Nonrenormalizability of the quantized Dirac-Einstein system*

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The coupled Dirac-Einstein system is quantized and shown to be one-loop nonrenormalizable: The diagrams with eight external fermions yield a divergence proportional to the fourth power of the fermion axial-vector current. The gravitational variables required to couple gravitation to fermions are the (sixteen) *vierbein* fields possessing local Lorentz as well as coordinate invariance. For gravitation coupled to bosons, the *vierbein* and metric formulations remain equivalent at the quantum level.

I. INTRODUCTION

Quantization of general relativity, the gauge field theory of massless spin-two fields, can be investigated by applying the covariant quantization methods appropriate to gauge theories. These methods bypass the difficulties of operator ordering and constraints which arise in canonical quantization. The two approaches are equivalent for electrodynamics, as well as for Yang-Mills theory as far as they have been compared there, and it is therefore of interest, in obtaining concrete results, to apply the covariant scheme to Einstein theory. As has been shown, Einstein theory can indeed be covariantly quantized, and the resulting quantum theory is unitary.[1] Feynman rules can be given, and renormalizability of the theory can be investigated. The infinities arising in one-loop diagrams must be regularized. A particularly convenient method is dimensional regularization; although equivalent to earlier regularization schemes, it preserves Ward identities, and has the practical advantage that in it tadpoles vanish.

To date, renormalizability has been investigated only at the one-loop level, with the following results: (1) Source-free general relativity (with or without cosmological term) is one-loop renormalizable,[1,2] but only due to a peculiar degeneracy of four-dimensional Riemann space, through which an otherwise nonrenormalizable counterterm is equivalent to renormalizable ones. (2) Brans-Dicke theory is nonrenormalizable even in the absence of sources,[2] at least in the absence of Higgs-type effects for the scalar component. (3) The graviton-scalar field system is nonrenormalizable.[1] (4) Coupled Maxwell-Einstein theory is nonrenormalizable,[2] despite cancellations of all but one *a priori* nonrenormalizable terms.

Before concluding that quantum theory and general relativity are incompatible, at least in a perturbative treatment, one must investigate the coupling of fermions to gravitation. (If it were renormalizable, one might argue that, a world with only spin-$\frac{1}{2}$ particles as basic being satisfactory, integer spins might be composite.) In this article we carry out this program and reach the same negative conclusion of one-loop nonrenormalizability as was found in the other coupled systems. For convenience, we take the fermions to be massless, but still four-component, so as to retain the essentials of electrons. The neutrino-graviton system can be treated along the same lines with appropriate γ_5 insertions.

Fermions differ in one basic respect from integer-spin systems in their coupling to gravitation. As was shown by Cartan, one cannot couple them directly to the metric (a simplified proof is given in Appendix A). Instead, one may introduce, with Weyl, a set of sixteen *vierbein* fields $e^a{}_\mu$ ($a=1, \ldots, 4$) at each space-time point; spinors can then be introduced in the local Minkowski frames which define the $e^a{}_\mu$. In the absence of spinors the *vierbein* and metric formulations are equivalent in the classical domain because the *vierbein* fields always occur in "squares" (the *vierbein* field is essentially the matrix square root of the metric). It is in fact an interesting question, which we settle in the affirmative, whether this equivalence carries over to the quantum domain. The *vierbein* components are in general sixteen in number; in addition to their ten (metric) symmetric components, they have six antisymmetric components, expressing the freedom of homogeneous transformations of the local Lorentz frames, which seems to introduce additional dynamical content, especially at the quantum level. The theory has two kinds of gauge invariances: the usual coordinate freedom (under which fermions behave as scalars, being defined only with respect to the local frames), and the local Lorentz rotations (under which the fermions transform as ordinary spinors). Both gauges must be fixed in the covariant quantization scheme by adding gauge-breaking terms. The coordinate gauge is fixed as in the metric formulation, by choosing the deDonder or harmonic conditions; the term breaking local Lorentz invariance is the sum of the squares of the antisymmetric *vierbein* components. Gauge breaking is, as usual, accompanied by unitarity-restoring Faddeev-Popov ghosts for each invariance. We shall have to determine if and how the antisymmetric *vierbein* components and their ghost companions disappear from the quantized theory.

Our analysis of the divergences will also differ from that used in the previous cases. There, an algorithm[1] could be applied to obtain the counter-Lagrangian directly as a function of various types of coefficients of the original Lagrangian. The present case does not, however, satisfy the requirements under which the algorithm holds, and extension of the algorithm seems complicated. Instead, we shall calculate explicitly the coefficients of all counterterms which contain eight (external) fermion fields η and no derivatives. Only the Feynman diagrams with eight external fermions can contribute to these counterterms. These diagrams are logarithmically divergent, which facilitates the calculation of their divergences within the dimensional-regularization framework. Moreover, the external fields in the total counter-Lagrangian satisfy the coupled Dirac-Einstein equations, since they represent all possible tree graphs ending on the loop. But as the field equations always involve differentiated fermions, these η^8 counterterms do not become equivalent to others. Had we chosen any other counterterms (with 0, 2, 4, or 6 external matter fields), such equivalences would occur and would therefore require the calculation of more and higher divergent diagrams. As it turns out, there is only one counterterm with eight external fermion fields and no derivatives. Its coefficient provides an excellent criterion for renormalizability: it is necessary for renormalization that this coefficient vanishes, and sufficient for nonrenormalizability that it is nonzero.

Throughout this article, we use the background-field method[3,4] in which one sums over all one-loop diagrams with a given number of external matter lines and any number of external *vierbein* lines. In this way, our counter-Lagrangian is generally covariant in the external fields.

In Sec. II we summarize the relevant aspects of the *vierbein* approach, and show that the quantized *vierbein* field is equivalent to the quantized metric Einstein field in the absence of fermions. In Sec. III, the fermion-*vierbein* system is quantized and its nonrenormalizability established. Some conclusions are given in the last section.

II. *VIERBEIN* FIELDS WITHOUT FERMIONS

In this section we first review some properties of *vierbein* fields and then investigate whether their self-interactions are renormalizable.

Following Weyl's prescription,[5] we erect at each space-time point $x=X$ a locally inertial Lorentz frame with orthogonal axes $\xi^a_X(x)$, $a=1,2,3,4$; this is a so-called *Vierbein* or tetrad. If space-time is parametrized by the coordinate system x^μ, then the *vierbein* fields relate the Lorentz axes to the coordinate axes at each point

$$e^a{}_\mu(x=X)=\left(\frac{\partial \xi^a_X(x)}{\partial x^\mu}\right)_{x=X} . \tag{1}$$

It follows that under coordinate transformations $x^\mu \to \bar{x}^\mu(x)$, this field transforms as a covariant world vector, whereas under homogeneous local Lorentz transformations L, it transforms as a Lorentz vector

$$e^{a\prime}{}_\mu(\bar{x})=\frac{\partial x^\nu}{\partial \bar{x}^\mu}\, e^a{}_\nu(x), \quad e^{a\prime}{}_\mu(x)=L^a{}_b(x)e^b{}_\mu(x) . \tag{2}$$

