilon .
\end{aligned} \tag{2.3}$$

It is convenient to define Z_μ, Z_g, Z_ϵ in such a way that C occurs as an overall factor in $\mathcal{L}$: i.e.,

$$\begin{aligned}
C\mu^2 Z_\mu &= C(\mu^2 + \delta\mu^2),\\
CgZ_g &= C^2(g + \delta g),\\
C\epsilon Z_\epsilon &= C^{1/2}(\epsilon + \delta\epsilon),
\end{aligned} \tag{2.4}$$

giving

$$\begin{aligned}
\mathcal{L}_{\rm sym} &= C[\tfrac{1}{2}(\partial^\mu\sigma)^2 + \tfrac{1}{2}(\partial^\mu\vec{\pi})^2\\
&\quad - \tfrac{1}{2}\mu^2 Z_\mu(\vec{\pi}^2 + \sigma^2) + \tfrac{1}{4} g Z_g(\sigma^2 + \vec{\pi}^2)^2],\\
\mathcal{L}_{\rm SB} &= -C\epsilon Z_\epsilon \sigma .
\end{aligned} \tag{2.5}$$

Finally, translate the σ field,

$$\sigma = s + (\xi + \delta\xi). \tag{2.6}$$

ξ and $\delta\xi$ are to be determined such that the vacuum expectation value $\langle s\rangle = 0$ order by order. Equation (2.1) becomes

$$\begin{aligned}
\mathcal{L} = C\{&\tfrac{1}{2}(\partial^\mu s)^2 + \tfrac{1}{2}(\partial^\mu\vec{\pi})^2 - \tfrac{1}{2}[\mu^2 Z_\mu - 3gZ_g(\xi+\delta\xi)^2]s^2 - \tfrac{1}{2}[\mu^2 Z_\mu - gZ_g(\xi+\delta\xi)^2]\vec{\pi}^2\\
&+ gZ_g(\xi+\delta\xi)s(s^2+\vec{\pi}^2) + \tfrac{1}{4}gZ_g(s^2+\vec{\pi}^2)^2 - [\mu^2 Z_\mu(\xi+\delta\xi) - gZ_g(\xi+\delta\xi)^3 + \epsilon Z_\epsilon]s\}.
\end{aligned} \tag{2.7}$$

In lowest order, we have in the standard way

$$\begin{aligned}
\epsilon &= g\xi^3 - \mu^2\xi,\\
m_\pi{}^2 &= \mu^2 - g\xi^2,\\
m_\sigma{}^2 &= \mu^2 - 3g\xi^2,\\
f_\pi &= \xi,\\
f_\pi m_\pi{}^2 &= -\epsilon .
\end{aligned} \tag{2.8}$$

In second order we first deal with the divergences. Write

$$\begin{aligned}
Z_{\mu,g} &= 1 + \delta Z_{\mu,g}\\
&= 1 + \Delta Z_{\mu,g}\,(\text{finite}) + DZ_{\mu,g}\,(\text{infinite});
\end{aligned} \tag{2.9}$$

choose[5]

$$\begin{aligned}
DZ_\mu &= 6g(1-\nu^2/\mu^2)B_0(\nu^2) + 6(g/\mu^2)A(\nu^2),\\
DZ_g &= 12gB_0(\nu^2),
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
A(\nu^2) &= i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\nu^2+i\epsilon},\\
B_0(\nu^2) &= i\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\nu^2+i\epsilon)^2} = \frac{d}{d\nu^2}A(\nu^2).
\end{aligned} \tag{2.11}$$

For any quantity calculated through second order, all divergences in loop graphs are canceled by the divergent DZ_μ and DZ_g. In what follows it is assumed that DZ_μ and DZ_g are disregarded and all integrals are rendered finite by subtracting the appropriate combination of $A(\nu^2)$ and $B_0(\nu^2)$.

