Renormalization of spinor and scalar electrodynamics with bilinear gauge conditions

Satish D. Joglekar

*Fermi National Accelerator Laboratory, Batavia, Illinois 60510**
and State University of New York at Stony Brook, Stony Brook, New York 11794

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We apply the recently developed technique of Ward-Takahashi identities for proper vertices in gauge theories to the problem of renormalization of electrodynamics—as a simple example of a gauge theory—when the gauge condition chosen is bilinear in fields. We show that spinor electrodynamics is renormalizable when the gauge condition is $f[A] \equiv (1/\sqrt{\alpha})(\partial_\mu A^\mu - \frac{1}{2}\xi A^2) = 0$, where ξ and α are real and arbitrary, and the parameter ξ is renormalized independently. We also show that scalar electrodynamics is renormalizable with the gauge condition $f[A] \equiv (1/\sqrt{\alpha})(\partial_\mu A^\mu - \frac{1}{2}\xi A^2 - \frac{1}{2}\eta\phi^*\phi) = 0$, where ξ and η are real and arbitrary. ξ and η must be renormalized independently.

I. INTRODUCTION

Renormalization of gauge theories (unbroken and spontaneously broken) has been discussed at length over the past few years. The earlier discussions on the renormalization of gauge theories have been based on the Ward-Takahashi (WT) identities for Green's functions.[1] Recently, renormalization of gauge theories has been discussed using the Ward-Takahashi identity for $\Gamma[\Phi]$, the generating functional of the one-particle irreducible (proper) vertices.[2] Since the renormalization procedure is stated in terms of proper vertices, use of the Ward-Takahashi identity for $\Gamma[\Phi]$ simplifies the discussion of renormalizability greatly. In the above-referenced discussions on renormalizability of gauge theories, the gauge conditions chosen to quantize the theory are linear in the fields. It is of some interest to see whether the proof of renormalizability goes through when the gauge condition chosen is bilinear in the fields[3] (that is, how far one can go if the gauge term is not to exceed four dimensions).

Here, we apply the method of Ref. 2, viz., the Ward-Takahashi identity for $\Gamma[\Phi]$, in order to carry out the renormalization of the simplest possible gauge theory. We work out the renormalization of an electromagnetic field interacting with a Dirac field or a complex scalar field. It is hoped that this exercise will help in the understanding of the renormalization of more complicated (e.g., non-Abelian) gauge theories in bilinear gauge conditions.

In Sec. II, we begin considering the Lagrangian for a free electromagnetic field with the gauge condition $f[A] \equiv (1/\sqrt{\alpha})(\partial_\mu A^\mu - \frac{1}{2}\xi A_\mu A^\mu) = 0$, with ξ and α as free parameters. Though the theory is trivial from the point of view of its physical content (S matrix), it is nontrivial from the point of view of renormalization. In fact the discussion of the renormalization of a free electromagnetic field makes it considerably simpler later to treat the interacting cases in this type of gauge. We note that, in this gauge ($\xi \neq 0$), there are $(A_\mu)^3$, $(A_\mu)^4$, and $(\bar{c}cA_\mu)$ vertices. ($\bar{c}$ and c are the Faddeev-Popov ghost fields.[4]) We obtain the Ward-Takahashi identity for proper vertices. We use the dimensional regularization. We analyze the divergences in $G^{-1}[\Phi]$ (the generating functional of proper vertices with two external ghosts) and in $\mathfrak{F}_\alpha[\Phi]$, which is essentially the expectation value of the gauge functional in the presence of external sources. Using the WT identity for $\Gamma[\Phi]$, we obtain relations among the divergences in $\Gamma[\Phi]$, $G^{-1}[\Phi]$, and $\mathfrak{F}_\alpha[\Phi]$ and show by an inductive proof that they can be removed by multiplicative renormalization on fields and parameters α and ξ. (We shall not state any specific renormalization conditions which determine the finite parts of renormalization constants.)

In Sec. III, we give the results of the one-loop calculation to carry out the renormalization program of Sec. II and verify the relations among the divergences obtained there.

In Sec. IV, we show that the 4-photon S-matrix amplitude vanishes in this gauge, as it should.

In Sec. V, we consider spinor electrodynamics. The extension from the noninteracting case is, more or less, straightforward. We prove the renormalizability of spinor electrodynamics and obtain the usual Ward identity between the renormalizations of the electron-photon vertex and the electron propagator.

In Sec. VI, we consider scalar electrodynamics. We find that in the gauge $f[A] \equiv (1/\sqrt{\alpha})(\partial_\mu A^\mu - \frac{1}{2}\xi A^2) = 0$ we cannot make proper vertices finite by multiplicative renormalizations on α and ξ (and fields, etc.). This is essentially because, in this case, $\mathfrak{F}_\alpha[\Phi]$ is such that its derivatives cannot be made finite to all orders by multiplicative renormalizations on α and ξ (and fields, etc.). However, we find that if we choose the gauge condition

$$f[A] \equiv \frac{1}{\sqrt{\alpha}}\left(\partial_\mu A^\mu - \tfrac{1}{2}\xi A^2 - \tfrac{1}{2}\eta\phi^*\phi\right) = 0\,,$$

and renormalize α, ξ, and η independently, the renormalization program goes through. This is explained in Sec. VI.

It is found that renormalization of ξ (or ξ and η) is (are) independent of those of other parameters and fields. From a practical point of view, such gauges would be more useful were the renormalizations on ξ and η dependent on other renormalization constants, for then certain simplications in the effective action could be made and maintained to all orders.

II. FREE ELECTROMAGNETIC FIELD

A. Preliminary

In the following we consider the Lagrangian for the electromagnetic field $A_\mu(x)$,

$$\mathcal{L}_0 = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

$\mathcal{L}_0$ is invariant under a local gauge transformation,

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\omega(x)\,. \tag{1}$$

We shall choose the nonlinear gauge function,

$$f[A] = +\frac{1}{\sqrt{\alpha}}\left(\partial_\mu A^\mu - \tfrac{1}{2}\xi A_\mu A^\mu\right). \tag{2}$$

Then the gauge term, to be added to $\mathcal{L}_0$, is given by

$$\mathcal{L}_{\text{gauge}} = -\tfrac{1}{2}\{f[A]\}^2\,.$$

Henceforth, we shall use a summation-integration convention (used, for example in Ref. 2). Thus, the gauge functional of Eq. (2) is

$$f_\alpha[A] = +\frac{1}{\sqrt{\alpha}}\left(\partial_i^\alpha A_i - \tfrac{1}{2}\xi_{ij}^\alpha A_i A_j\right)$$

$$[\partial_i^\alpha \equiv \partial_\mu\,\delta^4(x_\alpha - x_i),\ \ \xi_{ij}^\alpha \equiv \xi\,\delta^4(x_\alpha - x_i)\delta^4(x_i - x_j),\ \text{etc.}]\,. \tag{3}$$

As shown by Faddeev and Popov,[4] the Feynman rules for constructing Green's functions can be deduced from the effective Lagrangian,

$$\mathcal{L}_{\text{eff}}[A, c, \bar{c}] = \mathcal{L}_0 + \mathcal{L}_{\text{gauge}} + \bar{c}_\alpha M_{\alpha\beta}c_\beta\,, \tag{4}$$

where $\bar{c}_\alpha$ and c_β are fictitious, anticommuting complex scalar fields which generate the Faddeev-Popov ghost loops, and $M_{\alpha\beta}$ is given by

$$M_{\alpha\beta} = \frac{\delta f_\alpha[A]}{\delta A_i}\left(\frac{1}{e}\partial_i^\beta\right)e\sqrt{\alpha}$$
$$= \partial_i^\beta\left(\partial_i^\alpha - \xi_{ij}^\alpha A_j\right). \tag{5}$$

We note that there are $(A_\mu)^3$ and $(A_\mu)^4$ vertices arising out of the gauge term and a $\bar{c}cA_\mu$ vertex from the ghost term. These Feynman rules are given in Fig. 1. (The dashed lines denote the ghost lines; the wiggly lines denote photons.)

B. Ward-Takahashi identity for proper vertices

We shall deal with unrenormalized but dimensionally regularized quantities (in dimensions $4-\epsilon$). We shall use the notation of Ref. 2.

The generating functional of Green's functions is given by

$$W_F[J] = \int [dAdcd\bar{c}]\exp\{i(\mathcal{L}_{\text{eff}}[A, c, \bar{c}] + J_i A_i)\}\,. \tag{6}$$

As a result of gauge invariance, $W_F[J]$ satisfies the WT identity,[2] which in our specific case reads

$$\left\{-\frac{1}{\sqrt{\alpha}} f_\alpha\left[\frac{1}{i}\frac{\delta}{\delta J}\right] + J_i\,\partial_i^\beta M^{-1}{}_{\beta\alpha}\left[\frac{1}{i}\frac{\delta}{\delta J}\right]\right\} W_F[J] = 0\,. \tag{7}$$

$Z[J]$, the generating functional of connected Green's functions, is defined by

$$W_F[J] = \exp(iZ[J])\,.$$

We define

$$\Phi_i = \frac{\delta Z[J]}{\delta J_i}\,. \tag{8}$$

Then the functional $\Gamma[\Phi]$ defined by

$$\Gamma[\Phi] = Z[J] - J_i\Phi_i \tag{9}$$

generates the proper vertices. It follows from Eq. (9) that

$$J_i = -\frac{\delta\Gamma}{\delta\Phi_i}\,. \tag{10}$$

We go back to the WT identity of Eq. (7) and use the operator identity,

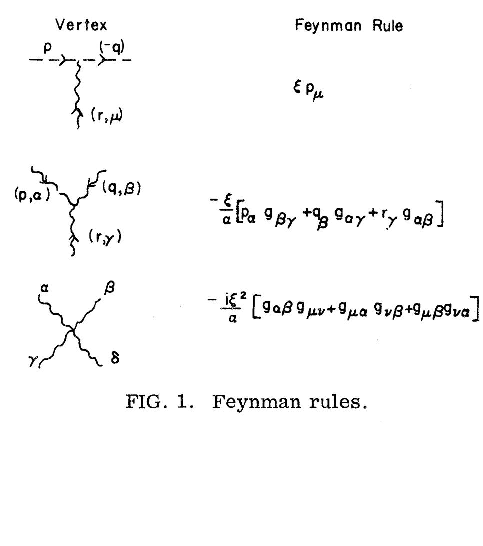

FIG. 1. Feynman rules.

$$B\left[\frac{1}{i}\frac{\delta}{\delta J}\right]e^{iZ[J]}=e^{iZ[J]}B\left[\Phi+\frac{1}{i}\frac{\delta}{\delta J}\right],$$

and thus obtain

$$-\frac{1}{\sqrt{\alpha}}f_\alpha\left[\Phi+\frac{1}{i}\frac{\delta}{\delta J}\right]\cdot 1 +\partial_i^\beta J_i M^{-1}{}_{\beta\alpha}\left[\Phi+\frac{1}{i}\frac{\delta}{\delta J}\right]\cdot 1=0. \quad (11)$$

Now,

$$\frac{\delta}{\delta J_i}=\frac{\delta\Phi_j}{\delta J_i}\frac{\delta}{\delta\Phi_j}\equiv-\Delta_{ij}[\Phi]\frac{\delta}{\delta\Phi_j}.$$

It can be shown that $\Delta_{ij}[\Phi]$ is the propagator when fields A_i are constrained to have expectation values Φ_i. Then, using Eq. (10), Eq. (11) becomes

$$-\frac{1}{\sqrt{\alpha}}f_\alpha\left[\Phi+i\Delta\frac{\delta}{\delta\Phi}\right]\cdot 1-\partial_i^\beta\frac{\delta\Gamma}{\delta\Phi_i}G_{\beta\alpha}[\Phi]=0, \quad (12)$$

where

$$G_{\beta\alpha}[\Phi]\equiv M^{-1}{}_{\beta\alpha}\left[\Phi+i\Delta\frac{\delta}{\delta\Phi}\right]\cdot 1$$

and can be shown to be the generating functional of proper vertices with two external ghost lines.