Fermion fields ψ can be introduced into general relativity[5] by describing them with respect to local Lorentz frames[6,7]; they are defined to be world scalars and transform as ordinary spinors under local Lorentz transformations of the *vierbein* frames ("Lorentz spinors"). The ordinary derivative $\partial_\mu \psi$ is a covariant world vector, but not a proper Lorentz spinor; a covariant derivative D_μ can be introduced such that $D_\mu\psi$ is a covariant world vector and a Lorentz spinor,

$$D_\mu \psi=\partial_\mu \psi+\tfrac{1}{2}\sigma^{ab}\,\omega_{\mu ab}\,\psi . \tag{3}$$

The numerical matrices σ^{ab} are the generators of the Lorentz group belonging to a given representation R ($R=I+\frac{1}{2}\sigma^{ab}\lambda_{ab}$ for small λ if the Lorentz matrix is $L^a{}_b=\delta^a{}_b+\lambda^a{}_b$); for example, for a Lorentz vector and a Lorentz spinor one has respectively

$$\begin{aligned}(\sigma^{ab})^c{}_d&=\eta^{ac}\delta^b_d-\eta^{bc}\delta^a_d ,\\ \sigma^{ab}&=\tfrac{1}{4}(\gamma^a\gamma^b-\gamma^b\gamma^a) ,\end{aligned} \tag{4}$$

where the Minkowski metric η^{ab} is defined by $p^a p^b \eta_{ab}=\vec{p}^2-p_0{}^2$. The γ^a are the usual constant Dirac matrices, $\gamma^a\gamma^b+\gamma^b\gamma^a=2\eta^{ab}$, which commute with the covariant derivative, $D_\mu\gamma^a=\gamma^a D_\mu$. The next step is to determine the symbol $\omega_{\mu ab}$ in Eq. (3).

From the definition in Eq. (1) it follows that the *vierbein* field is a matrix square root of the metric $g_{\mu\nu}$; indeed, with $(ds)^2=d\xi^a d\xi^b\eta_{ab}$, one has

$$e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu} \; . \tag{5}$$

Lorentz indices (denoted by latin characters) are raised and lowered by the Minkowski metric η^{ab} and η_{ab}, coordinate indices (indicated by Greek characters) by $g^{\mu\nu}$ and $g_{\mu\nu}$, while change of index-type is effected by $e^a{}_\mu$. The spin connection $\omega_{\mu ab}$ can easily be determined by requiring that not only the operations of index raising and lowering, but also the operations of changing index-type commute with covariant differentiation. In the same way as the affinity-Christoffel symbol relation follows from $g_{ij;k}=0$, one finds, using Eq. (4), from

$$e^a{}_{\mu\,;\,\nu} = \partial_\nu e^a{}_\mu - \Gamma^\sigma_{\mu\nu} e^a{}_\sigma + \omega_\nu{}^a{}_b e^b{}_\mu = 0 \tag{6}$$

that

$$\omega_{\mu ab} = [e_a{}^\nu(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) + \tfrac{1}{2} e_a{}^\rho e_b{}^\sigma (\partial_\sigma e_{c\rho} - \partial_\rho e_{c\sigma}) e^c{}_\mu]_{[ab]} \; . \tag{7}$$

The last symbol denotes antisymmetrization in (a, b), and the matrix $e_a{}^\mu$ is the inverse of $e^a{}_\mu$. According to the foregoing rules of raising and lowering of indices we have thus

$$e^a{}_\mu e_{a\nu} = g_{\mu\nu}, \quad e^{a\mu} e_{b\mu} = \delta^a_b, \quad e^{a\mu} e_a{}^\nu = g^{\mu\nu} \; . \tag{8}$$

From its curl structure, it is clear that $\omega_{\mu ab}$ is a covariant vector; under local Lorentz transformations it is not a tensor, but acquires an inhomogeneous term which is needed to make $D_\mu \psi$ a Lorentz spinor, as one can easily verify.

The Lagrangian density for the *vierbein* field must be a coordinate scalar density and a Lorentz scalar. One possibility is the usual Einstein action

$$\mathcal{L}^V(e) = -eR(g(e)) = \mathcal{L}^E(g) , \tag{9}$$

using the fact that $(-g)^{1/2} = e$, the determinant of $e^a{}_\mu$, while $g_{\mu\nu}$ is to be expressed in terms of $e^a{}_\mu$ by Eq. (5). An apparently distinct choice comes from considering the commutator

$$[D_\mu, D_\nu]\psi = \tfrac{1}{2} R_{\mu\nu ab} \sigma^{ab} \psi , \tag{10}$$

with

$$R_{\mu\nu ab}(\omega) = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac} \omega_\nu{}^c{}_b - \omega_{\nu ac} \omega_\mu{}^c{}_b \tag{11}$$

and defining the Lagrangian as

$$\mathcal{L}^V(e) = -e\, e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) \; . \tag{12}$$

However, insertion of Eq. (7) into Eqs. (11), (12) gives back Eq. (9). The only tensor that can be constructed from the *vierbein* field $e^a{}_\mu$ and its first and second derivatives and which is linear in its second derivatives is the tensor in Eq. (11); a proof is given in Appendix B. Therefore, the Lagrangian in Eq. (9) is unique under these requirements.[8]

The Lagrangian is quantized in two steps. First, using the background-field formalism, we consider the fields $(\bar{e}, \bar{g})$ in the Lagrangian

$$\bar{\mathcal{L}}^V(\bar{e}) = -\bar{e}\kappa^{-2} R(\bar{g}(\bar{e})), \quad \kappa^2 = 16\pi\gamma \tag{13}$$

as sums of classical ("background") fields (e, g) and quantum fields (c, h) according to

$$\bar{e}^a{}_\mu = e^a{}_\mu + \kappa c^a{}_\mu, \quad \bar{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu} \, . \tag{14}$$

The factors κ have been inserted to give the quantum fields canonical dimension (our units are $\hbar = c = 1$ and γ is the Newtonian constant). Henceforth, these quantum fields c and h can be considered just like other matter fields, such as photons and fermions, to move in the background field e. With our rules of lowering, raising, and changing index-type we have thus, for example, $c_{\mu\nu} \equiv e_{a\mu} c^a{}_\nu$, and covariant derivatives are always taken with respect to the background field e. Both sets (e, g) and $(\bar{e}, \bar{g})$ satisfy Eq. (5), hence

$$h_{\mu\nu} = c_{\mu\nu} + c_{\nu\mu} + \kappa c^a{}_\mu c_{a\nu} \, , \tag{15}$$

which shows that to first order in quantum fields, the quantized metric field is equal to the quantized symmetric *vierbein* field. Next, we expand $\mathcal{L}^V(e+c)$ in quantum fields c about the background field e:

$$\bar{\mathcal{L}}^V(\bar{e}) = \kappa^{-2}\mathcal{L}^V(e) + \kappa^{-1}\mathcal{L}^V_1(e, c) + \mathcal{L}^V_2(e, c) + \kappa O(c^3) , \tag{16}$$

and find, omitting total derivatives

$$\mathcal{L}^V_1(e, c) = 2c^a{}_\mu G_a{}^\mu(e) , \tag{17}$$

where $G_a{}^\mu = e_{a\nu} G^{\nu\mu}$, as before, and $G^{\mu\nu}$ is the symmetric Einstein tensor $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$. If and only if $G^{\mu\nu}=0$, does $\mathcal{L}^V_1$ vanish. For the second variational derivative of $\bar{\mathcal{L}}^V$ with respect to c we find

$$\mathcal{L}^V_2(e, c) = e c^a{}_\mu c^b{}_\nu (\eta_{ab} G^{\mu\nu}) + \mathcal{L}^E_2(h, g) , \tag{18}$$

where the term explicitly quadratic in c comes from the $h \sim cc$ part of Eq. (15) when inserted in $(\delta\mathcal{L}^E/\delta g_{\mu\nu}) h_{\mu\nu}$, while the last term is just the second variation with respect to g. In the latter, it is sufficient to replace h by its linear part $(c_{\mu\nu} + c_{\nu\mu})$. The cc term may be dropped altogether when the background field satisfies the classical equations, $G_{\mu\nu}=0$, as we assume from now on. Were integer spin sources also present, the combined action $I^V + I^M$ would still depend on the *vierbein* fields only through the combination $\bar{g}_{\mu\nu}(\bar{e})$; the cc term would have the coefficient $(G_{\mu\nu} + \frac{1}{2} T_{\mu\nu})$ and would therefore still vanish by the field equations resulting from $\mathcal{L}^V_1 + \mathcal{L}^M_1$. The contributions from $\mathcal{L}^M$ to $O(hh)$ then also depend only on the combination

