unchanged generates the tadpole counterterm $\tilde{\varphi}(m/\sqrt{\lambda})\delta m^2$ which is precisely needed to cancel the one-loop tadpole. For the kink, the shift is $\tilde{\varphi} \to \tilde{\varphi} + (m/\sqrt{\lambda})\tanh(xm/\sqrt{2})$. Hence the mass counterterm contributes to the energy of the kink by the amount

$$-\tfrac{1}{2}\delta m^2 \frac{m}{\lambda}\int_{-\infty}^{+\infty} dx\,[\tanh^2(x/\sqrt{2}) - 1]\ ,$$

which exactly cancels the divergent term of Eq. (A4). Collecting all the finite terms, one then arrives at Eq. (3.10).

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[1]See, for example, T. Skyrme, Nucl. Phys. 31, 556 (1962), or D. Finkelstein and J. Rubenstein, J. Math. Phys. 9, 1762 (1968), and references cited therein.

[2]R. Dashen, B. Hasslacher, and A. Neveu, preceding paper, Phys. Rev. D 10, 4114 (1974).

[3]T.-M. Yan, London Conference report (unpublished).

[4]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*,(McGraw-Hill, New York, 1953).

[5]Although Q defined by (4.6) looks positive-definite, one must remember that ψ is really an anticommuting object so that a real field can carry no charge.

[6]Similar equations have been considered by J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).

[7]This state is a self-charge-conjugate object like K_1^0. It has a partner (K_2^0) built on the kink with the negative sign in (3.6).

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Nonperturbative methods and extended-hadron models in field theory. III. Four-dimensional non-Abelian models*

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By analogy with the magnetic vortex of the Landau-Ginzburg theory of superconductivity, we construct extended models of hadrons with SU(2) Yang-Mills fields coupled to fermions and a spontaneous-symmetry-breaking scalar isospinor. These models are in four space-time dimensions.

Low- and medium-energy hadronic systems seem to behave as though they have a rich extended or composite nature, e.g., primitive bound-quark models give a surprisingly good description of static properties. More recently, the dynamics underlying the concept of duality has been given a concrete and beautiful interpretation in terms of the relativistic string. This has led to other models of hadrons as extended quantized systems, for example, the so-called MIT "bag" model.

Unfortunately all these systems as yet lack the flexibility and internal consistency of ordinary quantum field theory. In particular, string models work naturally only in certain unphysical space-time dimensions. There is a third possibility quite different from *ad hoc* extended models on the one hand and pointlike particles or constituent theories on the other, which is to find and study extended objects within a full field theory.

Quantum-mechanical bound states are notoriously hard to find in field theory, so one must take a more modest approach by finding classical solutions of finite energy and bounded spatial extent. This may be a good approximation, since, for example, in the dual model important features such as leading trajectories and degeneracy of states are semiclassical in nature. Such an approach has already been advocated by Nielsen and Olesen,[1] who consider the vortex solutions of the relativistic extension of the Landau-Ginzburg Lagrangian for superconducting metals[2] as an approximation to the dual string.

In the first two papers of this series,[3,4] we addressed ourselves to the formal problems of quantizing such classical solutions, and presented a simple two-dimensional model for an extended hadron using these techniques. We now wish to exhibit a four-dimensional model involving non-Abelian Yang-Mills fields. The superconducting vortex of Ref. 1 used Abelian fields and so generated a vortex line of infinite extent. To get rid of the end-point problem, one has to fix the ends of the vortex string, since there is an action principle that says that the action of the vortex is proportional to the area of the surface swept out by the vortex in space-time. So, if one works with Abelian gauge fields, the string must be ended on magnetic monopoles or closed on itself in a rotating loop to prevent collapse to a point. Giving the model more freedom by introducing non-Abelian gauge fields allows the possibility of a new object, an extended object in the spatial dimensions which closes on itself, in effect, a ball solution to the equations of a non-Abelian superconductor. Just as in the two-dimensional model of Ref. 4 and the superconducting vortex, the existence of a spontaneously broken symmetry is crucial for the existence of these classical solutions.