The condition for the vanishing of $\langle s\rangle$ to second order is

$$\epsilon(1+\delta Z_\epsilon)(1+\delta C) + (\xi+\delta\xi)(m_\pi{}^2 + \delta m_\pi{}^2) - 3g\xi[\bar{A}(m_\sigma{}^2) + \bar{A}(m_\pi{}^2)] = 0, \tag{2.12}$$

where δZ_ϵ, δC are the second-order parts of Z_ϵ and C, and where $m_\pi^2+\delta m_\pi^2$ is the shorthand for the second-order truncated part of the expression[6]

$$m_\pi^2+\delta m_\pi^2=C[\mu^2 Z_\mu - gZ_g(\xi+\delta\xi)^2]. \tag{2.13}$$

The integrals are

$$\bar{A}(x)=A(x)-A(\nu^2)-(x-\nu^2)B_0(\nu^2).$$

Next, writing the π propagator,

$$\begin{aligned} D_\pi^{-1}(s) &= s(1+\delta C)-(m_\pi^2+\delta m_\pi^2) \\ &\quad + g[5\bar{A}(m_\pi^2)+\bar{A}(m_\sigma^2)] \\ &\quad - 4g^2\xi^2\bar{B}(s;m_\pi^2,m_\sigma^2), \\ D_\pi^{-1} &= C(D_\pi^0)^{-1} \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} B(p^2;x,y) &= i\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(k-p)^2-x+i\epsilon][k^2-y+i\epsilon]}, \\ \bar{B}(p^2;x,y) &= B(p^2;x,y)-B_0(\nu^2). \end{aligned} \tag{2.15}$$

Renormalizing at the π mass gives

$$m_\pi^2\delta C-\delta m_\pi^2+g[5\bar{A}(m_\pi^2)+\bar{A}(m_\sigma^2) - 4g\xi^2\bar{B}(m_\pi^2;m_\pi^2,m_\sigma^2)]=0. \tag{2.16}$$

Using Eq. (2.16) to eliminate δm_π^2 from Eq. (2.12) gives a more transparent form of the stationary equation, Eq. (2.12):

$$\epsilon(1+\delta Z_\epsilon)+m_\pi^2(\xi+\delta\xi) - 4g^2\xi^3[B(m_\pi^2;m_\pi^2,m_\sigma^2)-B(0;m_\pi,m_\sigma^2)]=0. \tag{2.17}$$

We can identify the wave-function renormalization constant of the pion field Z_π through the relation

$$D_\pi(s)\underset{s\to m_\pi^2}{\sim}\frac{Z_\pi/C}{s-m_\pi^2}, \tag{2.18}$$

obtaining

$$Z_\pi=1+4g^2\xi^2\frac{d}{ds}B(s;m_\pi^2,m_\sigma^2)\Big|_{s=m_\pi^2}. \tag{2.19}$$

The renormalized field π^R is

$$\vec{\pi}^R=Z_\pi^{-1/2}\vec{\pi}^0=(Z_\pi/C)^{-1/2}\vec{\pi} \tag{2.20}$$

and the renormalized propagator D_π^R is

$$D_\pi^R(s)^{-1}=(s-m_\pi^2)-4g^2\xi^2\left[B(s;m_\pi^2,m_\sigma^2)-B(m_\pi^2;m_\pi^2;m_\sigma^2)-(s-m_\pi^2)\frac{d}{ds}B(s;m_\pi^2,m_\sigma^2)\Big|_{s=m_\pi^2}\right]. \tag{2.21}$$

In order to introduce the decay constant, f_π, we note the operator relation in this model

$$\begin{aligned} \partial_\mu\vec{A}^\mu &= -\epsilon^0\vec{\pi}^0 \\ &= -\epsilon Z_\epsilon C^{1/2}Z_\pi^{1/2}\vec{\pi}^R \end{aligned} \tag{2.22}$$

giving

$$f_\pi m_\pi^2=-\epsilon(1+\delta Z_\epsilon)(1+\tfrac{1}{2}\delta C)(1+\tfrac{1}{2}\delta Z_\pi). \tag{2.23}$$

This equation is one of the defining equations of our renormalization procedure in that we have chosen $\delta f_\pi=0$.