Now,

$$\sqrt{\alpha}\,f_\alpha\left[\Phi+i\Delta\frac{\delta}{\delta\Phi}\right]\cdot 1=\left[\partial_i^\alpha\left(\Phi_i+i\Delta_{ij}\frac{\delta}{\delta\Phi_j}\right)-\tfrac{1}{2}\xi_{ij}^\alpha\left(\Phi_i+i\Delta_{ik}\frac{\delta}{\delta\Phi_k}\right)\left(\Phi_j+i\Delta_{jl}\frac{\delta}{\delta\Phi_l}\right)\right]\cdot 1$$
$$=\partial_i^\alpha\Phi_i-\tfrac{1}{2}\xi_{ij}^\alpha(\Phi_i\Phi_j+i\Delta_{ij}).$$

Thus we obtain the WT identity for proper vertices:

$$G_{\beta\alpha}[\Phi]\,\partial_i^\beta\frac{\delta\Gamma[\Phi]}{\delta\Phi_i}=-\frac{1}{\alpha}\left\{\partial_i^\alpha\Phi_i-\tfrac{1}{2}\xi_{ij}^\alpha(\Phi_i\Phi_j+i\Delta_{ij}[\Phi])\right\}. \quad (13)$$

C. Expression for $G^{-1}{}_{\alpha\gamma}[\Phi]$

$G^{-1}{}_{\alpha\gamma}[\Phi]$ is the generating functional of the proper vertices with two ghost fields at α and γ. In order to carry through the renormalization program, we need to show that the renormalized $G^{-1}[\Phi]$ is a finite functional. Hence we need, first, to obtain an expression for $G^{-1}{}_{\alpha\gamma}[\Phi]$.

We have the identity

$$M_{\alpha\gamma}\left[\Phi+i\Delta\frac{\delta}{\delta\Phi}\right]M^{-1}{}_{\gamma\beta}\left[\Phi+i\Delta\frac{\delta}{\delta\Phi}\right]\cdot 1=\delta_{\alpha\beta}.$$

Using definitions of $M_{\alpha\gamma}$ [Eq. (3)] and $G_{\gamma\beta}$ [Eq. (12)], we get

$$\left[\partial_i^\alpha-\xi_{ij}^\alpha\left(\Phi_j+i\Delta_{jk}\frac{\delta}{\delta\Phi_k}\right)\right]\partial_i^\gamma G_{\gamma\beta}[\Phi]=\delta_{\alpha\beta}.$$

Using

$$\frac{\delta}{\delta\Phi_k}G_{\gamma\beta}[\Phi]=-G_{\gamma\xi}\frac{\delta G^{-1}{}_{\xi\eta}}{\delta\Phi_k}G_{\eta\beta},$$

we obtain

$$\left(\partial_i^\alpha\partial_i^\gamma-\xi_{ij}^\alpha\Phi_j\partial_i^\gamma+i\xi_{ij}^\alpha\partial_i^\eta\Delta_{jk}G_{\eta\zeta}\frac{\delta G^{-1}{}_{\zeta\gamma}}{\delta\Phi_k}\right)G_{\gamma\beta}=\delta_{\alpha\beta}.$$

Hence,

$$G^{-1}{}_{\alpha\gamma}[\Phi]=\partial_i^\alpha\partial_i^\gamma-\xi_{ij}^\alpha\Phi_j\partial_i^\gamma+i\xi_{ij}^\alpha\partial_i^\eta\Delta_{jk}G_{\eta\zeta}\frac{\delta G^{-1}{}_{\zeta\gamma}}{\delta\Phi_k}. \quad (14)$$

A diagrammatic representation for the last term is given in Fig. 2.

D. Renormalization transformations

To prove the renormalizability of the theory, we have to show that the derivatives of $\Gamma[\Phi]$ about its minimum can be rendered finite as $\epsilon\to 0$, by rescaling fields and parameters appearing in the Lagrangian $\mathcal{L}_{\rm eff}[\Phi,c,\bar{c}]$. We therefore define renormalized parameters and fields by the following renormalization transformations:

$$c=\tilde{Z}^{1/2}c^{(r)},\quad \Phi=\tilde{Z}^{1/2}\Phi^{(r)},$$
$$\frac{1}{\alpha}=\frac{W}{Z}\frac{1}{\alpha^{(r)}},\quad \xi=Y\tilde{Z}^{-1}Z^{1/2}\xi^{(r)}.$$

We also define

$$G_{\alpha\beta}[\Phi]=\tilde{Z}G^{(r)}_{\alpha\beta}[\Phi^r],$$
$$\Gamma[\Phi,\alpha,\xi]=\Gamma^{(r)}[\Phi^{(r)},\alpha^{(r)},\xi^{(r)}].$$

In the following, we shall always express everything in terms of renormalized quantities and drop the superscript (r). Thus the expression for (renormalized) $G^{-1}{}_{\alpha\gamma}[\Phi]$ becomes [from Eq. (14)]

$$G^{-1}{}_{\alpha\gamma}[\Phi]=\tilde{Z}\partial_i^\alpha\partial_i^\gamma-Y\xi_{ij}^\alpha\Phi_j\partial_i^\gamma+iY\xi_{ij}^\alpha\partial_i^\eta\Delta_{jk}G_{\eta\zeta}\frac{\delta G^{-1}{}_{\zeta\gamma}}{\delta\Phi_k}, \quad (15)$$

while the WT identity of Eq. (13) becomes

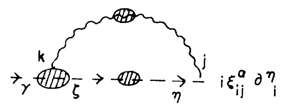

FIG. 2. Diagrammatic representation for the last term in Eq. (14).

$$G_{\beta\alpha}\partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i} = -\frac{1}{\alpha}\frac{W}{\tilde{Z}}\left(\partial_i^\alpha\Phi_i - \frac{Y}{2\tilde{Z}}\,\xi_{ij}^\alpha\Phi_i\Phi_j - \frac{iY}{2\tilde{Z}}\,\xi_{ij}^\alpha\Delta_{ij}\right)$$

$$\equiv -\frac{1}{\alpha}\,\mathfrak{F}_\alpha[\Phi]\ . \qquad (16)$$

To carry through the renormalization program, we start with the unperturbed Lagrangian expressed in renormalized fields,

$$\mathcal{L}_0 = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2\,,$$

and expand the proper vertices in terms of the loops the Feynman diagram contains. In each loop approximation we must determine the renormalization constants by a given prescription.

In the following, it will be assumed that renormalization constants Z, $\tilde{Z}$, W, Y are determined up to the $(n-1)$-loop approximation and that these make $G^{-1}{}_{\alpha\gamma}[\Phi]$, $\Gamma[\Phi]$, and $\mathfrak{F}_\alpha[\Phi]$ [defined in Eq. (16)] finite to each order, up to the $(n-1)$-loop approximation, in perturbation theory. We write, up to the $(n-1)$-loop approximation,

$$Z(\epsilon) = 1 + z_1(\epsilon) + z_2(\epsilon) + \cdots + z_{n-1}(\epsilon),\ \text{etc.}$$

Then we shall show that an appropriate choice of z_n, $\tilde{z}_n$, y_n, and w_n can be made to make $G^{-1}{}_{\alpha\gamma}[\Phi]$, $\Gamma[\Phi]$, and $\mathfrak{F}_\alpha[\Phi]$ finite up to the n-loop approximation.

E. Analysis of divergences in $G^{-1}{}_{\alpha\gamma}[\Phi]$ and $\mathfrak{F}_\alpha[\Phi]$

In order to show that $G^{-1}{}_{\alpha\gamma}[\Phi]$ and $\mathfrak{F}_\alpha[\Phi]$ can be made finite by appropriate choice of the renormalization constants in the n-loop approximation, we must show that various derivatives of $G^{-1}{}_{\alpha\gamma}[\Phi]$ and $\mathfrak{F}_\alpha[\Phi]$ at $\Phi=0$ (the minimum) in the n-loop approximation have received all the internal subtractions (the meaning of this statement will be clear soon), so that the divergences in these (those which are renormalization parts) are polynomials in external momenta and that therefore these can be removed by the local counterterms provided by the appropriate choices of the renormalization parameters.

We note from Fig. 1 that the Feynman rule at the $\bar{c}cA_\mu$ vertex is proportional to the momentum of the incoming ghost, so that in any proper vertex with two ghost lines there is a factor of p_μ for the incoming ghost of momentum p. This effectively decreases the degree of divergence (D) by one. Therefore, $G^{-1}{}_{\alpha\gamma}[\Phi]|_{\Phi=0}$ and $\delta G^{-1}{}_{\alpha\gamma}[\Phi]/\delta\Phi_k|_{\Phi=0}$ are renormalization parts but higher derivatives of $G^{-1}{}_{\alpha\gamma}[\Phi]$ are not renormalization parts. Also, only the first two derivatives of $\mathfrak{F}_\alpha[\Phi]$ at $\Phi=0$ are renormalization parts.

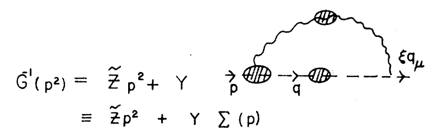

FIG. 3. Diagrammatic representation for $G^{-1}(p)$.

(A) We begin by considering $G^{-1}{}_{\alpha\gamma}[\Phi]|_{\Phi=0}$ in the n-loop approximation. We write this down in momentum space as Feynman diagrams (see Fig. 3).

Consider $[\Sigma]_n$, i.e., Σ in the n-loop approximation. The shaded blobs in $[\Sigma]_n$ contain at most $(n-1)$ loops, and the counterterms introduced up to the $(n-1)$-loop approximation provide the necessary subtractions for the subdiagrams in the blobs, making them finite by our hypothesis. However, $[\Sigma]_n$ needs further subtractions for the renormalization parts which are subdiagrams of $[\Sigma]_n$ and contain the rightmost vertex in $[\Sigma]_n$. On the basis of the remark made earlier about the $(\bar{c}cA_\mu)$ vertex, such renormalization parts arise only from two-particle cuts in $\delta G^{-1}{}_{\xi\gamma}/\delta\Phi_k$, the leftmost blob in $[\Sigma]_n$. See Fig. 4.

Thus these renormalization parts needing overall subtractions consist of the three-point proper vertex $\delta G^{-1}/\delta\Phi$ to various loop approximations [up to $(n-1)$ loops]. We shall show that the overall subtraction for such subdiagram is provided by y_r. Hence the additional internal subtractions needed by $[\Sigma]_n$ consist of $\sum_{r=1}^{n-1} y_r[\Sigma]_{n-r}$. Therefore,

$$[Y\Sigma]_n = \sum_{r=1}^{n-1} y_r[\Sigma]_{n-r}$$

has as its divergence a polynomial in momentum. Due to the Lorentz transformation property and the dimensions of $G^{-1}(p)$,

$$[Y\Sigma]_n^{\text{div}} = p^2K(\epsilon)\,.$$

(Here we note that there are no dimensional parameters in $\mathcal{L}_{\text{eff}}$.) Therefore, by choosing $\tilde{z}_n^{\text{div}} = K(\epsilon)$ we can make $[G^{-1}(p)]_n$ finite.

(B) Next, consider $\delta G^{-1}{}_{\alpha\gamma}[\Phi]/\delta\Phi_k|_{\Phi=0}$. From Eq. (15),

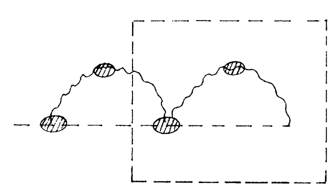

FIG. 4. Subdiagrams of $\Sigma(p)$ needing subtraction.

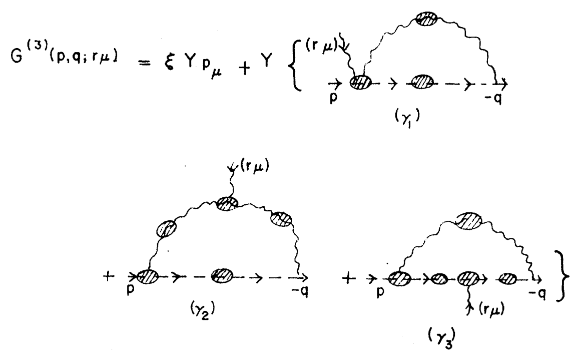

FIG. 5. Diagrammatic representation for $G^{(3)}(p, q; r\mu)$.

$$\left.\frac{\delta G^{-1}{}_{\alpha\gamma}}{\delta \Phi_k}\right|_{\Phi=0} = -Y\partial_i^\gamma \xi_{ik}^\alpha + iY\xi_{ij}^\alpha \partial_i^\eta \frac{\delta}{\delta\Phi_k} \Delta_{jm} G_{\eta\zeta} \left.\frac{\delta G^{-1}{}_{\zeta\gamma}}{\delta\Phi_m}\right|_{\Phi=0}. \tag{17}$$

We express the Fourier transform of Eq. (17) diagrammatically in Fig. 5.