$(c_{\mu\nu}+c_{\nu\mu})$. Thus, the *vierbein* and metric Lagrangians are also equivalent there. We quote[1] the form of $\mathfrak{L}_2^E(h)$:

$$\mathfrak{L}_2^E(g,h)=(-g)^{1/2}\left[-\tfrac{1}{4}h_{\alpha\beta;\mu}{}^2+\tfrac{1}{8}h_{;\alpha}{}^2+\tfrac{1}{2}(h_\mu-\tfrac{1}{2}h_{;\mu})^2+\tfrac{1}{2}h_{\alpha\beta}X_g^{\alpha\beta\mu\nu}h_{\mu\nu}\right]. \tag{19}$$

For convenience, we have defined $h_\mu \equiv h_\mu{}^\nu{}_{;\nu}$ and $h\equiv h_{\alpha\beta}g^{\alpha\beta}$, and recall that all covariant derivatives are with respect to the background fields g or e. The vertex X_g is given by

$$X_g^{\alpha\beta\mu\nu}=(R^{\alpha\mu\beta\nu}+g^{\alpha\beta}R^{\mu\nu}-g^{\alpha\mu}R^{\beta\nu}+P^{\alpha\beta\mu\nu}R), \tag{20}$$

where symmetrization in each pair $(\alpha\beta)$, $(\mu\nu)$, and under pair exchange is to be performed. The projection operator P will occur at later stages,

$$P^{\alpha\beta\mu\nu}=\tfrac{1}{4}(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}-g^{\alpha\beta}g^{\mu\nu}). \tag{21}$$

The second step in the quantization of the *vierbein* field is to deal with the invariances of the theory. The total action $\int dx\, \mathfrak{L}^V(\bar{e})$ possesses two separate invariances, under local Lorentz and under coordinate transformations. Restating these invariances in terms of gauge transformations on the quantum fields, the total Lagrangian $\bar{\mathfrak{L}}^V$ is invariant under the differential gauge transformations

$$\begin{aligned}c^a{}_\mu(x)&\to c^a{}_\mu(x)+\eta^\alpha{}_{;\mu}(e^a{}_\alpha+\kappa c^a{}_\alpha)+\eta^\alpha\kappa c^a{}_{\mu;\alpha}\\&\quad-\eta^\alpha\omega_\alpha{}^a{}_b(e^b{}_\nu+\kappa c^b{}_\nu),\\ e^a{}_\mu(x)&\to e^a{}_\mu(x)\end{aligned} \tag{22}$$

and under the algebraic gauge transformation

$$\begin{aligned}c^a{}_\mu(x)&\to c^a{}_\mu(x)+\lambda^a{}_b(x)(e^b{}_\mu+\kappa c^b{}_\mu),\\ e^a{}_\mu(x)&\to e^a{}_\mu(x).\end{aligned} \tag{23}$$

Note that only $\bar{\mathfrak{L}}^V$, but not $\mathfrak{L}_2^V$, is invariant under (22); $\mathfrak{L}_2^V$ *is* invariant under the homogeneous part of the transformation of Eq. (23). According to the Faddeev-Popov prescription, we must break these invariances by adding gauge-breaking terms to the Lagrangian, which in turn will introduce ghost particles. We fix the four coordinate gauges to be the usual deDonder (harmonic) gauges by adding

$$\mathfrak{L}^C=-\tfrac{1}{2}e\,(h_\mu-\tfrac{1}{2}h_{;\mu})^2. \tag{24}$$

Because $\mathfrak{L}^C$ is quadratic in quantum fields, only the linear part $c_{\mu\nu}+c_{\nu\mu}$ of $h_{\mu\nu}$ need be substituted in $\mathfrak{L}^C$. The local Lorentz invariance is broken by adding the purely algebraic term[9,10]

$$\mathfrak{L}^L=-\tfrac{1}{2}e\kappa^{-2}a_{\mu\nu}{}^2,\quad a_{\mu\nu}\equiv c_{\mu\nu}-c_{\nu\mu}. \tag{25}$$

This then is the only place where $a_{\mu\nu}$ occurs in the action in the absence of fermions. Note that $c_{\mu\nu}\equiv c_{a\mu}c^a{}_\nu$ transforms differently in its first and second index under Eqs. (22), (23). The ghost Lagrangian $\mathfrak{L}^G$ is obtained, as usual, by casting the gauge-breaking terms $\mathfrak{L}^C$ and $\mathfrak{L}^L$ in the form $-\frac{1}{2}\Gamma_i{}^2$, with Γ_i given by

$$\begin{aligned}\Gamma^C_a&=e^{1/2}e_a{}^\mu(h_\mu-\tfrac{1}{2}h_{;\mu}),\\ \Gamma^L_{[bd]}&=e^{1/2}e_b{}^\mu e_d{}^\nu(a_{\mu\nu}),\end{aligned} \tag{26}$$

and subjecting them to the ten gauge transformations in Eq. (22), (23) we find[11]

$$\mathfrak{L}^G=e[\zeta^*_\mu(D_\nu D^\nu\zeta_\mu-R_\mu{}^\nu\zeta_\nu)+\kappa^{-2}\vartheta^*_{\mu\nu}\vartheta_{\mu\nu}+\kappa^{-1}\vartheta^*_{\mu\nu}D_\nu\zeta_\mu] \tag{27}$$

after some irrelevant redefinitions of the fields ζ^* and ϑ^*; the indices of ζ and ϑ are some internal degrees of freedom and the noncovariance in *these* indices does not bother us. Since no conjugate vertex of the form $\zeta^*\vartheta$ is present, we may drop the term $\vartheta^*\zeta$ in Eq. (27), as it alone is insufficient for a closed-loop diagram containing ϑ and ζ fields. Closed-loop diagrams with a ghost consist therefore of purely ϑ or ζ loops.

The part of the total Lagrangian which contributes to one-loop diagrams consists of a ghost and a nonghost part. For the nonghost Lagrangian we have from Eqs. (18), (19), (24), and (25) and using the classical field equations $G_{\mu\nu}(g)=0$ to drop $\mathfrak{L}_1$,

$$\begin{aligned}\mathfrak{L}^{\rm NG}&=\mathfrak{L}_2+\mathfrak{L}^C+\mathfrak{L}^L\\&=e\,(-\tfrac{1}{2}h_{\alpha\beta;\mu}P^{\alpha\beta\rho\sigma}h_{\rho\sigma;\nu}g^{\mu\nu}+\tfrac{1}{2}h_{\alpha\beta}X_g^{\alpha\beta\rho\sigma}h_{\rho\sigma}\\&\quad-\tfrac{1}{2}\kappa^{-2}a_{\mu\nu}{}^2).\end{aligned} \tag{28}$$

We recall that it suffices to set, in $\mathfrak{L}^{\rm NG}$, $h_{\mu\nu}=c_{\mu\nu}+c_{\nu\mu}\equiv s_{\mu\nu}$. It follows that the symmetric *vierbein* and metric forms of $\mathfrak{L}^{\rm NG}$ coincide, having the same propagators and vertices, as do their respective ghosts.