In Ref. 4 we studied the classical kinklike solution of the two-dimensional Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_t\varphi)^2-\tfrac{1}{2}(\partial_x\varphi)^2+\tfrac{1}{2}m^2\varphi^2-\tfrac{1}{4}\lambda\varphi^4\,. \tag{1}$$

A striking feature of a kinklike solution is that $\langle\varphi\rangle$ undergoes a change of sign as it goes through the kink, which is thus the boundary between the two possible vacuum values $\langle\varphi\rangle=\pm m/\sqrt{\lambda}$. To extend such a feature to two space dimensions, for example, one could in (1) make φ complex, i.e., two real components, and look for a solution of the type $e^{in\theta}|\phi(r)|$, where $|\phi(r)|\to_{r\to\infty}m/\sqrt{\lambda}$, n an integer. However, one finds after a straightforward calculation that the energy of such a solution is infinite, due to divergent contributions at very large r's. This is related to the existence of a massless particle in such a theory [the Goldstone boson of the U(1) symmetry $\phi\to e^{i\omega}\varphi$].

One could expect that eliminating such massless scalar particles would cause the energy of a bounded classical solution to be finite. Indeed, the Higgs Lagrangian which also appears in the Landau-Ginzburg model of superconductivity does exactly this. The gauge field A behaves as shown in Fig. 1. We refer the reader to any standard textbook on superconductivity for the study of the classical solutions, and turn to four space-time dimensions.

We look for time-independent classical solutions with finite energy for an SU(2) Yang-Mills field coupled to scalar mesons with spontaneous symmetry breaking. Although we shall call it isospin, this internal-symmetry group would be more appropriate as a model for "color" symmetry. An extension of the model to SU(3) could be made using the work of Wu and Wu.[5] We require the symmetry breaking to be such that no color gauge meson will remain massless. This is best

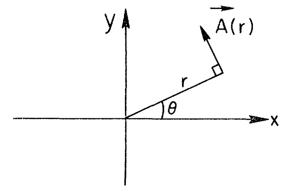

FIG. 1. Behavior of gauge field A.

achieved by using a scalar isospinor as first introduced by 't Hooft.[6]

We consider the Lagrangian

$$\mathcal{L}=-\tfrac{1}{4}(G^a_{\mu\nu})^2-\tfrac{1}{2}(D_\mu K^*)(D_\mu K)+\tfrac{1}{2}\mu^2K^*K-\tfrac{1}{4}\lambda(K^*K)^2\,, \tag{2}$$

where

$$G^a_{\mu\nu}=\partial_\mu W^a_\nu-\partial_\nu W^a_\mu+e\epsilon_{abc}W^b_\mu W^c_\nu\,,\qquad D_\mu=\partial_\mu-\tfrac{1}{2}ieW^a_\mu\tau^a\,, \tag{3}$$

where τ^a are the usual Pauli matrices. In the usual treatment of this Lagrangian, the scalar field K is translated by the constant $\binom{1}{0}\mu/\sqrt{\lambda}$, and after translation, all the W gauge mesons have the same nonzero mass; a global SU(2) symmetry remains.

We now look for static solutions of the classical equations of motion such that the direction of the breaking of the local gauge group varies with space. First, it is certainly consistent to choose W^a_0 to be zero. There remain nine gauge fields W^a_i, $i=1,2,3$. The solution is purely "magnetic." Since we want to avoid privileged directions in both ordinary space and isospin space, the natural thing to do is start from the Wu and Yang[7] ansatz

$$W^a_i=\epsilon_{iak}\frac{x^k}{r}g(r)\,,\quad r=(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2}\,,$$

which, for the Yang-Mills field alone, is compatible with the equations of motion. One must now find the spatial dependence of the scalar field K. It is not hard to realize that the choice

$$K=i\frac{\tau\cdot x}{r}\begin{pmatrix}1\\0\end{pmatrix}f(r)\,, \tag{4}$$

where f is a real function of r is also compatible with the equations of motion and the choice for W^a_i. One finds, after some algebra, that these choices give the Lagrangian density