As before, we need consider the internal subtractions needed to $[\gamma_1+\gamma_2+\gamma_3]_n$ for their only subdiagrams containing the rightmost vertex in each. It is easy to see that $[\gamma_2]_n$ and $[\gamma_3]_n$ do not have subdiagrams which are renormalization parts. However, $[\gamma_1]_n$ has such subdiagrams, which arise out of a two-particle cut in the proper $(\bar{c}cA_\mu A_\nu)$ vertex on the left. These fall into three categories, shown in Fig. 6.

As before, it is clear that the internal subtractions to $[\gamma_1+\gamma_2+\gamma_3]_n$ are provided by $\sum_{r=1}^{n-1} y_r[\gamma_1+\gamma_2+\gamma_3]_{n-r}$. Hence,

$$[Y(\gamma_1+\gamma_2+\gamma_3)]_n^{\text{div}} = p_\mu J(\epsilon).$$

(Here one must remember that each diagram is proportional to p_μ.) Therefore,

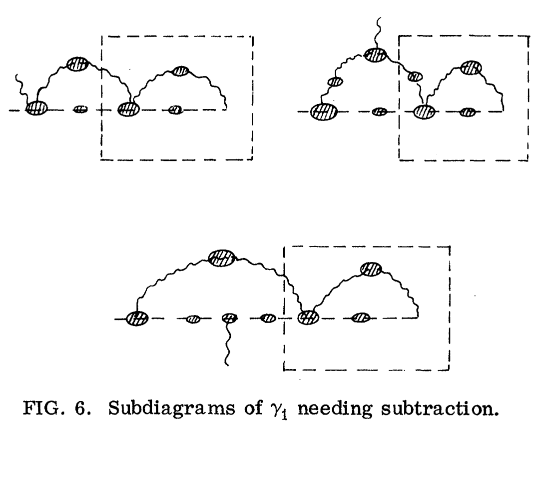

FIG. 6. Subdiagrams of γ_1 needing subtraction.

$$[G^{(3)}(p,q,r,\mu)]_n^{\text{div}} = p_\mu[\xi y_n(\epsilon) + J(\epsilon)] \tag{18}$$

and can be made finite by appropriate choice of $y_n(\epsilon)$.

(C) Consider divergences in $\mathfrak{F}_\alpha[\Phi]$. We note that for

$$\mathfrak{F}_\alpha[\Phi]\,|_{\Phi=0} = \frac{-iYW}{2\bar{Z}^2}\xi_{ij}^\alpha \Delta_{ij}$$

the Fourier transform is

$$\text{F.T.}\{\mathfrak{F}_\alpha[\Phi]\,|_{\Phi=0}\}_n = \frac{-iY\xi W}{\bar{Z}^2}\int [\Delta_{\mu\mu}(q^2)]_{n-1} d^4q\,, \tag{19}$$

where $\Delta_{\mu\nu}(q^2)$ is the photon propagator. Since the right-hand side must have dimensions (momentum)^2 and since there are no dimensional quantities in the integral that it can depend on, it must be zero in dimensional regularization.

Next, we consider the divergences in $\delta\mathfrak{F}_\alpha/\delta\Phi_k|_{\Phi=0}$ and $\delta^2\mathfrak{F}_\alpha/\delta\Phi_k\delta\Phi_l|_{\Phi=0}$:

$$\left.\frac{\delta\mathfrak{F}_\alpha}{\delta\Phi_k}\right|_{\Phi=0} = \frac{W}{\bar{Z}}\left[\partial_k^\alpha - \frac{Y}{2\bar{Z}}\xi_{ij}^\alpha \frac{\delta}{\delta\Phi_k}(i\Delta_{ij})\right]\Bigg|_{\Phi=0}, \tag{20}$$

$$\left.\frac{\delta^2\mathfrak{F}_\alpha}{\delta\Phi_k\delta\Phi_l}\right|_{\Phi=0} = -\frac{WY}{2\bar{Z}^2}\left[2\xi_{kl}^\alpha + \xi_{ij}^\alpha \frac{\delta^2}{\delta\Phi_k\delta\Phi_l}(i\Delta_{ij})\right]\Bigg|_{\Phi=0}. \tag{21}$$

We tabulate these in Fig. 7.

The constants A, B, and C are defined in each order in perturbation to be the overall divergences left in Δ, $\Box_{\mu\nu}$, and $\text{II}_{\mu\nu}$ (they are defined in Fig. 7), respectively, when all subtractions are performed on their subdiagrams which are renormalization parts. Note that we have not yet specified how the finite parts of A, B, and C (alternately those of Δ, $\Box_{\mu\nu}$, and $\text{II}_{\mu\nu}$) are to be defined. (For clarity we note that Δ, $\Box_{\mu\nu}$, and $\text{II}_{\mu\nu}$ are proper diagrams. The distinction between $\Box_{\mu\nu}$ and $\text{II}_{\mu\nu}$ is that when they are opened at the vertex denoted by a cross, $\text{II}_{\mu\nu}$ gives rise to a 4-photon *proper* vertex, while $\Box_{\mu\nu}$ gives rise to a 4-photon *improper* vertex.)

The subtractions needed for subdiagrams in the shaded blobs in $[\Delta]_n$, $[\Box_{\mu\nu}]_n$, and $[\text{II}_{\mu\nu}]_n$ (see Fig. 7) are provided by the counterterms already introduced in the Lagrangian up to the $(n-1)$-loop approximation. Thus, apart from an overall subtraction these need subtractions for the renormalization parts which are subdiagrams containing the leftmost vertex denoted by a cross. We tabulate these subdiagrams and subtractions needed to them in Fig. 8.

Thus, from Figs. 7 and 8, it follows that

(i) $[\Delta_\mu(p)]_n - \sum_{r=1}^{n-1} B_r[\Delta_\mu(p)]_{n-r} - \frac{1}{2}\sum_{r=1}^{n-1} C_r[\Delta_\mu(p)]_{n-r} - A_n p_\mu$ is finite ,

which can be written in the condensed form

$$[\Delta_\mu(p)(1-B-\tfrac{1}{2}C) - Ap_\mu]_n \text{ is finite ;} \tag{22}$$

(ii) $2[\Box_{\mu\nu}]_n - 2B_n g_{\mu\nu}$ is finite;

(iii) $[\Pi_{\mu\nu}]_n - \frac{1}{2}\sum_{r=1}^{n-1} C_r[\Pi_{\mu\nu}]_{n-r} - \sum_{r=1}^{n-1} C_r[\Box_{\mu\nu}]_{n-r} - 2\sum_{r=1}^{n-1} B_r[\Box_{\mu\nu}]_{n-r} - \sum_{r=1}^{n-1} B_r[\Pi_{\mu\nu}]_{n-r}$ is finite.

Adding (ii) and (iii) and writing in a condensed form, we see that

$$[(2g_{\mu\nu} + 2\Box_{\mu\nu} + \Pi_{\mu\nu})(1-B-\tfrac{1}{2}C)]_n \text{ is finite .} \tag{23}$$

Now,

$$\text{F.T.}\left\{\frac{\delta\mathfrak{F}_\alpha}{\delta\Phi_k}\bigg|_{\Phi=0}\right\} = i\left[\frac{W}{\tilde{Z}}p_\mu + \frac{\xi YW}{2\tilde{Z}^2}\Delta_\mu(p)\right] \tag{24}$$

and

$$\text{F.T.}\left\{\frac{\delta^2\mathfrak{F}_\alpha}{\delta\Phi_k\delta\Phi_l}\bigg|_{\Phi=0}\right\} = -\frac{WY\xi}{2\tilde{Z}^2}(2g_{\mu\nu} + 2\Box_{\mu\nu} + \Pi_{\mu\nu}) . \tag{25}$$

Comparing Eq. (22) with Eq. (24) and Eq. (23) with Eq. (25), it is clear that the two derivatives of $\mathfrak{F}_\alpha[\Phi]$ can be made finite simultaneously if we choose the factors $\xi YW/2\tilde{Z}^2$ and $W/\tilde{Z}$ appropriately [i.e., equal to $(1-B-\frac{1}{2}C)$ and $-\frac{1}{2}\xi A$, respectively.] However, since we would like (though it is not necessary) to determine the finite parts of W and Y by renormalization conditions on derivates of $\Gamma[\Phi]$ rather than of $\mathfrak{F}_\alpha[\Phi]$, we will state if differently. Suppose we have chosen w_r and y_r in the r-loop $(r<n)$ approximation by appropriate renormalization conditions. If they satisfy

$$\left[\frac{YW}{\tilde{Z}^2}\right]_r = [1-B-\tfrac{1}{2}C]_r \quad (0 \le r \le n-1) , \qquad \left[\frac{2}{\xi}\frac{W}{\tilde{Z}}\right]_r^{\text{div}} = -A_r^{\text{div}} \quad (0 \le r \le n-1) \tag{26}$$

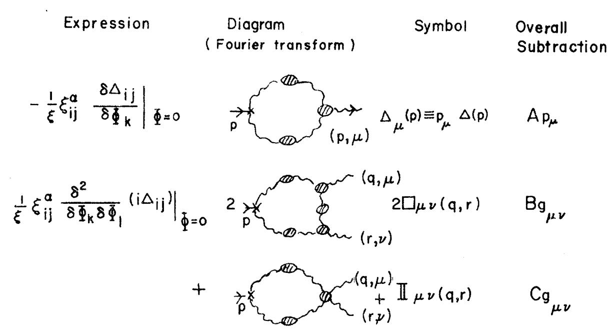

FIG. 7. Derivatives of $\mathfrak{F}_\alpha[\Phi]$ which are renormalization parts.

then we have to show that w_n and y_n chosen in the n-loop approximation will satisfy

$$\left[\frac{YW}{\tilde{Z}^2}\right]_n = [1-B-\tfrac{1}{2}C]_n , \qquad \left[\frac{2}{\xi}\frac{W}{\tilde{Z}}\right]_r^{\text{div}} = -A_n^{\text{div}} \tag{27}$$

and hence will make $\delta\mathfrak{F}_\alpha/\delta\Phi_i|_{\Phi=0}$ and $\delta^2\mathfrak{F}_\alpha/\delta\Phi_k\delta\Phi_l|_{\Phi=0}$ finite to the n-loop approximation.

Here we note the convention to define the finite parts of Δ_μ and $(2\Box_{\mu\nu}+\Pi_{\mu\nu})$. Once the finite parts of Y, W and Z have been chosen by a given set of renormalization prescriptions in the r-loop approximation, Eq. (26) then defines the finite part of $[1-B-\frac{1}{2}C]_r$ and hence that of $[2\Box_{\mu\nu}+\Pi_{\mu\nu}]_r$. The finite part of $[\Delta_\mu(p)]_r$ is so defined in the r-loop

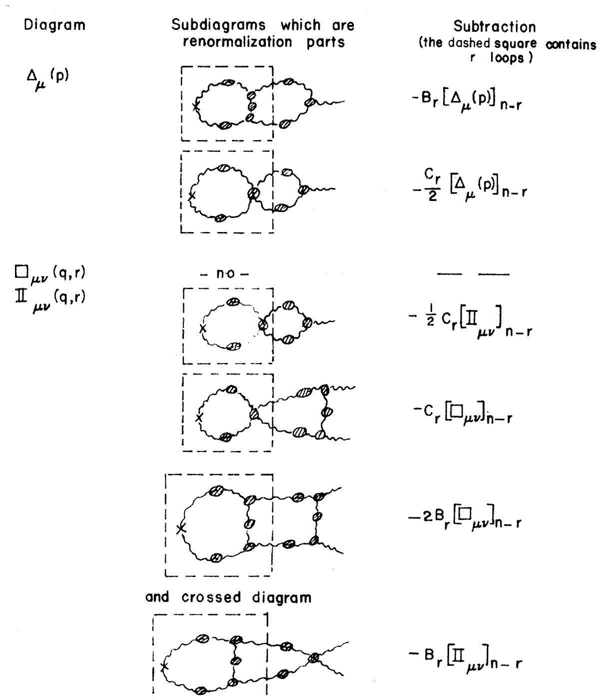

FIG. 8. Subtractions needed for $\Delta_\mu(p)$, $\Box_{\mu\nu}(q,r)$, $\Pi_{\mu\nu}(q,r)$.

approximation that the WT identity of Eq. (30) is satisfied by the finite parts.