This leaves the antisymmetric *vierbein* fields $a_{\mu\nu}$, and their corresponding ghost $\vartheta_{\mu\nu}$, neither of which propagates. Both are coupled (only) to e, but it is clear that they cancel in every diagram in which they appear: Each (being decoupled from other quantum fields) can only enter in a loop with no other internal particles but itself, and their contributions have opposite sign. (They also vanish separately, since diagrams $\sim\kappa^{2n}\int d^4p$ vanish in dimensional regularization.) Thus, the *vierbein* and metric formulation are identical; this is perhaps not surprising, since we took an action depending on the set of variables $h_{\mu\nu}$ and arbitrarily made the algebraic field redefinition in Eq. (15), which should not change physical results. Nevertheless, this exercise will be of use in Sec. III where fermions are considered, since $a_{\mu\nu}$ appears explicitly in the Dirac Lagrangian.

III. COUPLED FERMION-*VIERBEIN* FIELDS

The Lagrangian density for a massless spin-$\frac{1}{2}$ fermion in a gravitational field is given by the sum of the Dirac and Einstein Lagrangian densities

$$\mathcal{L}(\bar{e},\psi) = -\bar{e}\kappa^{-2}R(\bar{g}) - \bar{e}\bar{\psi}\gamma^a e_a{}^\mu D_\mu \psi , \tag{29}$$

where $\bar{g}$ is to be expressed in terms of $\bar{e}$ by Eq. (5), and the covariant derivative D is defined by Eq. (3). $\mathcal{L}$ is a world scalar density and a Lorentz scalar. Again, the fields $(\bar{e},\psi)$ are sums of background and quantum fields:

$$\bar{e}^a{}_\mu = e^a{}_\mu + \kappa c^a{}_\mu , \quad \psi = \kappa^{-1}\eta + \psi . \tag{30}$$

As in previous cases,[2] the factor κ^{-1} has been inserted in order that there will be no explicit κ dependence in the one-loop $\mathcal{L}_2$ [except in $\mathcal{L}^L$ in Eq. (25)] nor in $\Delta\mathcal{L}$ (except when $\mathcal{L}^L$ contributes). We expand $\mathcal{L}(e+\kappa c, \kappa^{-1}\eta+\psi)$ in quantum fields (c,ψ) around the background fields (e,η). For the first variational derivative, one has, omitting total divergences,

$$\mathcal{L}_1(e,c;\eta,\psi) = \kappa^{-1}e\{c^a{}_\mu[2G_a{}^\mu(e) + T_a{}^\mu(e,\eta)] - \bar{\eta}\gamma^\mu D_\mu(e)\psi - \bar{\psi}\gamma^\mu D_\mu(e)\eta\} . \tag{31}$$

The symbol D is from now on always with respect to e, and $\gamma^\mu = e_a{}^\mu\gamma^a$. $\mathcal{L}_1$ vanishes if and only if the classical field equations are satisfied by the background fields, namely,

$$G_{\mu\nu} = -\tfrac{1}{2}T_{\mu\nu}, \quad \gamma^\mu D_\mu \eta = (D_\mu\bar{\eta})\gamma^\mu = 0 . \tag{32}$$

The Einstein equations consist of a symmetric part, $G_{\mu\nu} = -\frac{1}{2}(T_{\mu\nu}+T_{\nu\mu})$ and an antisymmetric part, $T_{\mu\nu} - T_{\nu\mu} = 0$. The fermion stress tensor, $T_a{}^\mu = -\delta\mathcal{L}^D/\delta e^a{}_\mu$, is

$$T^a{}_\mu = -\bar{\eta}\gamma^a D_\mu\eta + \delta^a_\mu \bar{\eta}\gamma^\nu D_\nu\eta + \tfrac{1}{2}D_\lambda(\bar{\eta}\gamma^\lambda\sigma^{\mu a}\eta) + \tfrac{1}{2}D_\alpha[\bar{\eta}(\gamma^\mu\sigma^{a\alpha} + \gamma^a\sigma^{\mu\alpha})\eta] \tag{33}$$

and is *a priori* nonsymmetric, but it becomes symmetric, conserved, and traceless as a consequence of the Dirac equation $\gamma^\mu D_\mu\eta = 0$, which reduces it to the usual expression $T_{\mu\nu} = -\frac{1}{4}\bar{\eta}(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu)\eta$. Tracelessness is a consequence of the scale invariance of the massless Lagrangian.[12]

Using the results of Sec. II for the Einstein part, the total second variation $\mathcal{L}_2$ becomes, including gauge-breaking terms,

$$\mathcal{L}_2^{\text{tot}} = \mathcal{L}_2^{\text{NG}} + \mathcal{L}^G - e\bar{\psi}\gamma^\mu D_\mu\psi + \mathcal{L}^H(s,\psi,\eta) + \mathcal{L}^S(s,\eta) + \mathcal{L}^A , \tag{34}$$

where $\mathcal{L}_2^{\text{NG}}$ and $\mathcal{L}^G$ were defined in Eqs. (27), (28); $\mathcal{L}^H$ contains one quantized fermion and one quantized symmetric *vierbein* field $s_{\mu\nu} = c_{\mu\nu} + c_{\nu\mu}$, while $\mathcal{L}^S$ is quadratic in $s_{\mu\nu}$ and $\mathcal{L}^A$ contains at least one antisymmetric $a_{\mu\nu}$ field. For $\mathcal{L}^H$, one finds

$$\mathcal{L}^H = e(\bar{\psi}\gamma^m\sigma^{ab}\eta + \bar{\eta}\gamma^m\sigma^{ab}\psi)(\partial_a s_{bm}) - e(\bar{\eta}\gamma^m\partial_\mu\psi + \bar{\psi}\gamma^m\partial_\mu\eta)(s^\mu_m - \delta^\mu_m s_\lambda{}^\lambda) , \tag{35}$$

and for $\mathcal{L}^S$, omitting terms with $\partial_\mu\eta$ for reasons to be explained, one finds

$$\mathcal{L}^S = -e(\bar{\eta}\gamma^m\sigma^{ab}\eta) \times (s_{am}\partial^\rho s_{b\rho} + s_\lambda{}^\lambda\partial_b s_{am} + \tfrac{1}{2}s_b{}^\rho\partial_m s_{a\rho}) . \tag{36}$$

We have now that part of the total Lagrangian which contributes to one-loop diagrams, and if we could reduce its kinetic term to the form

$$\partial_\mu\phi_i W^{\mu\nu}_{ij}\partial_\nu\phi_j , \quad W^{\mu\nu}_{ij} = g^{\mu\nu}F_{ij} , \tag{37}$$

then we could apply the lemma of Ref. 1, and write down the counter-Lagrangian. It has been shown[13] that for fermions with nonderivative couplings the substitution $\psi \to -i\gamma_\mu\partial_\mu\xi$ does indeed lead to a kinetic term of the form of Eq. (37). However, in our case the fermions have derivative coupling to the *vierbein* fields, and even after the $\psi\to\xi$ substitution the matrix W does not factorize as in Eq. (37). The lemma is not applicable, and we have therefore adopted another strategy.