We add in passing that the following Ward identity relates these quantities:

$$f_\pi m_\pi^2=-(\xi+\delta\xi)D_\pi^R(0)^{-1}[1+\tfrac{1}{2}(\delta C-\delta Z_\pi)]. \tag{2.24}$$

Now we turn to scattering graphs. The $\pi_i\pi_j-\pi_k\pi_l$ amplitude is decomposed in the standard way:

$$A=A^s\delta_{ij}\delta_{kl}+A^t\delta_{ik}\delta_{jl}+A^u\delta_{il}\delta_{jk}. \tag{2.25}$$

The tree graphs are

$$T^s=\frac{m_\pi^2-m_\sigma^2}{\xi^2}\,\frac{s-m_\pi^2}{s-m_\sigma^2} \tag{2.26}$$

and similar expressions for T^t and T^u. In second order there are numerous loop graphs, denoted L, which we do not give. They can be found in Ref. 3 and implicitly from our paper II. We give the σ propagator in our language since it enters in scattering amplitudes and it demonstrates how we handle the finite parts of the counter terms. Thus,

$$\Delta_\sigma^{-1}(s)=s(1+\delta C)-(m_\sigma^2+\delta m_\sigma^2)+g[3\bar{A}(m_\pi^2)+3\bar{A}(m_\sigma^2)]-6g^2\xi^2[3\bar{B}(s;m_\sigma^2,m_\sigma^2)+\bar{B}(s;m_\pi^2,m_\pi^2)],$$

where

$$m_\sigma^2+\delta m_\sigma^2=(1+\delta C)[\mu^2 Z_\mu-3gZg(\xi+\delta\xi)^2], \qquad \Delta_\sigma^{-1}(s)=C\Delta_\sigma^0(s)^{-1}.$$

We wish to note that loop graphs have the following ξ dependence:

$$L=\frac{1}{\xi^4}\hat{L}(m_\pi^2, m_\sigma^2, s, t). \tag{2.27}$$

In second order there are also counterterm graphs K which are

$$K^s=\left[\frac{m_\pi^2-m_\sigma^2}{\xi^2}(1-\delta C+\Delta Z_g+2\delta Z_\pi)\left(\frac{s-m_\pi^2-\mu^2\Delta Z_\mu+g\xi^2\Delta Z_g+2\xi g\delta\xi}{s-m_\sigma^2-\mu^2\Delta Z_\mu+3g\xi^2\Delta Z_g+6\xi g\delta\xi}\right)\right]_{\text{second-order part}}$$

$$=T^s(2\delta Z_\pi-\delta C)+K_g^s\Delta Z_g+K_\mu^s\Delta Z_\mu+K_\xi^s\frac{\delta\xi}{\xi}, \tag{2.28}$$

and similar expressions for K^t and K^u. All second-order graphs S are $S=K+L$. The isospin amplitudes in the s channel are

$$A^{I=0}=3A_s+A_t+A_u,$$
$$A^{I=1}=A_t-A_u,$$
$$A^{I=2}=A_t+A_u.$$

We make a partial-wave analysis and calculate phase shifts from the Padé form:

$$A_l^I=\frac{(T_l^I)^2}{T_l^I-S_l^I}. \tag{2.29}$$

Unitarity reads

$$\text{Im}A_l^I=\frac{1}{32\pi}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}|A_l^I|^2. \tag{2.30}$$

So far we have given three equations [(2.16), (2.17), and (2.23)] to determine the five quantities ΔZ_μ, ΔZ_g, δZ_ϵ, $\delta\xi$, and δC. The quantity δC is inessential in that even though δZ_ϵ, ΔZ_g, and $\delta\xi$ depend on δC, no physical quantities do—as we will discuss below. We must find one more relation to fix the essential counterterms. This is done in two ways as described in the Introduction:

(i) Renormalize at $A_{l=0}^{I=0}$ at an energy close to threshold, s_0 (method I). The needed relation is then $S_{l=0}^{I=0}(s_0)=0$. More explicitly,

$$\left[T^I(2\delta Z_\pi-\delta C)+K_g^I\Delta Z_g+K_\mu^I\Delta Z_\mu+K_\xi^I\frac{\delta\xi}{\xi}+\frac{1}{\xi^4}\hat{L}\right]_{s=s_0}=0. \tag{2.31}$$

We wish to show for this method that the perturbation series is a power series in $1/\xi^2$ (including the first two terms). To see this, one can verify that for an n-line S-matrix element the tree graphs $T^{(n)}\sim1/\xi^{n-2}$ (masses, momenta fixed) and the one-loop graphs $L^{(n)}\sim1/\xi^n$. The second-order counterterm graphs $K^{(n)}\sim1/\xi^n$ also. This is because we renormalize at the tree value of physical quantities giving counterterm defining equations of the form $K^{(n)}+L^{(n)}=0$ [Eqs. (2.16), (2.17), (2.23), and (2.31)]. Hence $K^{(n)}$ has the ξ dependence of $L^{(n)}$. Since we renormalize at f_π, $f_\pi=\xi$, and the expansion parameter is also $1/f_\pi^2$.