The higher derivatives of $\mathcal{F}_\alpha[\Phi]$ are not renormalization parts. The proof that they become finite in the n-loop approximation once Eq. (26) are satisfied proceeds similarly.

F. Proof of renormalizability

Consider the inverse photon propagator $\Gamma_{\mu\nu}(p)$. Because of the Lorentz transformation property and the fact that there are no dimensional parameters in the theory, it follows that

$$[\Gamma_{\mu\nu}(p)]_n^{\text{div}} = (g_{\mu\nu}p^2 - p_\mu p_\nu)N^t(\epsilon) + p_\mu p_\nu N^l(\epsilon).$$

Define $\Gamma^{(t)}$ and $\Gamma^{(l)}$ by

$$\Gamma_{\mu\nu}(p) = (g_{\mu\nu}p^2 - p_\mu p_\nu)\Gamma^{(t)}(p) + p_\mu p_\nu \Gamma^{(l)}(p); \tag{28}$$

then

$$[\Gamma^{(t)}(p)]_n^{\text{div}} = N^t(\epsilon), \quad [\Gamma^{(l)}(p)]_n^{\text{div}} = N^l(\epsilon).$$

We shall choose z_n and w_n such that $N^t(\epsilon)$ and $N^l(\epsilon)$, respectively, become finite. As shown earlier, we can choose $\tilde{z}_n$ such that $[G^{-1}(p)]_n$ is finite. Then we have to show that a proper choice of y_n can be made so that the $(\bar{c}cA_\mu)^3$, $(A_\mu)^3$, and $(A_\mu)^4$ vertices become finite and the relations (27) are satisfied.

To this end, we consider the WT identity of Eq. (16). We consider successive derivatives of this identity at $\Phi=0$ and equate the quantities on both sides in the n-loop approximation.

(A) Differentiate the WT identity with respect to Φ_k and set $\Phi=0$. We obtain

$$G_{\beta\alpha}\partial_i^\beta \frac{\delta^2\Gamma}{\delta\Phi_i\delta\Phi_k} = -\frac{1}{\alpha}\frac{W}{\tilde{Z}}\left[\partial_k^\alpha - \frac{iY}{2Z}\xi_{ij}^\alpha \frac{\delta(\Delta_{ij})}{\delta\Phi_k}\right]\Bigg|_{\Phi=0}. \tag{29}$$

Writing this in momentum space, using the Fourier transforms defined earlier [see Eqs. (24) and (28)], we obtain

$$G(p^2)p_\nu p^2\Gamma^{(l)}(p^2) = -\frac{1}{\alpha}\left[\frac{W}{\tilde{Z}}p_\nu + \xi\frac{YW}{2\tilde{Z}^2}\Delta_\nu(p)\right]. \tag{30}$$

The left-hand side of Eq. (30) is finite in the n-loop approximation (with $\tilde{z}_n$ and w_n already chosen), so that the right-hand side of (30) is finite, and

$$-\left[\frac{W}{\tilde{Z}}p_\nu + \xi\frac{YW}{2\tilde{Z}^2}\Delta_\nu(p)\right]_n = -\left[\frac{W}{\tilde{Z}}\right]_n p_\nu - \frac{\xi}{2}\sum_{r=0}^{n-1}\left[\frac{YW}{\tilde{Z}^2}\right]_r [\Delta_\nu(p)]_{n-r}$$

$$= -\left[\frac{W}{\tilde{Z}}\right]_n p_\nu - \frac{\xi}{2}\sum_{r=0}^{n-1}[1-B-\tfrac{1}{2}C]_r[\Delta_\nu(p)]_{n-r} \quad \text{[by Eq. (26)]}$$

$$= -\left[\frac{W}{\tilde{Z}}\right]_n p_\nu - \frac{\xi}{2}A_n p_\nu + \text{finite terms} \quad \text{[by Eq. (22)]}.$$

Therefore,

$$[A]_n^{\text{div}} = -\frac{2}{\xi}\left[\frac{W}{\tilde{Z}}\right]_n^{\text{div}}. \tag{31}$$

(B) Next, differentiate the WT identity with respect to Φ_k and Φ_l and set $\Phi=0$. We obtain

$$G_{\beta\alpha}\partial_i^\beta\frac{\delta^3\Gamma}{\delta\Phi_i\delta\Phi_k\delta\Phi_l} + \frac{\delta}{\delta\Phi_k}(G_{\beta\alpha})\partial_i^\beta\frac{\delta^2\Gamma}{\delta\Phi_i\delta\Phi_l} + (k \leftrightarrow l) = \frac{1}{\alpha}\frac{WY}{2\tilde{Z}^2}\left[2\xi_{lk}^\alpha + \xi_{ij}^\alpha\frac{\delta^2(i\Delta_{ij})}{\delta\Phi_l\delta\Phi_k}\right]. \tag{32}$$

Let us define

$$\text{F.T.}\left\{\frac{\delta^3\Gamma}{\delta\Phi_i\delta\Phi_k\delta\Phi_l}\right\} = -i\Gamma^{(3)}_{\alpha\mu\nu}(p,q,r). \tag{33}$$

Then the left-hand side of Eq. (32) has the diagrammatic representation shown in Fig. 9.

Now, $[\Gamma^{(3)}_{\alpha\mu\nu}(p,q,r)]_n^{\text{div}}$ must be a polynomial linear in external momenta $(p+q+r=0)$ and a Bose-symmetric Lorentz tensor. This implies that

$$[\Gamma^{(3)}_{\alpha\mu\nu}(p,q,r)]_n^{\text{div}} = D(\epsilon)(g_{\alpha\mu}r_\nu + g_{\alpha\nu}q_\mu + g_{\mu\nu}p_\alpha). \tag{34}$$

As shown in Eq. (18),

$$[G^{(3)}(p,q;r,\mu)]_n^{\text{div}} = p_\mu[\xi y_n + J(\epsilon)] \equiv a(\epsilon)p_\mu. \tag{35}$$

Similarly,

$$[G^{(3)}(p,q;r,\nu)]_n^{\text{div}} = a(\epsilon)p_\nu. \tag{36}$$

Hence Eq. (32), in momentum space, becomes

$$\frac{D}{p^2}\,p_\alpha(g_{\alpha\mu}r_\nu+g_{\alpha\nu}q_\mu+g_{\mu\nu}p_\alpha)+\frac{a}{\alpha}\,\frac{p_\mu r_\nu}{p^2}+\frac{a}{\alpha}\,\frac{p_\nu q_\mu}{p^2}$$

$$=\left[\frac{YW\xi}{2\alpha\tilde{Z}^2}\left[2g_{\mu\nu}+2\Box_{\mu\nu}(q,r)+\Pi_{\mu\nu}(q,r)\right]\right]_n^{\text{div}}+\text{finite terms}$$

$$=\left[\frac{\xi WY}{\alpha\tilde{Z}^2}\right]_n^{\text{div}}g_{\mu\nu}+\frac{\xi}{2\alpha}\sum_{r=0}^{n-1}\left[\frac{YW}{\tilde{Z}^2}\right]_r\left[2g_{\mu\nu}+2\Box_{\mu\nu}+\Pi_{\mu\nu}\right]_{n-r}+\text{finite terms}\,.$$

Using Eq. (26), this equals

$$\left[\frac{\xi WY}{\alpha\tilde{Z}^2}\right]_n^{\text{div}}g_{\mu\nu}+\frac{\xi}{2\alpha}\sum_{r=0}^{n-1}\left[1-B-\tfrac{1}{2}C\right]_r\left[2g_{\mu\nu}+2\Box_{\mu\nu}+\Pi_{\mu\nu}\right]_{n-r}+\text{finite terms}\,.$$

Using Eq. (23), it equals

$$\left[\frac{\xi WY}{\alpha\tilde{Z}^2}\right]_n^{\text{div}}g_{\mu\nu}-\frac{\xi}{\alpha}\left[1-B-\tfrac{1}{2}C\right]_n g_{\mu\nu}+\text{finite terms}\,. \qquad (37)$$

Expressing Eq. (37) as a function of p and q, we obtain

$$D[-(p+q)_\nu p_\mu+p_\nu q_\mu+g_{\mu\nu}p^2]-\frac{a}{\alpha}\,p_\mu(p+q)_\nu+\frac{a}{\alpha}\,p_\nu q_\mu=\frac{\xi}{\alpha}\,p^2\left\{\left[\frac{WY}{\tilde{Z}^2}\right]_n^{\text{div}}-\left[1-B-\tfrac{1}{2}C\right]_n\right\}g_{\mu\nu}+\text{finite terms}\,.$$

Comparing coefficients of $p_\mu q_\nu$ and $g_{\mu\nu}p^2$, we obtain

$$D=-\frac{a}{\alpha}+\text{finite terms}$$

$$=\frac{\xi}{\alpha}\left\{\left[\frac{WY}{\tilde{Z}^2}\right]_n^{\text{div}}-\left[1-B-\tfrac{1}{2}C\right]_n\right\}+\text{finite terms}\,. \qquad (38)$$

But, from Eq. (18), $a\equiv\xi y_n(\epsilon)+K_n(\epsilon)$ can be made finite with the appropriate choice of $y_n(\epsilon)$. Then $D(\epsilon)$ becomes finite and

$$\left[1-B-\tfrac{1}{2}C\right]_n^{\text{div}}=\left[\frac{YW}{\tilde{Z}^2}\right]_n^{\text{div}}\,. \qquad (39)$$

As remarked earlier, finite part of $(1-B-\frac{1}{2}C)$ will be defined such that

$$\left[1-B-\tfrac{1}{2}C\right]_n=\left[\frac{YW}{\tilde{Z}^2}\right]_n\,. \qquad (40)$$

(C) We shall consider, finally, the WT identity differentiated thrice with respect to Φ_k, Φ_l, and Φ_m, and we shall set $\Phi=0$. With choices of renormalization constants in the n-loop approximation already made, all vertices entering the equation are made finite except (possibly) the $(A_\mu)^4$ vertex. From this equation it follows trivially that the $(A_\mu)^4$ vertex is also finite. Thus all renormalization parts of $\Gamma[\Phi]$ are shown to become finite.

Thus we have shown that if renormalization constants are chosen up to the $(n-1)$-loop approximation such that $\Gamma[\Phi]$, $G^{-1}{}_{\alpha\beta}[\Phi]$, and $\mathfrak{F}_\alpha[\Phi]$ are finite in the limit $\epsilon\to 0$ in the loop approximation, then renormalization constants z_n, $\tilde{z}_n$, w_n, and y_n can be chosen to make $\Gamma[\Phi]$, $G^{-1}{}_{\alpha\beta}[\Phi]$, and $\mathfrak{F}_\alpha[\Phi]$ finite up to the n-loop approximation. For $n=1$, this is trivially true if we choose $z_0=\tilde{z}_0=w_0=y_0=1$. Hence the proof by induction is complete.

We shall present the results of the one-loop calculation in Sec. III.

III. RESULTS OF ONE-LOOP CALCULATION

In this section we state the results of the one-loop calculation to verify the relation between divergences [See Eq. (27)] in $G^{-1}{}_{\alpha\beta}[\Phi]$, $\Gamma[\Phi]$, and $\mathfrak{F}_\alpha[\Phi]$.