Consider the class of one-loop diagrams with $2n$ external fermion lines. If only symmetric *vierbein* lines are propagated in the loops, these diagrams do not depend on κ and are finite for $n>4$, logarithmically divergent for $n=4$, and diverge more rapidly for $n<4$. This follows from Eqs. (35), (36) and simple power counting. To what terms in the counter-Lagrangian can such diagrams contribute? The list of all *a priori* possible, Lorentz scalar and world scalar density terms of canonical dimension four is long; to quote a few,

$$\Delta\mathcal{L} = (e/\epsilon)\Big[\alpha_1 R_{\mu\nu}{}^2 + \alpha_2 R^2 + \alpha_3(\bar{\eta}\gamma_\mu D_\nu\eta)^2 + \alpha_4(\bar{\eta}\gamma^\mu D_\mu\eta)^2 + \alpha_5 R_{\mu\nu\rho\sigma}(\bar{\eta}\sigma^{\mu\nu}\eta)(\bar{\eta}\sigma^{\rho\sigma}\eta) + \alpha_6 R^{\mu\nu}(\bar{\eta}\gamma_\mu D_\nu\eta) + \alpha_7\prod_{i=1}^{4}(\bar{\eta}F_i\eta) + \alpha_8(\bar{\eta}F_1\eta)^2(\bar{\eta}\gamma_\mu D_\nu\eta) + \alpha_9\kappa^2(\bar{\eta}D_\mu\eta)^2(\bar{\eta}F\eta)^2 + \cdots\Big] . \tag{38}$$

The matrices F_i contain γ matrices, but no derivatives. On the mass shell, $T_{\mu\nu}\sim\bar{\eta}\gamma_\mu D_\nu\eta$ and $R_{\mu\nu} = -\frac{1}{2}T_{\mu\nu}$, $R=0$ while $\gamma^\mu D_\mu\eta = 0$; hence, the terms with α_2 and α_4 vanish, while α_1, α_3, and α_6 are equivalent. Explicit κ^2-dependent terms such as α_9 can only come from loops involving $a_{\mu\nu}$ propagators $\sim\kappa^2$. They can never be reduced to κ^2-independent ones because the field equations do not involve κ.

Our strategy is now to focus on those loops having eight external fermions and no internal antisymmetric *vierbein* fields, hence no dependence on κ and no derivatives.[14] They, and only they, contribute, by dimensions, to terms of the type α_7 (further matter-line attachments would lead to convergent terms, while use of the field equations cannot mix α_7 with other terms, since the field equations always involve derivatives). In our gauge, ghosts do not couple to external fermions, and so do not contribute to α_7. Finally, as we saw, the $a_{\mu\nu}$-dependent parts of $\mathcal{L}_2^{\mathrm{NG}}$ are $\sim\kappa^2$ and can be disregarded. This attack is useful only if α_7 is nonzero, for otherwise one would have to analyze the more complicated terms to settle the renormalizability issue.

We now outline the calculation, starting from the three-point and the four-point (seagull) vertices (V_3, V_4) defined by $\mathcal{L}^H$ and $\mathcal{L}^S$ of (35) and (36). In each, the *vierbein* fields are internal (and only the symmetric ones are needed), while both seagull fermions are external, as is one of the V_3 fermions. The second fermion in V_3 is then part of an internal fermion propagator S_F, which is necessarily connected to a similar V_3 to produce a net S-matrix insertion ("Compton insertion") of the form $S^H \sim V_3 S_F V_3$, in which the incoming, internal, and outgoing fermions form an uninterrupted line. Each η^8 diagram then consists of a box diagram with four internal *vierbein* propagators into which are inserted all possible permutations of four seagull and Compton insertions, as exemplified in Fig. 1. Let us now count powers of the internal momentum p. The four graviton propagators contribute p^{-8}, while each seagull always has one derivative (p) on one of the emerging graviton legs. Compton insertions also have one net p, since each V_3 has one derivative $[S^H \sim V_3 S_F V_3 \sim p(-\not{p}/p^2)p \sim p]$. Thus every diagram is at most logarithmically divergent $(\sim\int d^4p\, p\, p^4/p^8)$. But this means that we can ***neglect all explicit external momenta*** k_i in the diagram, since they clearly lead to finite integrals when in the numerator, and likewise no denominator translation need be performed so that we may just write $(p^{-2})^4$ for the four internal graviton lines. At the end, we need merely average the four numerator momenta $p_\alpha p_\beta p_\gamma p_\delta$ giving the usual product of Kronecker deltas. A further advantage is that we may drop

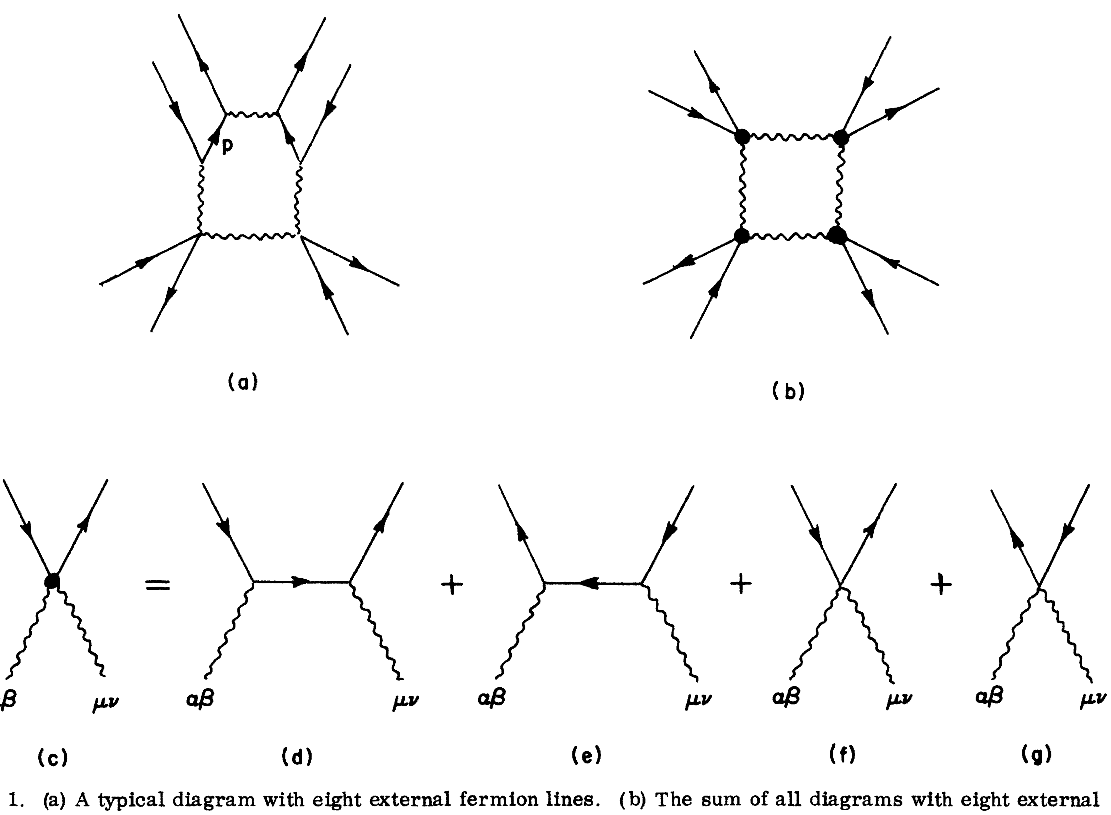

FIG. 1. (a) A typical diagram with eight external fermion lines. (b) The sum of all diagrams with eight external fermion lines in terms of the continuous fermion-line insertion. (c) The continuous fermion-line insertion as a sum of Compton [diagrams (d), (e)] and seagull [diagrams (f), (g)] insertions. Diagrams (f) and (g) differ topologically, owing to their momentum dependence.