(ii) Renormalize at σ mass (method II). Renormalizing at m_σ is not well defined since it is unstable, but we approximate the condition as follows. We calculate $\delta_{l=0}^{I=0}$ from Eq. (2.29) as a function of s and ΔZ_g. We then adjust ΔZ_g until the most rapid variation in s of the phase shift occurs at $s=m^2$; i.e.,

$$\left.\frac{d^2\delta_0^0}{ds^2}\right|_{s=m_\sigma^2}=0. \tag{2.32}$$

With this condition in place of Eq. (2.31) our argument about $1/\xi^2$ being an expression parameter no longer holds. However, as the interaction is turned off, $g\to0$ ($\xi\to\infty$, m_π, m_σ fixed), this condition is consistent with the σ approaching the zero-width approximation. $\Delta Z_g\sim1/\xi^2$ for large ξ but contains higher terms in $1/\xi^2$. Quantities that do not depend on ΔZ_g, such as δZ_ϵ, $\delta\xi/\xi$, and δZ_π, have a $1/\xi^2$ dependence.

We now return to δC to show that its value does not enter in any physical quantity. First choose $\delta C=0$ and solve for the counterterms denoted $\Delta\tilde{Z}_g$, $\Delta\tilde{Z}_\mu$, $\delta\tilde{\xi}/\xi$, and $\delta\tilde{Z}_\epsilon$. Then for $\delta C\neq0$ the values of these quantities are easily seen to be

$$\begin{aligned}\Delta Z_g&=\Delta\tilde{Z}_g+\delta C,\\ \Delta Z_\epsilon&=\delta\tilde{Z}_\epsilon-\tfrac{1}{2}\delta C,\\ \frac{\delta\xi}{\xi}&=\frac{\delta\tilde{\xi}}{\xi}-\tfrac{1}{2}\delta C,\\ \Delta Z_\mu&=\Delta\tilde{Z}_\mu.\end{aligned} \tag{2.33}$$

If we then calculate any physical quantity by using the expressions on the left-hand side of Eq. (2.33), δC drops out, as can be verified on all the relevant equations in this section. This is of course a consequence of C being an overall factor in $\mathcal{L}$.

How then should we choose δC? Looking at $\mathcal{L}_{\text{SB}}$, Eq. (2.5), and Eq. (2.23) we have

$$\mathcal{L}_{\text{SB}}=-C\epsilon Z_\epsilon\sigma=f_\pi m_\pi^2(C/Z_\pi)^{1/2}\sigma. \tag{2.34}$$

Hence, if we choose $C=Z_\pi$, then the symmetry-breaking parameter in $\mathfrak{L}_{SB}$ is $f_\pi m_\pi{}^2(=-\epsilon)$ to second order.

We now give the BL choice of counterterms in our language. Their choice can be deduced from their Eqs. (3.3), (4), (7), (8), and (9). Their propagators $\hat{\Delta}_\pi, \hat{\Delta}_\sigma$ are what we call D_π^0, Δ_σ^0. Let us choose our cutoff mass $\nu^2=m_\pi{}^2$, and choose $\delta\xi=0$ ($\delta f_\pi\neq 0$) and $C=1$, as Basdevant and Lee did:

$$(D_\pi^0)^{-1}=s-m_\pi{}^2-\mu^2\Delta Z_\mu{}^2+g\Delta\tilde{Z}_g\xi^2 +g\overline{A}(m_\sigma{}^2)-4g^2\xi^2\overline{B}(s;m_\pi{}^2,m_\sigma{}^2)\,,$$
$$(\Delta_\sigma^0)^{-1}=s-m_\sigma{}^2-\mu^2\Delta\tilde{Z}_\mu+3g\Delta\tilde{Z}_g\xi^2+3g\overline{A}(m_\sigma{}^2) -6g^2\xi^2[3\overline{B}(s;m_\sigma{}^2,m_\sigma{}^2)+\overline{B}(s;m_\pi{}^2,m_\pi{}^2)]\,. \tag{2.35}$$