A. The inverse photon propagator

The diagrams of Fig. 10 contribute to the inverse photon propagator in the one-loop approximation. We use dimensional regularization to compute

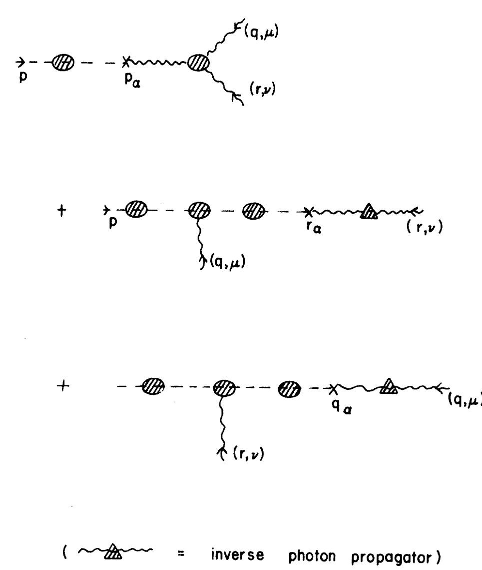

FIG. 9. Left-hand side of Eq. (32).

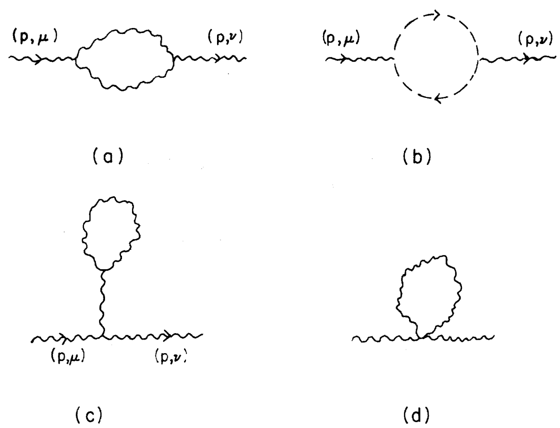

FIG. 10. One-loop diagrams for the inverse photon propagator.

these and state the divergences in units of

$$I \equiv \frac{-i}{(2\pi)^4}\left[\int \frac{d^4q}{q^4}\right]_{\text{pole}} = \frac{\pi^2}{(2\pi)^4}\frac{2}{\epsilon}\ . \tag{41}$$

Let m(a) denote the diagram of Fig. 10(a) evaluated with the usual Feynman rules; let Div{m(a)} denote the terms in m(a) which have a pole in ϵ. We find

$$\text{Div}\{\text{m(a)}\} = \frac{i\xi^2(2\alpha^2-3\alpha+3)}{2\alpha^2} I p_\mu p_\nu - i\xi^2 I[\tfrac{1}{3}(g_{\mu\nu}p^2 - p_\mu p_\nu) - \tfrac{1}{4}g_{\mu\nu}p^2]\ ,$$

$$\text{Div}\{\text{m(b)}\} = i\xi I[\tfrac{1}{3}(g_{\mu\nu}p^2 - p_\mu p_\nu) - \tfrac{1}{4}g_{\mu\nu}p^2]\ ,$$

$$\text{m(c)} = \text{m(d)} = 0\ .$$

Therefore,

$$\text{Div}\{i\Gamma_{\mu\nu}(p)\} = \frac{i\xi^2(2\alpha^2-3\alpha+3)}{2\alpha^2} I p_\mu p_\nu\ . \tag{42}$$

If we write $Z=1+z$, $W=1+w$, etc. to the one-loop approximation, the counterterm is

$$-iz(g_{\mu\nu}p^2 - p_\mu p_\nu) - \frac{i}{\alpha}(w-z)p_\mu p_\nu\ . \tag{43}$$

Hence, we find that the following choices will make the renormalized inverse propagator finite:

$$\text{Div}\{z\} = 0\ ,\quad \text{Div}\{w\} = \frac{\xi^2}{2\alpha}(2\alpha^2 - 3\alpha + 3)I\ . \tag{44}$$

Here we see that the transverse part is unrenormalized to the one-loop approximation, while the longitudinal part is renormalized.

B. The inverse ghost propagator

The diagrams of Fig. 11 contribute to the inverse ghost propagator. We find

$$\text{Div}\{\text{m(a)}\} = -\tfrac{1}{4}i\xi^2(3-\alpha)Ip^2\ ,\quad \text{m(b)} = 0\ . \tag{45}$$

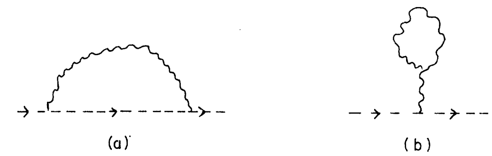

FIG. 11. One-loop diagrams for the inverse ghost propagator.

If we write $\tilde{Z} = 1+\tilde{z}$, the divergence in the inverse ghost propagator will be canceled by the counterterm $-ip^2\tilde{z}$ if we choose

$$\text{Div}\{\tilde{z}\} = -\tfrac{1}{4}(3-\alpha)\xi^2 I\ . \tag{46}$$

C. The ghost-ghost-photon ($\bar{c}cA_\mu$) vertex

The diagrams of Fig. 12 contribute to the proper vertex. It is found that

$$\text{Div}\{\text{m(a)}\} = \tfrac{3}{4}\xi^3 I p_\alpha\ ,$$

$$\text{Div}\{\text{m(b)}\} = -\tfrac{1}{4}\xi^3 I p_\alpha\ .$$

Thus,

$$\text{Div}\{\text{m(a)} + \text{m(b)}\} = \tfrac{1}{4}(3-\alpha)\xi^3 I p_\alpha\ . \tag{47}$$

The counterterm is $y\xi p_\alpha$. Hence we choose

$$\text{Div}\{y\} = -\tfrac{1}{4}(3-\alpha)\xi^2 I\ . \tag{48}$$

Thus far, we have determined the divergent parts of the renormalization constants. Now we shall verify the relations between divergent parts of Δ_μ, $\Box_{\mu\nu}$, $\Pi_{\mu\nu}$, $\Gamma^{(3)}_{\alpha\mu\nu}$, and $\Gamma^{(4)}_{\alpha\beta\mu\nu}$.

D. $\Delta_\mu(p)$

The diagram for $\Delta_\mu(p)$ is shown in Fig. 7. It is found that

$$\text{Div}\{A\} = \frac{3\xi}{2\alpha}(\alpha^2 - \alpha + 2)I\ . \tag{49}$$

Then from Eqs. (49), (46), and (44) we easily verify the second of Eqs. (27), viz.,

$$\text{Div}\{A\} = -\frac{2}{\xi}\,\text{Div}\{w - \tilde{z}\}\ . \tag{50}$$

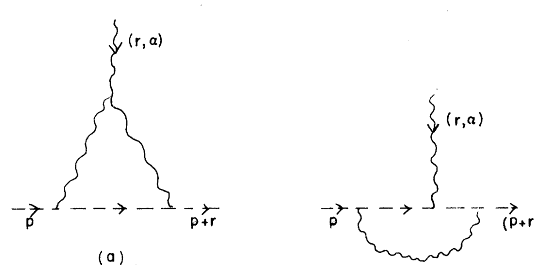

FIG. 12. One-loop diagrams for the ghost-ghost-photon vertex.

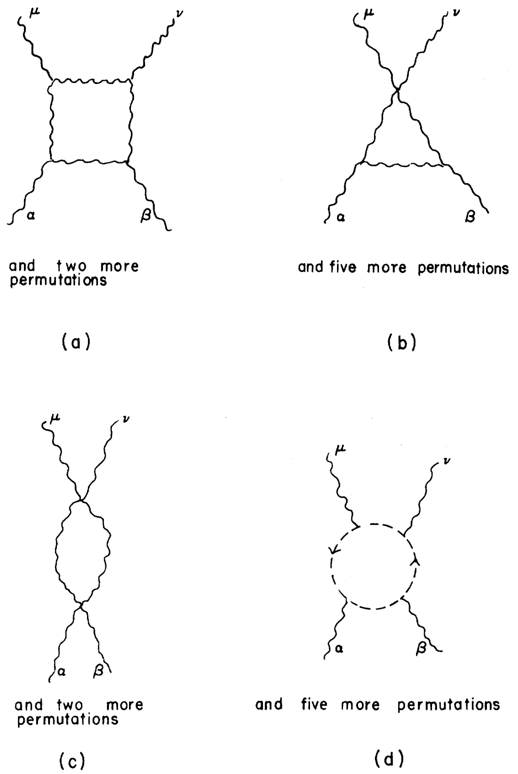

FIG. 13. One-loop diagrams for the $(A_\mu)^4$ vertex.

E. $\Box_{\mu\nu}(q,r)$ and $\Pi_{\mu\nu}(q,r)$

These are defined in Fig. 7. Here, we obtain

$$\text{Div}\{B\} = \frac{3\xi^2(1+\alpha)}{2\alpha} I\,,$$
$$\text{Div}\{C\} = \frac{3\xi^2(3+\alpha^2)}{2\alpha} I\,. \tag{51}$$

Therefore, we can verify the first of Eqs. (27), viz.,

$$-\text{Div}\{B + \tfrac{1}{2}C\} = \text{Div}\{y + w - 2\bar{z}\}\,. \tag{52}$$

F. The 4-photon vertex

The diagrams of Fig. 13 contribute to the 4-photon vertex $\Gamma^{(4)}_{\alpha\beta\mu\nu}$. The results are

$$\text{Div}\{\text{m(a)}\} = \frac{5i\xi^4}{4\alpha} I A_{\mu\nu\alpha\beta}\,,$$

$$\text{Div}\{\text{m(b)}\} = \frac{-2i\xi^4(\alpha+2)}{\alpha^2} I A_{\mu\nu\alpha\beta}\,,$$

$$\text{Div}\{\text{m(c)}\} = \frac{-i\xi^4(17+2\alpha+5\alpha^2)}{4\alpha^2} I A_{\mu\nu\alpha\beta}\,,$$

$$\text{Div}\{\text{m(d)}\} = \frac{-i\xi^4}{4} I A_{\mu\nu\alpha\beta}\,,$$

where

$$A_{\mu\nu\alpha\beta} = g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}\,. \tag{53}$$

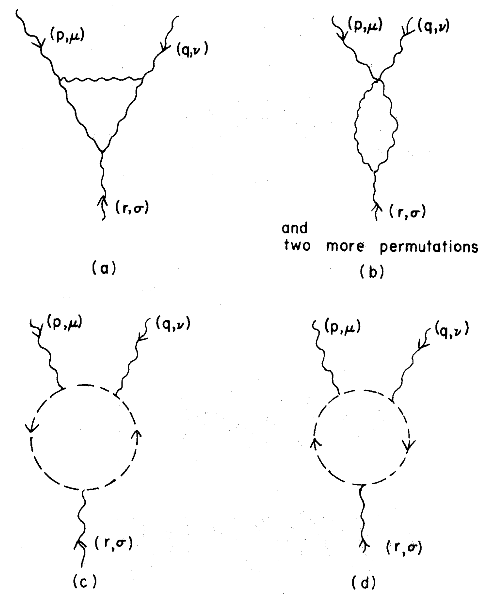

FIG. 14. One-loop diagrams for the $(A_\mu)^3$ vertex.

Hence, we can verify that the counterterm $[-i(2y+w-2\bar{z})(\xi^2/\alpha)A_{\mu\nu\alpha\beta}]$ cancels the divergence in the 4-point vertex.

G. 3-photon proper vertex

The diagrams of Fig. 14 contribute to the three-photon proper vertex. The results are

$$\text{Div}\{\text{m(a)}\} = -\frac{\xi^3}{\alpha^2}\left(\frac{3}{4}+\alpha\right) I F_{\mu\nu\sigma}\,,$$

$$\text{Div}\{\text{m(b)}\} = \frac{\xi^3}{\alpha^2}\left(\frac{9}{4} - \frac{\alpha}{2} + \frac{5}{4}\alpha^2\right) I F_{\mu\nu\sigma}\,, \tag{54}$$

$$\text{Div}\{\text{m(c)}+\text{m(d)}\} = -\frac{\xi^3}{4} I F_{\mu\nu\sigma}\,,$$

where

$$F_{\mu\nu\sigma} = p_\mu g_{\nu\sigma} + q_\nu g_{\mu\sigma} + r_\sigma g_{\mu\nu}\,.$$

Then it is easy to verify that the total divergence in the $(A_\mu)^3$ vertex is canceled by the counterterm

$$-\tfrac{1}{2}\xi(y+w-\bar{z})F_{\mu\nu\sigma}\,.$$

IV. THE S MATRIX

In this section, we shall show that the renormalized three- and four-photon S-matrix elements vanish.