those terms in $\mathcal{L}^H$, $\mathcal{L}^S$ in which the derivatives act on η, as they give convergent terms. This includes the $\bar\psi\gamma^m\partial_\mu\eta$ part of (35) and a term $\bar\eta\gamma^m\partial_\mu\eta$ [see Eq. (36)]. The $\bar\eta\gamma^m\partial_\mu\psi$ part of $\mathcal{L}^H$ remains, of course, since ψ carries internal momentum p. Despite the absence of its adjoint, it will contribute to the Compton insertion $V_3S_FV_3$ when contracted with the appropriate $(\bar\psi\gamma\sigma\eta)$ part of the other term in V_3.

We shall next define a net $\bar\eta\eta$ two-external fermion insertion which combines both Compton and seagull insertions [Fig. 1(c)]; it depends on two pairs of graviton indices to be contracted with two graviton propagators. It is straightforward to obtain for the Compton part[15]

$$S^H_{b_2\mu_2;b_1\mu_1}=(\tfrac{1}{16})\bar\eta(t_{b_2\mu_2}\not p t_{b_1\mu_1}-t_{b_1\mu_1}\not p t_{b_2\mu_2})p^{-2}, \tag{39}$$

where

$$t_{b\mu}\equiv\gamma_b p_\mu+\gamma_\mu p_b\ . \tag{40}$$

S^H is not only symmetric in each of its graviton pairs, but antisymmetric in pair interchange. The latter property results from crossing, since change of internal momentum flow direction (which changes the sign of p) corresponds to interchange of the two attached gravitons. Similarly, we find for the seagull contribution (plus its crossed partner)

$$S^S_{b_2\mu_2;b_1\mu_1}=(-\tfrac{1}{16})\bar\eta(q_{b_2b_1}\eta_{\mu_2\mu_1}+q_{\mu_2b_1}\eta_{b_2\mu_1}+q_{b_2\mu_1}\eta_{\mu_2b_1}+q_{\mu_2\mu_1}\eta_{b_2b_1}), \tag{41}$$

with

$$q_{b\mu}\equiv\gamma_b\not p\gamma_\mu-\gamma_\mu\not p\gamma_b=-q_{\mu b}\ . \tag{42}$$

Note that $q_{b\mu}$ is conserved ($p_b q_{b\mu}=0$). The symmetries of S^S are the same as those of S^H. The two-fermion insertion of Fig. 1(c) is thus given by the simple form

$$S^T_{b_2\mu_2;b_1\mu_1}\equiv(S^H+S^S)_{b_2\mu_2;b_1\mu_1}=(-\tfrac{1}{16})\bar\eta(q_{b_2b_1}\Pi_{\mu_2\mu_1}+q_{\mu_2b_1}\Pi_{b_2\mu_1}+q_{b_2\mu_1}\Pi_{\mu_2b_1}+q_{\mu_2\mu_1}\Pi_{b_2b_1})\eta, \tag{43}$$

where $\Pi_{ab}=\Pi_{ba}$ is the transverse ($p_a\Pi_{ab}=0$) projection operator

$$\Pi_{ab}=\eta_{ab}-p_ap_bp^{-2},\quad \Pi_{ab}\Pi^b{}_c=\Pi_{ac},\quad \Pi^a{}_a=3\ . \tag{44}$$

Note that $\Pi_a{}^b q_{bc}=q_{ac}$. A useful property of each S is its tracelessness

$$S^T_{b_2\mu_2;\lambda\lambda}=S^T_{\lambda\lambda;b_1\mu_1}=0 \tag{45}$$

in each pair of graviton indices, which is a direct consequence of the antisymmetry and conservation of q; it is also conserved on each of its indices. Since the graviton propagator $D_{\mu\nu,\alpha\beta}(p)$ has the form $p^{-2}(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha}-\delta_{\mu\nu}\delta_{\alpha\beta})$, we see that its last term will never contribute, while the other two are equivalent by the symmetry in each pair of indices of S^T. Thus, the total diagram, which has the matrix structure $S^TDS^TDS^TDS^TD$, is proportional to the simple expression[16]

$$\Delta\mathcal{L}=\int d^4p\,p^{-8}\operatorname{tr}[S^T(p)]^4\ . \tag{46}$$

We evaluate this trace in two steps (dropping over-all numerical factors henceforth). First we take the matrix (S^TS^T). It has the form

$$(S^T)^2{}_{b_3\mu_3;b_1\mu_1}=S^T_{b_3\mu_3;b\mu}S^T_{b\mu;b_1\mu_1}=(\bar\eta q_{b_3b}\eta)(\bar\eta q_{bb_1}\eta)\Pi_{\mu_3\mu_1}+\cdots, \tag{47}$$

where the appropriate symmetrization in each pair is to be taken.

Finally, we take the trace of $(S^T)^2(S^T)^2$. From the properties of q and Π given above, it is clear that its form is

$$\operatorname{tr}(S^T)^4=\alpha\operatorname{tr}q^4+\beta(\operatorname{tr}q^2)^2, \tag{48}$$

where $\operatorname{tr}q^2\equiv q_{\alpha\beta}q_{\beta\alpha}$ and $\operatorname{tr}q^4\equiv q_{\alpha\beta}\cdots q_{\delta\alpha}$. Explicit calculation yields $\alpha=11$, $\beta=3$. We can find the exact structure of (48) rather easily by invariance arguments. First note that each q_{ab} of Eq. (42) can be written as

$$\bar\eta q_{ab}\eta=2\epsilon_{a\alpha db}p^\alpha A^d,\quad A^d=\bar\eta\gamma^d\gamma_5\eta, \tag{49}$$

where A^d is the axial-vector current. Next, we perform the p integration as follows:

$$\int d^4p\,p^{-8}p^\alpha p^\beta p^\gamma p^\delta=\tfrac{1}{24}(\delta^{\alpha\beta}\delta^{\gamma\delta}+\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma})I\ , \tag{50}$$

with the basic dimensionally regularized integral I defined by

$$I=\int\frac{d^4p}{(p^2+M^2)^2}=\left(\frac{1}{\epsilon}\right). \tag{51}$$

The $(\operatorname{tr}q^2)^2$ contribution in (48) is obviously positive: Using (49) and (50), we find

$$\int\frac{d^4p}{p^8}(\operatorname{tr}q^2)^2=\left(\frac{40}{\epsilon}\right)(A^aA^b\eta_{ab})^2\ . \tag{52}$$

Similarly, $\operatorname{tr}q^4$ is also positive; here (49) and (50) yield

$$\int\frac{d^4p}{p^8}(\operatorname{tr}q^4)=\left(\frac{20}{\epsilon}\right)(A^aA^b\eta_{ab})^2\ . \tag{53}$$