Their choice of ΔZ_g and ΔZ_μ is then

$$\Delta\tilde{Z}_g=-\overline{A}(m_\sigma{}^2)/\xi^2\,,$$
$$\Delta\tilde{Z}_\mu=-4g^2\xi^2\overline{B}(m_\pi{}^2;m_\sigma{}^2,m_\sigma{}^2)\,. \tag{2.36}$$

[The quantities $\Delta\tilde{Z}_g$, $\Delta\tilde{Z}_\mu$ are those obtained by setting $C=1$ and are related to ΔZ_g, ΔZ_μ by Eq. (2.33).] With this choice, the π propagator has a pole at the tree π mass and the σ propagator has a pole at a point away from the tree σ mass. If we had normal-ordered in the Lagrangian, $\overline{A}(m^2)$ would not have appeared, and then Eq. (2.35) would give $\Delta\tilde{Z}_g=0$.

III. NUMERICAL RESULTS

We first discuss the results for the renormalization in which we fix $\pi\pi$ amplitude at a low energy point (called method I). Figure 1 shows what the amplitudes look like near threshold. The linear Weinberg[7] amplitude is given for the sake of comparison. We have fixed Z_g such that

$$\mathrm{Re}S_{l=0}^{I=0}(E=2.2m_\pi)=0$$

(S = second order). As a consequence, S_0^0 is very close to zero for all energies below threshold (note the relative scales between tree and second order in Fig. 1). The $I=2$ amplitude S_0^2 is an order of magnitude smaller, also with zeros in this region. The curve labeled "box" is the sum of all box graphs and is given to show the suppression of the cusp that occurs when all other second-order graphs are added to it. Noting the change in scale, the suppression in the cusp structure is 2 to 3 orders of magnitude. This procedure (method I) is the same as for paper II. We pick the physical value of f_π, 0.095 GeV, and choose m_σ to give the best δ_0^0 phase shift. We then vary f_π over a range to show the dependence of the results on f_π.

Our results are given, along with phase-shift analysis,[8-11] in Fig. 2 and in Table I. The δ_0^2 phase shift is heading for large negative values for both values of f_π. By increasing f_π, the onset of that disaster is delayed to higher energies but does not qualitatively change its behavior. The three states generated in the P and D waves are in the same mass ranges as states found in the SU_3 σ model in paper II.

Our fit to δ_0^0 is comparable or perhaps better than the fit in paper II. The dotted curve is an interesting result. This is the phase shift calcu-

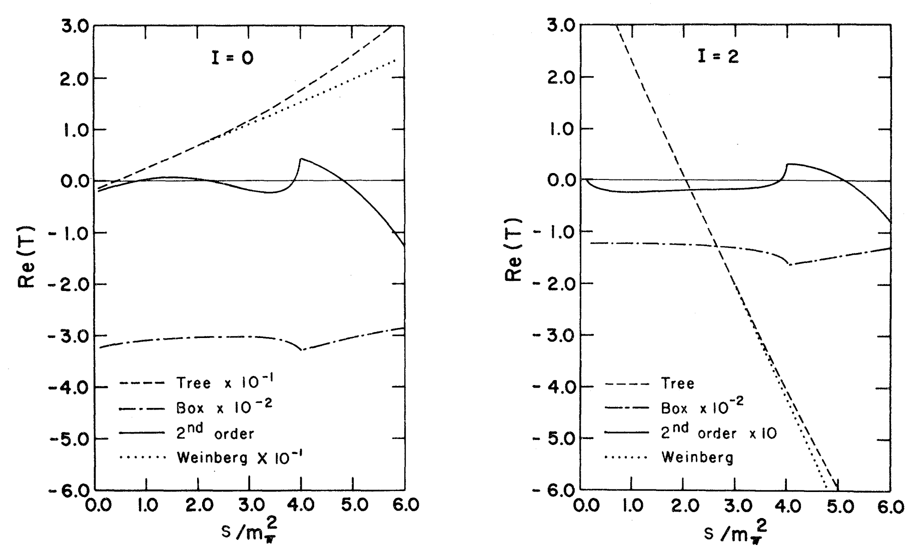

FIG. 1. $\pi\pi$, $I=0,2$, $l=0$ amplitudes in the low-energy region for method I. We renormalized the $I=0$ amplitude at $E=2.2m_\pi$. The linear Weinberg (Ref. 7) amplitudes are shown. Curves labeled "box" are the sum of box graphs. Note the scale factors for various functions.