First let us note that the polarization vector $\epsilon_\mu(p)$ of a physical photon ($p^2=0$) of momentum p satisfies $p\cdot\epsilon = 0$. ($\epsilon_\mu(p) \equiv \langle p|A_\mu(0)|0\rangle$. With the linear gauge condition $\partial_\mu A^\mu = 0$ it immediately follows that $p\cdot\epsilon=0$. In our case the gauge condition

$$f_\alpha[A] = \frac{1}{\sqrt{\alpha}}(\partial_\mu A^\mu - \tfrac{1}{2}\xi A_\mu A^\mu) = 0$$

means that matrix elements of $f_\alpha[A]$ between physical states vanish: $\langle p|f_\alpha[A]|0\rangle = 0$. This translates into $\epsilon^\mu[p_\mu - a\Delta_\mu(p)] = 0$, where a is some constant Since $\Delta_\mu(p) = p_\mu \Delta(p)$, it follows that $p\cdot\epsilon = 0$.)

It is easy to see that the three-photon amplitude vanishes on the mass shell. Three photons of momenta p, q, and r(with $p+q+r=0$) can be on the mass shell only when $p = \alpha q = \beta r$ for some α and β. Thus there is only one independent 4-vector. Any tensor with three Lorentz indices constructed out of it vanishes when dotted with polarization vectors.

Finally, we wish to show that the 4-photon amplitude vanishes on the mass shell. Since the amplitude is a truncated Green's function, it is easier to use the WT identity for $Z[J]$, the generating functional of the connected Green's functions. Referring back to Eq. (16), we can write the WT identity for $Z^{(r)}[J^{(r)}]$ in terms of renormalized quantities, dropping the superscript (r):

$$G_{\beta\alpha}[J]\partial_i^\beta J_i = \frac{1}{\alpha}\frac{W}{\tilde{Z}}\left[\partial_i^\alpha \frac{\delta Z}{\delta J_i} - \frac{Y}{2\tilde{Z}}\xi_{ij}^\alpha\left(\frac{\delta Z}{\delta J_i}\frac{\delta Z}{\delta J_j} - \frac{i\delta^2 Z}{\delta J_i \delta J_j}\right)\right]. \tag{55}$$

Differentiating with respect to J_k, J_l, and J_m and setting $J=0$, we obtain

$$\frac{\delta^2 G_{\beta\alpha}}{\delta J_k \delta J_m}\partial_i^\beta + (\text{two permutations of } k,l,m) - \frac{1}{\alpha}\frac{W}{\tilde{Z}}\partial_i^\alpha \frac{\delta^4 Z}{\delta J_i \delta J_k \delta J_m \delta J_l}$$
$$= -\frac{1}{\alpha}\frac{WY}{2\tilde{Z}^2}\xi_{ij}^\alpha\left(\frac{\delta^3 Z}{\delta J_i \delta J_k \delta J_l}\frac{\delta^2 Z}{\delta J_j \delta J_m} + \text{permutations}\right) + \frac{1}{\alpha}\frac{iYW}{2\tilde{Z}^2}\xi_{ij}^\alpha \frac{\delta^5 Z}{\delta J_i \delta J_j \delta J_k \delta J_l \delta J_m}\,. \tag{56}$$

We show the Fourier transform of Eq. (56) in Fig. 15. (A shaded box stands for a connected truncated Green's function.)

The first term (and its permutations) does not contribute when dotted with polarization vectors since it is proportional to $s_\lambda(q_\mu, r_\nu)$. The first term on the right-hand side does not contribute because it does not have a pole at $p^2=0$. One can verify that (at least) in the one-loop calculation the last term does not have a pole at $p^2=0$ that would contribute with on-mass-shell photons. Therefore, from Eq. (56) it follows that

$$\lim_{p^2,q^2,r^2,s^2\to 0} p^2 q^2 r^2 s^2 \epsilon^\mu(q)\epsilon^\nu(r)\epsilon^\lambda(s) p^\alpha G^{(4)}_{\alpha\mu\nu\lambda}(p,q,r,s)$$
$$\equiv p^\alpha T^{(4)}_{\alpha\mu\nu\lambda}\,\epsilon^\mu(q)\epsilon^\nu(r)\epsilon^\lambda(s) = 0\,. \tag{57}$$

Here $G^{(4)}_{\alpha\mu\nu\lambda}(p,q,r,s)$ is a connected 4-photon Green's function.

Equation (57) is just the statement of gauge invariance of the T matrix under an arbitrary gauge transformation $A_\mu(x) \to A_\mu(x) - (1/e)\partial_\mu\omega(x)$. Since an $\omega(x)$ exists which can change $f[\alpha,\xi,A(x)]$ to $f[\alpha,\xi+d\xi,A(x)]$, it follows, in particular, that

$$\frac{\partial}{\partial\xi}[T^{(4)}_{\alpha\mu\nu\lambda}(p,q,r,s)\epsilon^\alpha(p)\epsilon^\mu(q)\epsilon^\nu(r)\epsilon^\lambda(s)] = 0\,. \tag{58}$$

Since we know that $T^{(4)}_{\alpha\mu\nu\lambda} = 0$ at $\xi = 0$, Eq. (58) tells us that

$$T^{(4)}_{\alpha\mu\nu\lambda}(p,q,r,s)\epsilon^\alpha(p)\epsilon^\mu(q)\epsilon^\nu(r)\epsilon^\lambda(s) = 0 \tag{59}$$

for any ξ.

V. RENORMALIZATION OF SPINOR ELECTRODYNAMICS

In this section we shall consider a Dirac field (electron) interacting with the electromagnetic field quantized with the same gauge condition [of Eq. (3)]. We shall show that we can remove the divergences in all the proper vertices by multiplicative renormalizations on the electron field and electric charge e, in addition to the renormalizations done in Sec. II, and by choosing a mass counterterm δm. We shall be brief.

A. Preliminary

The Lagrangian (in terms of unrenormalized fields and parameters) is

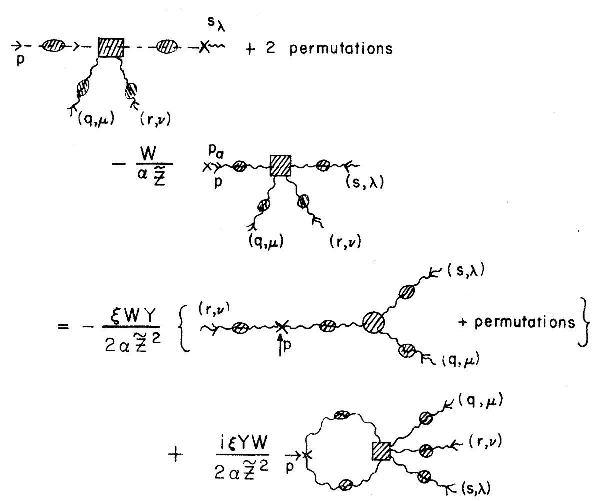

FIG. 15. Diagrammatic representation for Eq. (56). A shaded box stands for connected truncated Green's functions.

$$\mathcal{L}[\psi,\bar{\psi},A_\mu]=\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\partial\!\!\!/-eA\!\!\!/-m)\psi\,. \tag{60}$$

$\mathcal{L}$ is invariant under the local gauge transformations

$$\begin{aligned}&A_\mu(x)\to A_\mu(x)-\frac{1}{e}\partial_\mu\omega(x)\,;\\&\psi(x)\to e^{-i\omega(x)}\psi(x)\,,\\&\bar{\psi}(x)\to e^{i\omega(x)}\bar{\psi}(x)\,.\end{aligned} \tag{61}$$

We note that $M_{\alpha\beta}[A]$ of Eq. (5) is still unchanged and hence $\mathcal{L}_{\text{eff}}[\psi,\bar{\psi};A_\mu;c,\bar{c}]$ is given by

$$\mathcal{L}_{\text{eff}}[\psi,\bar{\psi};A_\mu;c,\bar{c}]=\mathcal{L}[\psi,\bar{\psi},A_\mu]+\mathcal{L}_{\text{gauge}}+\bar{c}_\alpha M_{\alpha\beta}c_\beta\,. \tag{62}$$

We note that there are no basic ghost-electron vertices.

The generating functional of the Green's functions is now constructed by introducing sources (corresponding to fermion fields) η_α and $\bar{\eta}_\beta$. They anticommute among themselves and with the electron field. We have

$$\begin{aligned}W_F[J,\eta,\bar{\eta}]=\int&[dA\,d\psi\,d\bar{\psi}\,dc\,d\bar{c}]\\&\times\exp\{i(\mathcal{L}_{\text{eff}}[\psi,\bar{\psi};A;c,\bar{c}]\\&\qquad+J_iA_i+\bar{\psi}_i\eta_i+\bar{\eta}_i\psi_i)\}\,.\end{aligned} \tag{63}$$

We define fields $\bar{\chi}$ and χ, the expectation values of electron fields $\bar{\psi}$ and ψ, by

$$\bar{\chi}_\alpha=-\frac{\delta Z}{\delta\eta_\alpha}\,,\quad\chi_\beta=\frac{\delta Z}{\delta\bar{\eta}_\beta}\,. \tag{64}$$

We also define

$$Z[J,\bar{\eta},\eta]=-i\ln W[J,\eta,\bar{\eta}]\,.$$

We define the generating functional of proper vertices by

$$\Gamma[\Phi;\bar{\chi},\chi]=Z[J,\eta,\bar{\eta}]-\bar{\chi}_i\eta_i-\bar{\eta}_i\chi_i-J_i\Phi_i\,. \tag{65}$$

The inverse propagator for the electron field in the presence of external sources is

$$s^{-1}{}_{ij}=\frac{\delta^2\Gamma}{\delta\chi_i\delta\bar{\chi}_j}\,, \tag{66}$$

while the propagator s_{ij} is given by

$$s_{ij}=-\frac{\delta^2Z}{\delta\bar{\eta}_i\delta\eta_j}\equiv-\bar{s}_{ji}\,. \tag{67}$$

B. WT identities

Let us obtain the WT identity for $W_F[J,\eta,\bar{\eta}]$. Following the procedure of Ref. 2 and noting the transformation properties of fields [Eq. (61)], we obtain the following identity:

$$\left\{-\frac{1}{\sqrt{\alpha}}f_\alpha\left[\frac{1}{i}\frac{\delta}{\delta J}\right]+\left[J_i\partial_i^\beta+ie\zeta_{ij}^\beta\left(\bar{\eta}_i\frac{\delta}{\delta\bar{\eta}_j}+\eta_i\frac{\delta}{\delta\eta_j}\right)\right]M^{-1}{}_{\beta\alpha}\left[\frac{1}{i}\frac{\delta}{\delta J_i},-\frac{1}{i}\frac{\delta}{\delta\eta},\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\right]\right\}W_F[J,\eta,\bar{\eta}]=0$$

[where $\zeta_{ij}^\beta=\delta^4(x_\beta-x_i)\delta^4(x_\beta-x_j)\delta_{ij}$].