In obtaining these results, we express products of two ϵ symbols in terms of Kronecker δ to obtain the identities

$$\begin{aligned}(\epsilon^a{}_{\alpha rb}\epsilon^b{}_{\beta sa})(\epsilon^c{}_{\gamma td}\epsilon^d{}_{\delta uc})\Delta^{\alpha\beta\gamma\delta}&=20\Delta_{rstu}\ ,\\(\epsilon^a{}_{\alpha rb}\epsilon^b{}_{\beta sc}\epsilon^c{}_{\gamma td}\epsilon^d{}_{\delta ua})\Delta^{\alpha\beta\gamma\delta}&=10\Delta_{rstu}\ ,\end{aligned} \tag{54}$$

where

$$\Delta^{\alpha\beta\gamma\delta}=\delta^{\alpha\beta}\delta^{\gamma\delta}+\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma} .$$

The final result for the η^8 part of $\Delta\mathcal{L}$ is then the contact self-interaction term

$$\Delta\mathcal{L}(\eta^8)=\lambda(e/\epsilon)(A^aA^b\eta_{ab})^2 , \tag{55}$$

where λ is a nonvanishing (but numerically uninteresting) constant and the dependence on external *vierbein* fields is, as dictated by general covariance, through e. It implies that the Dirac-Einstein system is nonrenormalizable[17] at the one-loop level, since $\Delta\mathcal{L}$ cannot be absorbed by a field renormalization, and is unaffected by the classical field equations.

IV. CONCLUSIONS

General relativity, in its *vierbein* formulation, can be quantized covariantly, and is equivalent to the quantized metric approach when matter is absent and in the presence of integer spin sources. The coordinate gauge invariance is broken by adding gauge-breaking terms of the usual harmonic type involving only the symmetric part of the quantized *vierbein* fields. The local Lorentz invariance is broken by adding algebraic gauge-breaking terms involving only the antisymmetric *vierbein* fields. The corresponding ghosts are a vector and an antisymmetric tensor. The symmetric *vierbein* Lagrangian plus the coordinate ghost are equivalent to metric theory. Neither the antisymmetric *vierbein* field nor its ghost propagates, and in fact they cancel each other, as they should since the theory can also be described without these variables.

When fermions are included, antisymmetric *vierbein* fields become coupled to them; however, nonrenormalizability of the coupled system can be demonstrated without discussing the antisymmetric *vierbein* fields. We saw that there is a divergent counterterm proportional to the fourth power of the axial-vector current, which remains quite distinct from other combinations of the background spinors, such as the Dirac stress tensor, even after use of the classical field equations.

Nonrenormalizability of the Dirac-Einstein system (at least in the presently available perturbative framework) is quite disturbing, since spin-$\frac{1}{2}$ fields are basic building blocks of matter. It is, of course, possible that future nonperturbative techniques or some new improved variant of Einstein theory[18] will resolve this impasse. There also remains one orthodox possibility we have not explored. It is known[19] that the definition of minimal gravitational coupling is ambiguous for spin $\frac{1}{2}$, depending on whether one expresses it in first- or second-order formulation with respect to the *vierbein* field. We have employed the latter, which differs from the former by an effective contact interaction $\mathcal{L}^C(\bar{e},\psi)\sim\kappa^2\bar{e}(\bar{\psi}\gamma_a\gamma_5\psi)^2$. This term contributes additional (κ-independent) vertices to $\mathcal{L}_2$, which could lead to $(1/\epsilon)A^4$ as well as other additions to $\Delta\mathcal{L}$. It also alters the background field equations by nonderivative terms in both $T^a{}_\mu(\eta)$ and the Dirac equation, which will lead to mixing between η^8 and other types of counterterms. We have not investigated the complicated algebra involved, in which all divergent terms now have to be faced. It may be that all counterterms in η finally reduce here to the combination $T^a{}_\mu(\eta)$, as they did in the Maxwell case[2] (which does ***not*** mean for either system that it is renormalizable).[20] A conclusive result would obviously be of value.

ACKNOWLEDGMENTS

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APPENDIX A: GROUP THEORY AND SPINORS

Spinors cannot be introduced into general relativity by coupling them to the metric $g_{\mu\nu}$; instead, as was shown in Sec. II, one can use *vierbein* fields for this purpose. Although this result is a fact well known to both mathematicians[7] and physicists, we include here a simple proof. If spinors were on an equal footing with, say, vectors, then they would have to transform under the coordinate transformations $x^\mu\to\bar{x}^\mu(x)$ according to a representation of the matrix group $(\partial\bar{x}^\mu/\partial x^\nu)$. This is the group GLR(4) of all real regular 4×4 matrices, and as we will show, there exist no representation of GLR(4) which reduces for its subgroup of Lorentz transformations to the usual spinor representations.

A particularly simple proof is to consider the group SO(2) of rotations around the z axis; this group is a subgroup both of the Lorentz group and of SLR(2), all real 2×2 matrices with determinant one; SLR(2) itself is a subgroup of GLR(4):

$$\mathrm{SO}(2)\subset L(3,1), \quad \mathrm{SO}(2)\subset\mathrm{SLR}(2)\subset\mathrm{GLR}(4) . \tag{A1}$$

If there is a spinor representation of GLR(4), then it is at the same time a representation of SLR(2), and a double-valued representation of SO(2). All representations of SLR(2) are, however, well known; the Lie group

$$M=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\exp(\alpha_iK_i), \quad \alpha\delta-\beta\gamma=1 \tag{A2}$$

has as generators

$$K_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad K_-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad K_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{A3}$$

which form the familiar Lie algebra of angular momentum

$$[K_0,K_+]=K_+,\quad [K_0,K_-]=-K_-,\quad [K_+,K_-]=K_0\tag{A4}$$

[its complexification yields SU(2), just as the complexification of SO(4) yields the Lorentz group; so, strictly speaking, one determines the spectrum of angular momentum in physics not from SU(2) but from SLR(2)]. The essential point now is that in SLR(2) rotations around the z axis are generated by (K_+-K_-) and not by K_0. Only for the group SO(3) would the generator of the 2×2 representation be given by K_0 and lead to double-valued representations. A rotation of about the z axis in SLR(2) is thus given by

$$R(\omega)=\exp[\omega(K_+-K_-)],\tag{A5}$$

and this is a single-valued function for all representations of the Lie algebra in Eq. (A4), $R(0)=R(2\pi)$.

It is interesting to note that, although it lacks any spinor representations, the goup GLR(n) is doubly connected for $n>2$. Since each element of GLR(n) can be written as the direct product of a triangular matrix times an element of SO(n), and since the triangular matrices can be shrunk continuously to the unit matrix, the topology of GLR(n) is that of SO(n). The proof is completed by noting that SO($n>2$) is doubly connected and SO(2) infinitely connected. Note that SO(3) and the Lorentz group do have spinor representations and are doubly connected. It is thus necessary, but not sufficient, for a group G to have multivalued representations, that it be multiply connected.

Actually, no multivalued representations for SLR(n) exist at all. Its covering group is therefore not a matrix group, since a matrix group always has one faithful representation: itself.

Also, SLR(4) does not have any multivalued representations. For the proof, note that the topologies of SLR(n) and SLC(n) are the same, the one being obtained by complexification of the other. Since, however, the topology of SLC(n) is that of SU(n) (again by the argument of triangular matrices) and SU(n) is simply connected, no multivalued representations for SU(n) exist, and hence none do for SLC(n) and SLR(n).