lated from the phase of the π form factor. This is the form factor for π coupled to a scalar source. In an exact solution, this phase must agree with the four-point-function phase shift, but not at any finite order. Even though both are in the one-loop approximation, there is, in some sense, less of the dynamics (and much less labor) in the form factor. In a calculation to the n-loop order, the form factor phase gives a phase shift "intermediate in order." If this argument is valid, the closeness of these two curves indicates good convergence for this phase shift. The curve marked BL is taken from Ref. 3. The change in the trend of the $I=0$ data is apparent since this curve looked better in 1970.

We have not checked the crossing-symmetry conditions for partial waves.[12] However, since our second-order amplitude is so small in the Mandelstam triangle, and since tree graphs satisfy those conditions, we believe the violations of crossing symmetry relations will be small.

The S-wave scattering lengths are also shown in Table I. It is instructive to use the $I=0$ scattering length to illustrate the effect from the two methods of renormalization.

The tree scattering amplitude has a pole at $s=m_\sigma^2$ and the Padé approximant pushes the σ pole way out, a few hundred MeV, in the complex s plane. Renormalization method I requires the scattering length essentially unchanged from the tree value, which differs from the Weinberg value on account of the finiteness of the σ mass. Quali-

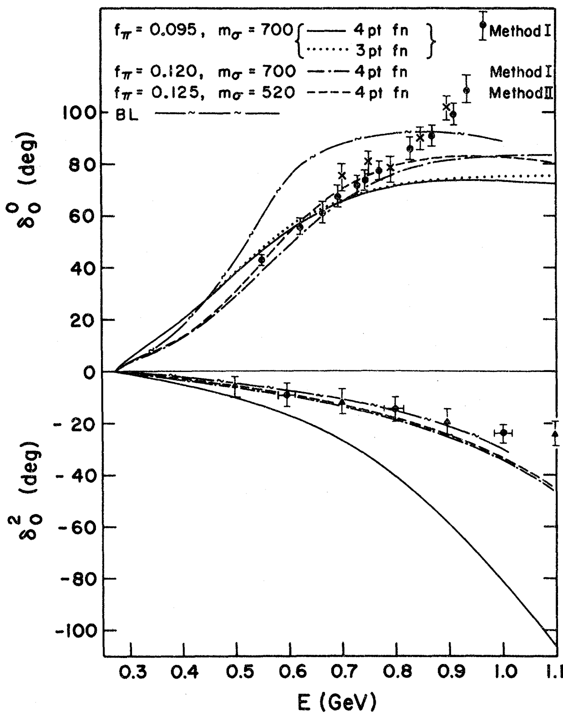

FIG. 2. $\pi\pi$, $I=0, 2$, $l=0$ phase shifts. Methods I and II refer to the renormalization procedure (see text). Four-point function means calculated from scattering amplitude. Three-point function means extracted from π scalar form factor. Curve marked "BL" is taken from Ref. 3. For $I=0$ the dots are taken from Ref. 8 and the crosses from Ref. 9. For $I=2$ the dots are taken from Ref. 10 and the triangles from Ref. 11.

TABLE I. Tree and second-order quantities. Set $\alpha=1$ to get quantities calculated with the $f_\pi=0.095$ as indicated. We chose $C=Z_\pi$.