$$\mathcal{L}=-g'^2-\left(\frac{2}{r}gg'+\frac{1}{r^2}g^2\right)-\frac{2}{r^2}g^2-\frac{2}{r}eg^3-\frac{e^2}{2}g^4-\tfrac{1}{2}f'^2-f^2\left(\frac{1}{r}+\frac{e}{2}g\right)^2+\tfrac{1}{2}\mu^2f^2-\tfrac{1}{4}\lambda f^4\,, \tag{5}$$

where we use primes to indicate the operation d/dr. The Hamiltonian becomes

$$H=4\pi\int_0^\infty r^2dr\left[g'^2+\frac{2}{r^2}g^2+\frac{2}{r}eg^3+\frac{e^2}{2}g^4+\tfrac{1}{2}f'^2+f^2\left(\frac{1}{r}+\frac{e}{2}g\right)^2-\tfrac{1}{2}\mu^2f^2+\tfrac{1}{4}\lambda f^4\right]\,. \tag{6}$$

Minimizing this Hamiltonian yields coupled second-order nonlinear equations for f and g:

$$g''+\frac{2}{r}g'-\frac{2}{r^2}g-\frac{3}{r}eg^2-e^2g^3-\frac{e}{2r}f^2-\frac{e^2}{4}f^2g=0\ ,$$
$$f''+\frac{2}{r}f'-\frac{2}{r^2}f-\frac{2e}{r}fg-\frac{e^2}{2}fg^2+\mu^2f-\lambda f^3=0\ . \tag{7}$$

When these equations are satisfied, K and W satisfy the field equations obtained from (2). Equations (7) cannot be integrated analytically but can be easily studied on a computer. We will need boundary data at $r=0$ and $r=\infty$ and they follow from requiring the energy to be finite. They are $f(0)=g(0)=0$ and $f(\infty)=\mu/\sqrt{\lambda}$, $g(\infty)=0$. One finds empirically that there is only one such solution which is displayed in Fig. 2. For this solution $g(r\to\infty)\sim -2/er$. Asymptotically the solution approaches that found by Wu and Yang. It is a pure gauge: The energy density vanishes exponentially as r goes to infinity, as it should since we have no massless fields. This is, of course, in contrast to 't Hooft's magnetic-monopole solution,[8] for which $g\sim -1/er$ and the energy density does not vanish exponentially.

By the rescaling

$$g\to\frac{\mu}{e}g\ ,\ f\to\frac{\mu}{e}f\ ,\ r\to\frac{r}{\mu}$$

H can be rewritten as

$$H=4\pi\frac{\mu}{e^2}\int_0^\infty r^2dr\left[g'^2+\frac{2}{r^2}g^2+\frac{2}{r}g^3+\tfrac{1}{2}g^4+\tfrac{1}{2}f'^2+f^2\left(\frac{1}{r}+\tfrac{1}{2}g\right)^2-\tfrac{1}{2}f^2+\tfrac{1}{4}\alpha f^4\right]\ , \tag{8}$$

with $\alpha=\lambda/e^2$.

For any r, $f(r)$ and $g(r)$ are now of order unity. Hence the mass of the classical solution is of order μ/e^2. This is a general feature of classical solutions. If we want the mass to be small then $e^2\gg 1$ and handling the quantization in a systematic fashion will require formal progress in computing strong-coupling limits.

Before adding fermions to the model, we want to discuss the stability properties of our solution. There are two possible kinds of stability, one we call topological and the other is the usual energetic one. For the Lagrangian (1) the kink is manifestly stable for topological reasons: Its decay would require flipping the value of $\langle\varphi\rangle$ from $+m/\sqrt{\lambda}$ to $-m/\sqrt{\lambda}$ (or vice versa) over half of space, which is forbidden by an infinite-energy barrier. For the two-space dimensional case of the Landau-Ginzburg Lagrangian, stability analysis is made more complex by the existence of gauge transformations. However, the gauge transformation that would make the field φ real is $e^{-in\theta}$ which is singular at the origin: Any gauge transformation which goes to $e^{-in\theta}$ at infinity has to be singular at some finite distance. This is because U(1) is a multiply-connected group. Hence the infinite-energy argument applies again.