Going through steps analogous to those of Sec. II B, we obtain

$$-\frac{1}{\alpha}f_\alpha\left[\Phi+i\Delta\frac{\delta}{\delta\Phi}\right]\cdot1+\left\{\frac{\delta\Gamma}{\delta\Phi_i}\partial_i^\beta+e\zeta_{ij}^\beta\left[\frac{\delta\Gamma}{\delta\chi_i}\left(\chi_j+is_{jk}\frac{\delta}{\delta\bar{\chi}_k}\right)-\frac{\delta\Gamma}{\delta\bar{\chi}_i}\left(\bar{\chi}_j+i\bar{s}_{jk}\frac{\delta}{\delta\chi_k}\right)\right]\right\}G_{\beta\alpha}[\Phi,\bar{\chi},\chi]=0\,. \tag{68}$$

Thus, the WT identity for the generating functional of proper vertices $\Gamma[\Phi,\bar{\chi},\chi]$ is

$$\begin{aligned}\left[\delta_{\eta\beta}\frac{\delta\Gamma}{\delta\Phi_i}\partial_i^\beta+e\zeta_{ij}^\beta\left(\chi_j\frac{\delta\Gamma}{\delta\chi_i}-\bar{\chi}_j\frac{\delta\Gamma}{\delta\bar{\chi}_i}\right)\delta_{\eta\beta}+ie\zeta_{ij}^\beta\left(s_{jk}\frac{\delta G^{-1}{}_{\xi\eta}}{\delta\bar{\chi}_k}G_{\beta\xi}\frac{\delta\Gamma}{\delta\chi_i}-\bar{s}_{jk}\frac{\delta G^{-1}{}_{\xi\eta}}{\delta\chi_k}G_{\beta\xi}\frac{\delta\Gamma}{\delta\bar{\chi}_i}\right)\right]G_{\eta\alpha}[\Phi,\bar{\chi},\chi]\\=-\frac{1}{\alpha}[\partial_i^\alpha\Phi_i-\tfrac{1}{2}\xi_{ij}^\alpha(\Phi_i\Phi_j+i\Delta_{ij})]\,.\end{aligned} \tag{69}$$

C. Renormalization transformations

In addition to the renormalization transformations defined in Sec. II D, we define the following renormalizations on the fields $\chi,\bar{\chi}$ and on the electric charge e:

$$\chi=Z_\chi{}^{1/2}\chi^{(r)}\,,\quad\bar{\chi}=Z_\chi{}^{1/2}\bar{\chi}^{(r)}\,,$$

$$e=e^{(r)}XZ_\chi{}^{-1}Z^{-1/2}\,,$$

and

$$G_{\alpha\beta}[\Phi,\chi,\bar{\chi};\alpha,\xi,e]=\tilde{Z}G_{\alpha\beta}^{(r)}[\Phi^{(r)},\chi^{(r)},\bar{\chi}^{(r)};\alpha^{(r)},\xi^{(r)},e^{(r)}]\,,$$

$$\Gamma[\Phi,\chi,\bar{\chi};\alpha,\xi,e]=\Gamma^{(r)}[\Phi^{(r)},\chi^{(r)},\bar{\chi}^{(r)};\alpha^{(r)},\xi^{(r)},e^{(r)}]\,.$$

In the following we shall express everything in terms of renormalized quantities and drop the superscript (r).

The WT identity of Eq. (69) becomes

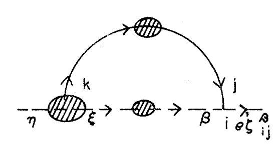

FIG. 16. Diagrammatic representation for $K_{\eta i}[\Phi,\chi,\bar{\chi}]$.

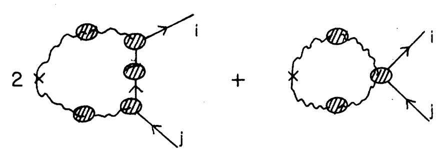

FIG. 17. Diagrammatic representation for $\delta^2\mathfrak{F}_\alpha/\delta\bar\chi_i\delta\chi_j|_{\Phi=\chi=\bar\chi=0}$.

$$\left[\delta_{\eta\beta}\partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i} + \frac{eX}{Z_\chi}\zeta_{ij}^\beta\left(\chi_j\frac{\delta\Gamma}{\delta\chi_i} - \bar\chi_j \frac{\delta\Gamma}{\delta\bar\chi_i}\right)\delta_{\eta\beta} - \frac{X}{Z_\chi}\left(\bar K_{\eta i}\frac{\delta\Gamma}{\delta\bar\chi_i} - K_{\eta i}\frac{\delta\Gamma}{\delta\chi_i}\right)\right] G_{\eta\alpha} = -\frac{1}{\alpha}\mathfrak{F}_\alpha\,, \quad (70)$$

where

$$\bar K_{\eta i} \equiv + ie\zeta_{ij}^\beta\, \bar s_{jk} \frac{\delta G^{-1}{}_{\xi\eta}}{\delta\chi_k} G_{\beta\xi}\,,$$

$$K_{\eta i} = ie\zeta_{ij}^\beta\, s_{jk}\frac{\delta G^{-1}{}_{\xi\eta}}{\delta\bar\chi_k} G_{\beta\xi}\,, \quad (71)$$

$$\mathfrak{F}_\alpha = + \frac{W}{\bar Z}\left[\partial_i^\alpha\Phi_i - \frac{Y}{2\bar Z}\,\xi_{ij}^\alpha(\Phi_i\Phi_j + i\Delta_{ij})\right].$$

A diagrammatic representation for $K_{\eta i}$ is shown in Fig. 16.

D. Analysis of divergences in $G^{-1}[\Phi,\chi,\bar\chi]$, $\mathfrak{F}_a[\Phi,\chi,\bar\chi]$ and $K_{\eta i}[\Phi,\chi,\bar\chi]$,

1. $G^{-1}[\Phi,\chi,\bar\chi]$

Referring back to the discussion of Sec. II E (A) (see Figs. 3 and 4), we need only worry about the internal subtractions for renormalization parts containing the rightmost vertex in Fig. 3. In introducing the fermion fields we do not introduce any such additional renormalization parts, since any diagram with two ghost lines and two or more fermion lines has a superficial degree of divergence, $D \leq -1$. Thus the discussion of Secs. II E (A) and II E (B) goes through.

2. $\mathfrak{F}_a[\Phi,\chi,\bar\chi]$

Here, too, we do not introduce any new renormalization parts in derivatives of $\mathfrak{F}_\alpha[\Phi,\chi,\bar\chi]$ which contain the leftmost vertex denoted by a cross. (See Figs. 7 and 8.) Here, too, any subdiagram containing this vertex and two or more fermion lines has $D \leq -1$. Hence, the discussion of Sec. II E (C) goes through unchanged. A similar discussion, as applied to $\delta^2\mathfrak{F}_\alpha/\delta\bar\chi_i\delta\chi_j|_{\Phi=\chi=\bar\chi=0}$ (which is represented by Fig. 17) shows that it becomes finite to the n-loop approximation once the appropriate choices of renormalization constants up to the $(n-1)$-loop approximation have been made according to Eq. (26).

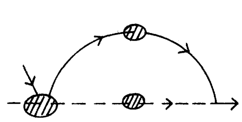

FIG. 18. Diagrammatic representation for $[\delta K_{\eta i}/\delta\chi_m]_{\Phi=\chi=\bar\chi=0}$.

3. $K_{\eta i}[\Phi,\chi,\bar\chi]$

We shall show that $K_{\eta i}$ and $\bar K_{\eta i}$ become finite to the n-loop approximation once the proper vertices to the $(n-1)$-loop approximation have been made finite. It is clear that the lowest derivative of $K_{\eta i}$ which is nonzero at $\phi=\chi=\bar\chi=0$ is $\delta K_{\eta i}/\delta\chi_j|_{\Phi=\chi=\bar\chi=0}$, since $K_{\eta i}[\phi=0=\chi=\bar\chi]=0$. The first derivative is shown in Fig. 18.

The blobs in Fig. 18 are made finite by renormalization counterterms introduced up to the $(n-1)$-loop approximation, and the diagram needs subtractions for renormalization parts containing the rightmost vertex. But there are no such renormalization parts. A suspected renormalization part shown in Fig. 19 is not a renormalization part because the leftmost vertex on the ghost line within this subdiagram must be a $\bar c c A_\mu$ vertex and it contains a factor of external momentum q_μ (external to this subdiagram). Thus this subdiagram has $D=-1$. Furthermore, $\delta K_{\eta i}/\delta\chi_j|_{\Phi=\chi=\bar\chi=0}$ is itself not a renormalization part ($D=-1$) and hence it becomes finite in the n-loop approximation once the counterterms are chosen up to the $(n-1)$-loop approximation. A similar discussion goes through for higher derivatives of $K_{\eta i}$ and also for $\bar K_{\eta i}[\Phi,\chi,\bar\chi]$.

E. Proof of renormalizability

Here we shall deal only with the new renormalization part, the $\bar\psi\psi A_\mu$ vertex. The discussion for the remaining renormalization parts proceeds parallel to the discussion in Sec. II F and will not be repeated here.

We assume that renormalization constants and a mass counterterm have been chosen up to the $(n-1)$-loop approximation making all the proper vertices finite up to the $(n-1)$-loop approximation. We assume that the proper choice of z_n, w_n, $\tilde z_n$, $(z_\chi)_n$, and $(\delta m)_n$ has been made making the photon, ghost, and electron propagators finite to the n-loop approximation. We shall show that it is possible to make the $\bar c c A_\mu$ vertex finite with appropriate choices of x_n and that we may choose $x_n=(z_\chi)_n$ if we have chosen $x_r=(z_\chi)_r$ $(0\leq r\leq n-1)$, yielding the usual Ward identity.

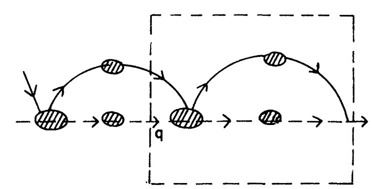

FIG. 19. A suspected renormalization part of diagram in Fig. 18.

Differentiate the WT identity of Eq. (70) with respect to χ_m and $\bar{\chi}_n$ and set $\Phi=\chi=\bar{\chi}=0$. We get

$$\partial_i^\beta \frac{\delta^3\Gamma}{\delta\Phi_i\delta\chi_m\delta\bar{\chi}_n} G_{\beta\alpha} - e\frac{X}{Z_\chi}\left(\zeta_{im}^\beta \frac{\delta^2\Gamma}{\delta\bar{\chi}_n\delta\chi_i} + \zeta_{in}^\beta \frac{\delta^2\Gamma}{\delta\chi_m\delta\bar{\chi}_i}\right) G_{\beta\alpha} - \frac{X}{Z_\chi} G_{\eta\alpha}\left(\frac{\delta\bar{K}_{\eta i}}{\delta\bar{\chi}_n}\frac{\delta^2\Gamma}{\delta\chi_m\delta\bar{\chi}_i} + \frac{\delta K_{\eta i}}{\delta\chi_m}\frac{\delta^2\Gamma}{\delta\bar{\chi}_n\delta\chi_i}\right) = -\frac{1}{\alpha}\frac{\delta^2\mathfrak{F}_\alpha}{\delta\chi_m\delta\bar{\chi}_n} . \tag{72}$$

Equating the n-loop divergence on both sides, we obtain that

$$\partial_i^\beta \left[\frac{\delta^3\Gamma}{\delta\Phi_i\delta\chi_m\delta\bar{\chi}_n}\right]_n^{\text{div}} [G_{\beta\alpha}]_0 + e\left[\frac{X}{Z_\chi}\right]_n^{\text{div}} [+\zeta_{im}^\beta S^{-1}{}_{in} - \zeta_{in}^\beta S^{-1}{}_{mi}]_0 [G_{\beta\alpha}]_0 \text{ is finite} , \tag{73}$$

since

$$\left[\frac{\delta K_{\eta i}}{\delta\bar{\chi}_m}\bigg|_{\Phi=0=\chi=\bar{\chi}}\right]_n^{\text{div}} = 0 = \left[\frac{\delta^2\mathfrak{F}_\alpha}{\delta\chi_m\delta\bar{\chi}_n}\bigg|_{\Phi=0=\chi=\bar{\chi}}\right]_n^{\text{div}} , \text{ etc.}$$

It is clear from Eq. (73) that if we choose x_n such that

$$\left[\frac{X}{Z_\chi}\right]_n^{\text{div}} \text{ is finite} \tag{74}$$

we will have

$$\left[\frac{\delta^3\Gamma}{\delta\Phi_i\delta\chi_m\delta\bar{\chi}_n}\right]_n^{\text{div}} \text{ finite} .$$

Further, if we have chosen $x_r=(z_\chi)_r$ $(0\le r\le n-1)$, then Eq. (74) gives

$$\text{Div}\{x_n\} = \text{Div}\{(z_\chi)_n\}$$

and we may choose the finite part of x_n such that $x_n=(z_\chi)_n$.

VI. A COMPLEX SCALAR FIELD INTERACTING WITH AN ELECTROMAGNETIC FIELD

In this section we shall discuss the renormalization of a complex scalar field interacting with an electromagnetic field when the gauge condition chosen is bilinear. We shall consider only the unbroken version of the theory ($\mu^2{}_{\text{ren}}>0$).