APPENDIX B: TENSORS CONSTRUCTED FROM *VIERBEIN* FIELDS

The relation between *vierbein* fields and tensors, as well as the discussion of the possible invariants of given differential order constructed from them, was given in Ref. 21. We include here a simplified derivation.

One can construct other tensors from *vierbein* fields algebraically, for example, the determinant $e=(-g)^{1/2}$, the inverse $e_a{}^\mu$, or by contraction with $\epsilon_{\mu\nu\rho\sigma}$. As for tensor analysis, the first derivatives of $e^a{}_\mu$ alone can never produce a tensor, since they can be made to vanish locally by choosing appropriate frames. Consider

$$S^a_{\mu\nu}\equiv(\partial_\mu e^a{}_\nu+\partial_\nu e^a{}_\mu),\quad A^a_{\mu\nu}=(\partial_\mu e^a{}_\nu-\partial_\nu e^a{}_\mu).\tag{B1}$$

Under the coordinate transformations

$$x^\alpha=a^\alpha+b^\alpha{}_\mu\bar{x}^\mu+\tfrac{1}{2}c^\alpha_{\mu\nu}\bar{x}^\mu\bar{x}^\nu+\tfrac{1}{6}d^\alpha_{\mu\nu\rho}\bar{x}^\mu\bar{x}^\nu\bar{x}^\rho,\tag{B2}$$

S transforms as

$$\bar{S}^a_{\mu\nu}(\bar{x})=\frac{\partial x^\alpha}{\partial\bar{x}^\mu}\frac{\partial x^\beta}{\partial\bar{x}^\nu}S^a_{\alpha\beta}(x)+\frac{\partial^2x^\alpha}{\partial\bar{x}^\mu\partial\bar{x}^\nu}e^a{}_\alpha(x)$$

and can be made to vanish at $\bar{x}=0$ by choosing $c^\alpha{}_{\mu\nu}$ appropriately. The A complex is a coordinate tensor, as follows from its curl structure, but transforms under Lorentz transformations as

$$\bar{A}^a_{\mu\nu}(x)=L^a{}_b(x)\bar{A}^b_{\mu\nu}(x)+\left(\frac{\partial}{\partial\bar{x}^\mu}L^a{}_b(x)\right)e^b{}_\nu(x)-\left(\frac{\partial}{\partial\bar{x}^\nu}L^a{}_b(x)\right)e^b{}_\mu(x).\tag{B3}$$

The Lorentz matrix L can be written as $L=e^{H(x)}$, where $H(x)$ is antisymmetric. Choosing $L(0)=I$, one has

$$L^a{}_b(x)=\delta^a{}_b+x^\mu(J_\mu)^a{}_b+\tfrac{1}{2}x^\mu x^\nu(K_{\mu\nu})^a{}_b+O(x^3),\tag{B4}$$

where the constant matrices J are antisymmetric. $\bar{A}(x)$ vanishes at $x=0$ if

$$(I\partial_\mu+J_\mu)^a{}_be^b{}_\nu-(I\partial_\nu+J_\nu)^a{}_be^b{}_\mu=0,\tag{B5}$$

where $I^a{}_b=\delta^a{}_b$. By choosing $(J_\mu)^a{}_b=\omega_\mu{}^a{}_b$, and adding the appropriate Christoffel symbols in each term in Eq. (B5) (they cancel due to the antisymmetry in $\mu\nu$), one finds that Eq. (B5) is indeed satisfied, since it is equal to $e^a{}_{\nu;\mu}-e^a{}_{\mu;\nu}$, which is zero according to Eq. (6).

The second-order derivatives of $e^a{}_\mu$ can be expressed in terms of $\partial_\rho S^a_{\mu\nu}$ and $\partial_\rho A^a_{\mu\nu}$. There are 160 elements $\partial_\kappa\partial_\lambda e^a{}_\mu$, but choosing d in Eq. (B2) appropriately, one can eliminate 80 elements, while proper choice of K in Eq. (B4) eliminates another 60 [$(K_{\mu\nu})^a{}_b$ is antisymmetric in (ab) and given by $\partial_\mu\partial_\nu H(0)+\frac{1}{2}(J_\mu J_\nu+J_\nu J_\mu)$]. This leaves at most 20 independent elements; there are, however, 20 elements $R_{\mu\nu ab}$ defined by Eq. (11), and we conclude that the only tensor which can be constructed out of $e^a{}_\mu$ and is linear in the second derivatives of $e^a{}_\mu$ is the curvature $R_{\mu\nu ab}$ (and, of course, its contractions).

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[6]As was first shown by Cartan (Ref. 7) spinors cannot be accommodated in a purely metric formulation; a simplified proof is given in Appendix A.

[7]E. Cartan, *Leçons sur la Théorie des Spineurs* (Hermann, Paris, 1938), Vol. II.

[8]Relaxation of linearity in the second derivatives leads to a Lagrangian depending nonlinearly on the curvature and its derivatives (see Ref. 21).

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[11]$\mathcal{L}^G$ is defined as the generic form $\mathcal{L}^G = \varphi^{a*}(\delta\Gamma_a/\delta\eta^b)\varphi^b$, where in our case η are the ten gauge parameters $(\eta^\alpha, \lambda^a{}_b)$ and (φ^*, φ) are the ten ghost fields, and for one-loop purposes any dependence of $(\delta\Gamma/\delta\eta)$ on quantum fields may be neglected.

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[14]One might expect *a priori* that for massless fields only their derivatives can be present in the counterterms. This is the case for scalars and photons because their free-field symmetries (under $\varphi \to \varphi$ + constant) are respected by gravitational coupling, since there no connections are introduced ($D_\mu\varphi = \partial_\mu\varphi$, $D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$). However, coupling of spinors breaks this symmetry through the spin connection terms.

[15]We have used $S^H = i^2\mathcal{L}^H\mathcal{L}^H$ and $S^S = i\mathcal{L}^S$.

[16]The trace is only over the index pairs (b_i, μ_i) of S^T in Eq. (43). Dirac matrices remain sandwiched between $\bar\eta$ and η.

[17]Strictly speaking, renormalizability only demands that $\Delta\mathcal{L}$ be one of a *finite number* of counterterms (to all orders).

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Scalar- and matter-dominated cosmologies in Schwinger's scalar-tensor theory of gravity*

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A scalar-tensor theory of gravity, suggested by Schwinger, is reviewed. This theory is like the Brans-Dicke one in that the gravitational constant is time-dependent, and in fact coincides with it whenever the scalar field is weak. It is argued that the apparent discrepancy between the observed mass density of galaxies and the density required by standard cosmologies with $q_0 \sim 1$ may be resolved by supposing that we are still in an era in which the scalar field is the dominant source of gravity. Solutions in such an era are discussed, and it is seen that they can be characterized by an increasing gravitational constant. Finally, the more conventional matter-dominated regime is treated.

I. INTRODUCTION

In the past decade there has been considerable interest, both theoretically and experimentally, in the idea that a portion of the gravitational interaction is mediated by helicity-zero particles. Such ideas seem to have originated with Jordan,[1] but the related version due to Brans and Dicke[2,3] has enjoyed the greatest discussion. Their ideas were based on an attempt to implement Mach's principle, and resulted in a theory in which the gravitational "constant" is time-dependent.