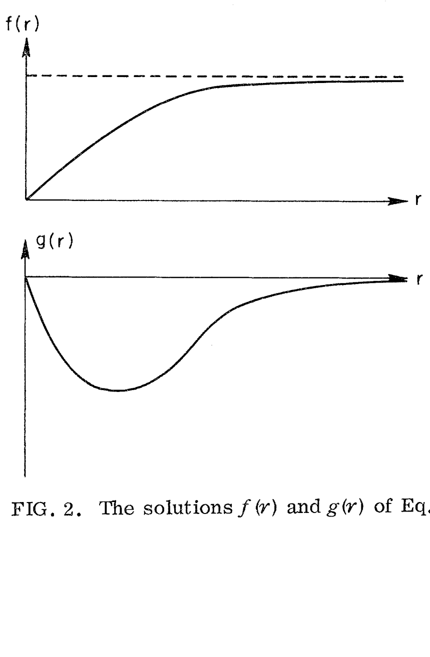

FIG. 2. The solutions $f(r)$ and $g(r)$ of Eq. (7).

The case of three space dimensions is quite different. We remark that one can multiply Eq. (4) by $\cos\omega/2+i(\tau\cdot x/r)\sin\omega/2$, where $\omega=\omega(r)$ is to be specified. This is a unitary matrix, hence an SU(2) gauge transformation. Choosing $\omega(r)$ to vanish fast enough at $r=0$ and to go to π exponentially as $r\to\infty$, one obtains a gauge transform which is continuous and differentiable everywhere. It undoes at infinity the complicated dependence of the isospin on space which appears in Eq. (4). One sees here the crucial topological difference between the circle symmetry of the superconducting vortex and the spherical one of the present case. Hence, even if our solution is classically stable against small oscillations, which we have not checked due to the complexity of such an analysis, it cannot be absolutely stable due to topological considerations, and can decay quantum mechanically by tunneling. Quantum-mechanical stability could only be achieved in the strong-coupling limit, where it could become lighter than the Yang-Mills and scalar bosons, a coherent state which is the lowest-energy state in the theory. This emphasizes the importance of developing strong-coupling approximations.

We now turn to the introduction of fermions. We will use four-component Dirac spinors, each component of which will be an isospin. Take γ_0 to be diagonal. It is straightforward to check that taking the "large" components to be

$$e^{-i\omega t}\begin{bmatrix}\begin{pmatrix}0\\ i\end{pmatrix}\\ \begin{pmatrix}-i\\ 0\end{pmatrix}\end{bmatrix}u(r)\,,\quad u\text{ real} \tag{9}$$

and the "small" components to be

$$e^{-i\omega t}\, i\,\frac{\tau\cdot x}{r}\begin{bmatrix}\begin{pmatrix}0\\ i\end{pmatrix}\\ \begin{pmatrix}-i\\ 0\end{pmatrix}\end{bmatrix}d(r)\,,\quad d\text{ real} \tag{10}$$

is compatible with the Dirac equation and the above choice (3) for the gauge field. We want to solve

$$(\partial^2+\tfrac{1}{4}e^2W^aW^a+\tfrac{1}{2}\tau^a\epsilon_{ijk}\sigma_k G^a_{ij})u=-(\omega^2-m^2)d \tag{11}$$

and get the radial equations

$$u''+\frac{2}{r}u'-e^2g^2u-(m^2-\omega^2)u-e\left(g'+\frac{2}{r}g\right)u=0\,,$$
$$d''+\frac{2}{r}d'-\frac{2}{r^2}d-e^2g^2d-(m^2-\omega^2)d+eg'd-\frac{2e}{r}gd=0\,, \tag{12}$$

with the connecting equation between the "up" and "down" components

$$d=\frac{1}{m+\omega}(-u'+egu)\,,$$
$$u=\frac{1}{m-\omega}\left(-d'-2\frac{d}{r}-egd\right)\,. \tag{13}$$

In particular, the fermion current produced by such a Dirac field is of the correct form to produce a Yang-Mills field of the Wu-Yang type. Thus, in the terminology of paper II we can "occupy" such fermion states and still have a self-consistent solution.