	Method I ($m_\sigma=700$ MeV) $f_\pi=0.095\alpha$		Method II ($m_\sigma=520$ MeV) $f_\pi=0.125$		
ϵ (GeV^3)	$-0.001\,81\alpha$			$-0.002\,38$	
$\delta\epsilon$ (GeV^3)	$-0.000\,12\alpha^{-1}$			$-0.000\,09$	
$\delta\xi$ (GeV)	$0.000\,13\alpha^{-1}$			$0.000\,07$	
$\delta\xi^0$ $(=\delta\xi+\frac{1}{2}\xi\delta C)$ (GeV)	$-0.006\,34\alpha^{-1}$			$-0.004\,82$	
$Z_\pi^{1/2}-1$	$-0.068\,12\alpha^{-2}$			$-0.039\,34$	
		(Weinberg)	(Tree 2nd)	(Padé)	(Weinberg)
a_0^0	$(0.175\alpha^{-2}+0.004\alpha^{-4})$	$(0.147\alpha^{-2})$	(0.119-0.019)	(0.103)	(0.092)
a_0^2	$-(0.0398\alpha^{-2}+0.000\alpha^{-4})$	$(-0.042\alpha^{-2})$	(−0.023-0.001)	(−0.024)	(−0.026)
	$f_\pi=0.095$	$f_\pi=0.120$		$f_\pi=0.125$	
m_ρ (MeV)	740	980		840	
m_f (MeV)	1190	1375		1140	
$m_{I=2,l=2}$ (MeV)	1525	1875		1480	

tatively the complex σ pole should be roughly of distance m_σ^2 away from the threshold in the complex s plane to exert the same amount of influence. Method II requires the real part of the pole position to be the same as m_σ^2. Therefore, the σ pole becomes farther away in the complex s plane from the threshold and exerts less influence on the low energy region. The net result is that the Padé scattering length is closer to the Weinberg value but the second-order correction is larger.

For method II, we fix Z_g by demanding $d^2\delta_0^0/ds^2=0$ for $s=m_\sigma^2$ (tree), which is close in spirit to renormalizing at the σ mass. In this fit we allow m_σ and f_π to vary to find a best fit to δ_0^0 and δ_0^2. Comparing this fit with method I ($f=0.095$), method II gives an improvement in δ_0^2 and gives P and D wave spectra about the same, at the cost of losing a fit to f_π. If we compare method II with method I ($f=0.120$), we see that the S-wave phase shifts are about the same, but the ρ and f are better for method II.

These two methods give a sample of the ambiguity in the model to this order of approximation. With three methods of determining counterterms now explored — methods I and II, and BL — we can draw one simple conclusion: For a reasonable fit to δ_0^0, δ_0^2 is governed almost solely by the value of f_π.

Table I illustrates some of the features of our renormalization procedure. Note that the f_π dependence of quantities is indicated by the factor α. Since we renormalize at f_π, $\xi=f_\pi$. Further, the symmetry-breaking parameter in the Lagrangian $C\epsilon Z_\epsilon=\epsilon$ for $C=Z_\pi$ as we showed above. $\delta\xi^0$ is the correction to $\langle\sigma^0\rangle$ where σ^0 is the field in Eq. (2.1). Note $\delta\xi \ll \delta\xi^0$. To see why, we combine Eqs. (2.17), (2.19), and (2.23) to give

$$\frac{\delta\xi}{\xi}=4g^2\xi^2\left\{[B(m_\pi^2;m_\pi^2,m_\sigma^2)-B(0;m_\pi^2,m_\sigma^2)]/m_\pi^2-\frac{d}{ds}B(s;m_\pi^2,m_\sigma^2)\bigg|_{s=m_\pi^2}\right\}$$

and we see that the leading term m_π^2/m_σ^2 cancels. This equation is in fact the second-order part of the Ward identity (2.24) for $C=Z_\pi$.

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[1]L.-H. Chan and R. Haymaker, preceding paper, Phys. Rev. D 10, 4143 (1974), referred to as paper II.

[2]A partial list of references on the subject can be found in Ref. 1.

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[4]C plays no part in determining physical quantities and can be set equal to 1 to simplify the discussion. It is introduced in order to have $\mathcal{L}_{SB}=m_\pi^2 f_\pi\sigma$ to second order. It is understood that all equations in this section are truncated to the appropriate order; terms higher than second order are always dropped.

[5]Had we normal-ordered the Lagrangian, there would be no divergence DZ_μ.

[6]δm_π^2 and δm_σ^2 are not the total corrections to the π and σ masses; we renormalize at m_π, and the shift in m_σ gets additional loop contributions.

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