The contribution H_f of the fermion to the Hamiltonian is found to be

$$H_f=4\pi\int_0^\infty r^2dr[-2ud'+2egdu+(m-\omega)u^2-(m+\omega)d^2]\,, \tag{14}$$

where m is the fermion mass. The appropriate boundary data are now $u(\infty)=d(\infty)=0$,

$$u(0)=\text{constant},\ u'(0)=0\,,\ d(0)=0\,.$$

With these boundary conditions, the equations for u and d constitute an eigenvalue equation for ω. Therefore only a discrete set of fermion states can be occupied. The normalization on the eigenfunctions u and d is

$$4\pi\int_0^\infty r^2dr(u^2+d^2)=1 \tag{15}$$

for each occupied state. The coupled nonlinear equations for f, g, u, and d are sufficiently complex that we have not been able to study them in any detail. It seems likely, however, that the solution developed above for the W and K fields will serve as a "magnetic" well to trap fermions. We hope to return to the fermion case in a future publication.

This model can be taken as a prototype of quark confinement by very heavy color gauge mesons. Note that, because of the special dependence of isospin on space, our solutions have zero total isospin (or color). The symmetry-breaking scalar isospinor could be regarded as only a phenomenological entity, just as the Higgs scalar in the Landau-Ginzburg model is only an approximate description of composite Cooper pairs.

One of the authors (B.H.) would like to express his appreciation to Dr. Carl Kaysen for the hospitality extended to him by the Institute for Advanced Study.

APPENDIX

There are, of course, other more general solutions to the four-dimensional Yang-Mills gauge field, coupled to isospin scalars with spontaneous symmetry breaking. Some of these may have topological stability and we display them simply for completeness.

Allow fields of the form $W_i\equiv W_i^a\tau^a$. Define $\not{x}=\tau_a x_a$. Then the ansatz

$$W_i=\left(\frac{x_i}{r^2}\not{x}-\frac{1}{3}\tau_i\right)f(r)+\epsilon_{iak}\frac{x_k}{r}\tau_a g(r)+\tau_i h(r)$$

together with scalar fields of the form

$$K=\left[k_1(r)\frac{\not{x}}{r}+ik_2(r)\right]\begin{pmatrix}1\\ 0\end{pmatrix}$$

satisfy the equations of motion. The $(D_uK^*)(D_uK)$ piece of the Lagrangian gives

$$\tfrac{1}{2}k_1'^2+\tfrac{1}{2}k_2'^2+\frac{k_1^2}{r^2}-k_2'(\tfrac{1}{3}k_1f+\tfrac{1}{2}k_1h)+k_1'(\tfrac{1}{3}k_2f+\tfrac{1}{2}k_2h)+2\frac{k_1}{r}(\tfrac{1}{2}k_1g-\tfrac{1}{6}k_2f+\tfrac{1}{2}k_2h)-\tfrac{1}{4}(k_1^2+k_2^2)(\tfrac{1}{3}f^2+\tfrac{3}{2}h^2+g^2)\,,$$

while the $(G_{\mu\nu})^2$ piece becomes

$$\frac{3}{2}\left(\frac{g}{r}+g'+\frac{2}{g}f^2+\tfrac{1}{3}fh+h^2\right)^2+\frac{1}{2}\left(-g'+\frac{g}{r}+g^2+\tfrac{1}{3}f^2-fh\right)^2$$

$$+\left(\frac{g}{r}+g'-\frac{2}{g}f^2+\tfrac{1}{3}fh+h^2\right)\left(-g'+\frac{g}{r}+g^2+\tfrac{1}{3}f^2-fh\right)+\left(-\frac{f}{r}-\tfrac{1}{3}f'+h'-\tfrac{2}{3}fg-gh\right)^2 .$$

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