The Lagrangian in terms of unrenormalized fields is

$$\mathcal{L} = |(\partial_\mu - ieA_\mu)\phi|^2 - \mu^2\phi^*\phi + \tfrac{1}{4}\lambda(\phi^*\phi)^2 - \tfrac{1}{4}\mathfrak{F}_{\mu\nu}\mathfrak{F}^{\mu\nu} . \tag{75}$$

$\mathcal{L}$ is invariant under the electromagnetic gauge transformation

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\omega(x) ;$$

$$\phi(x) \to e^{-i\omega(x)}\phi(x) , \quad \phi^*(x) \to e^{i\omega(x)}\phi^*(x) .$$

We shall show the following:

(i) With a simple counterexample, if we choose the previous gauge function $f[A]=(1/\sqrt{\alpha})(\partial_\mu A^\mu - \tfrac{1}{2}\xi A^2)$ it is not possible to make proper vertices finite by renormalization on fields and parameters ξ, e, α, μ^2, and λ.

(ii) However, if we choose the gauge function $f'=(1/\sqrt{\alpha})(\partial_\mu A^\mu - \tfrac{1}{2}\xi A^2 - \tfrac{1}{2}\eta\phi^*\phi)$ and renormalize parameter η independently, all the proper vertices can be made finite.

A. The gauge function $f=(1/\sqrt{\alpha})(\partial_\mu A^\mu - \tfrac{1}{2}\xi A^2)$

Since this discussion is similar to that in Sec. V, we shall be brief.

Let us introduce two sources K_a ($a=1$ or 2; x_a) corresponding to fields ϕ_a [$\phi_1\equiv\phi^*(x)$, $\phi_2\equiv\phi(x)$]. The generating functional of the Green's functions is given by

$$W[J,K] = \int [dA_i d\phi_a dc\,d\bar{c}] \exp\{i(\mathcal{L}_{\text{eff}}[A_i,\phi_a,c,\bar{c}] + J_iA_i + K_a\phi_a)\} ,$$

$$Z[J,K] \equiv -i\ln W[J,K] . \tag{76}$$

As before, we define expectation values Ψ_a for field ϕ_a in presence of external sources by

$$\Psi_a = \frac{\delta Z[J,K]}{\delta K_a} .$$

We note that under an infinitesimal gauge transformation

$$A_i \to A_i - \frac{1}{e}\partial_i^\beta\omega_\beta ; \quad \phi_a \to \phi_a + i\zeta_{ab}^\beta\phi_b\omega_\beta$$

$$[\zeta_{ab}^\alpha \equiv \delta^4(x_\alpha - x_b)\delta^4(x_\alpha - x_a)\zeta_{ab} ; \ \zeta_{11} = -\zeta_{22} = 1 ;$$

$$\zeta_{12} = \zeta_{21} = 0] . \tag{77}$$

Following the derivation of the WT identity, we obtain the following WT identity for $\Gamma[\Phi,\Psi]$:

$$G_{\beta\alpha}\left[\partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i} - e\zeta_{dc}^\sigma\left(\Psi_c\delta_{\beta\sigma} - iP_{ca}G_{\sigma\xi}\frac{\delta G^{-1}{}_{\xi\beta}}{\delta\Psi_a}\right)\frac{\delta\Gamma}{\delta\Psi_d}\right] = -\frac{1}{\alpha}[\partial_i^\alpha\Phi_i - \tfrac{1}{2}\xi_{ij}^\alpha(\Phi_i\Phi_j + i\Delta_{ij})] \tag{78}$$

(where $P_{ca}\equiv -\delta^2 Z/\delta K_c\delta K_a$ = propagator of ϕ field).

To exhibit the difficulty, let us consider Eq. (78) up to the one-loop approximation. Since there are no basic ($\bar{c}c\phi$) vertices, the term

$ieP_{ca}G_{\sigma\xi}(\delta G^{-1}{}_{\xi\beta}/\delta\Psi_a)\zeta^{\sigma}_{dc}$ contains at least two loops and hence will be dropped in this consideration.

Introduce the renormalization transformation identical to those of Sec. II D; in addition to the renormalization of e and the scalar field,

$$\Psi = Z_\psi{}^{1/2}\Psi^{(r)}\,, \quad e = e_r XZ_\psi{}^{-1}Z^{-1/2}\,. \tag{79}$$

Then expressing Eq. (78) to the one-loop approximation [in terms of renormalized quantities, dropping superscript (r)], we obtain

$$G_{\beta\alpha}\left(\partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i} - e\frac{X}{Z_\psi}\zeta^\beta_{dc}\Psi_c\frac{\delta\Gamma}{\delta\Psi_d}\right) = -\frac{1}{\alpha}\frac{W}{\tilde{Z}}\left[\partial_i^\alpha\Phi_i - \frac{Y}{2\tilde{Z}}\xi^\alpha_{ij}(\Phi_i\Phi_j + i\Delta_{ij})\right]. \tag{80}$$

Let us write $Z=1+z$, $W=1+w$, etc.

Suppose we have chosen z, w, $\delta\mu^2$, z_ψ, and $\tilde{z}$ to make the photon, scalar, and ghost propagators finite to the one-loop approximation. Then we shall show that charge renormalization alone cannot remove the divergence in the $(\phi^*\phi A_\mu)$ vertex. Essentially, this happens because the $(\phi^*\phi A_\mu)$ vertex in the one-loop approximation has a divergence proportional to the photon momentum in addition to the divergence of the form of the bare vertex.

Differentiating Eq. (80) with respect to Ψ_a and Ψ_e and setting $\Phi=0=\Psi$ (the vacuum expectation values), we obtain

$$G_{\beta\alpha}\partial_i^\beta\frac{\delta^3\Gamma}{\delta\Phi_i\delta\Psi_a\delta\Psi_e} - e(x-z_\psi)G_{\beta\alpha}\left(\zeta^\beta_{da}\frac{\delta^2\Gamma}{\delta\Psi_e\delta\Psi_d} + \zeta^\beta_{de}\frac{\delta^2\Gamma}{\delta\Psi_a\delta\Psi_d}\right) = \frac{i}{2\alpha}\left(\frac{WY}{\tilde{Z}^2}\right)\xi^\alpha_{ij}\frac{\delta^2\Delta_{ij}}{\delta\Psi_a\delta\Psi_e}\,. \tag{81}$$

Let us choose $a=1$, $e=2$ in Eq. (81).

Remembering that the propagators are made finite to the one-loop approximation, we may equate the divergence on both sides of Eq. (81) in that approximation:

$$[G_{\beta\alpha}]_0\partial_i^\beta\left[\frac{\delta^3\Gamma}{\delta\Phi_i\delta\Psi_a\delta\Psi_e}\right]_1^{\text{div}} - e(x-z_\psi)[G_{\beta\alpha}]_0\left[\zeta^\beta_{da}\frac{\delta^2\Gamma}{\delta\Psi_e\delta\Psi_d} + (a \leftrightarrow e)\right]_0 = \frac{i}{2\alpha}\left[\xi^\alpha_{ij}\frac{\delta^2\Delta_{ij}}{\delta\Psi_a\delta\Psi_e}\right]_1^{\text{div}} + \text{finite terms}. \tag{82}$$

Let us define

$$\text{F.T.}\left\{\frac{\delta^3\Gamma}{\delta\Phi_i\delta\Psi_a\delta\Psi_e}\right\} = i\Gamma^{(3)}_\mu(p;q,r).$$

Then

$$[\Gamma^{(3)}_\mu(p;q,r]_1^{\text{div}} = (q+r)_\mu b(\epsilon) + (q-r)_\mu c(\epsilon)\,,$$

$$\text{F.T.}\left\{\frac{\delta^2\Gamma}{\delta\Psi_d\delta\Psi_e}\right\}_0 = r^2-\mu^2\,,$$

$$\text{F.T.}\left\{\frac{\delta^2\Gamma}{\delta\Psi_a\delta\Psi_d}\right\} = q^2-\mu^2\,,$$

$$\text{F.T.}\{G_{\beta\alpha}\}_0 = \frac{1}{p^2}\,,$$

$$\text{F.T.}\left\{\xi^\alpha_{ij}\frac{\delta^2\Delta_{ij}}{\delta\Psi_a\delta\Psi_e}\right\}_1^{\text{div}} \equiv \Sigma(\epsilon)\,.$$

Then Eq. (82) becomes

$$b(\epsilon) + \frac{q^2-r^2}{(q+r)^2}c(\epsilon) - e(x-z_\psi)\frac{(r^2-q^2)}{(r+q)^2} = \frac{1}{2\alpha}\Sigma(\epsilon) + \text{finite terms}. \tag{83}$$

Thus, we can make $c(\epsilon)$ finite by choosing $x=z_X$ +finite terms. Also,

$$b(\epsilon) = \frac{1}{2\alpha}\Sigma(\epsilon) + \text{finite terms}.$$

An explicit calculation of $\Sigma(\epsilon)$ shows that it is divergent. Hence, it follows that the $\phi^*\phi A_\mu$ vertex will necessarily contain divergence. [We show the graphs contributing to $\Sigma(\epsilon)$ in Fig. 20.]

We may express this in another way. We saw in Eq. (83) that a derivative of $\mathfrak{F}_\alpha[\Phi,\Psi]$, viz.,

$$\left.\frac{\delta^2\mathfrak{F}_\alpha}{\delta\Psi_a\delta\Psi_e}\right|_{\Phi=0,\,\Psi=0}\,,$$

contained (nonrenormalizable) divergence, and by virtue of WT identity there must be divergence in its left-hand side which must come from a proper vertex. The reason why $\mathfrak{F}_\alpha[\Phi,\Psi]$ cannot be made finite, as against the previous two cases, is that the derivatives of $\mathfrak{F}_\alpha[\Phi,\Psi]$ at $\Phi=\Psi=0$ need additional internal subtractions (in addition to those shown in Fig. 8; see Sec. II E). For example, we may consider $\Delta_\mu(p)$ defined in Fig. 7. The additional subtractions needed are shown in Figs. 21(a) and 21(b). Clearly these subtractions cannot be expressed as

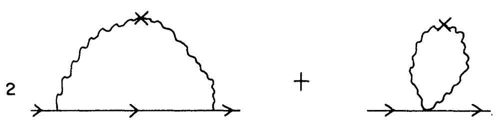

FIG. 20. Diagrammatic representation of Σ.

(divergent constant)$\times \Delta_\mu(p)$,

unlike the subtractions in Fig. 8. These are rather generated out of derivatives of a loop consisting of a scalar propagator. See Fig. 21(c). This suggests that we modify the gauge functional to $f' = (1/\sqrt{\alpha})(\partial_\mu A^\mu - \frac{1}{2}\xi A^2 - \frac{1}{2}\eta\phi^*\phi)$. Then we may be able to make

$$f_\alpha\left[\Phi + i\Delta\frac{\delta}{\delta\Phi}\,,\ \Psi + iP\frac{\delta}{\delta\Psi}\right]\cdot 1$$

finite.

B. The gauge condition $f' = (1/\sqrt{\alpha})(\partial_\mu A^\mu - \frac{1}{2}\xi A^2 - \frac{1}{2}\eta\phi^*\phi)$

Let us choose the gauge functional

$$f'_\alpha[A,\Phi] = \frac{1}{\sqrt{\alpha}}(\partial_i^\alpha A_i - \tfrac{1}{2}\xi_{ij}^\alpha A_i A_j - \tfrac{1}{2}\eta_{ab}^\alpha \phi_a\phi_b). \tag{84}$$

In this section, we shall use Hermitian fields

$$\phi_1 = \frac{\phi+\phi^*}{2}\,,\quad \phi_2 = \frac{\phi-\phi^*}{2i}$$

[so that $\eta_{ab}^\alpha = \delta^4(x_\alpha - x_a)\delta^4(x_a - x_b)\eta_{ab}$, $\eta_{11} = \eta_{22} = 1$, $\eta_{12} = \eta_{21} = 0$].

Since the last term is gauge-invariant, $M_{\alpha\beta}$ of Eq. (5) is unchanged, i.e., the ghost Feynman rules are unchanged. The Feynman rules for $\phi^*\phi A_\mu$, $(\phi^*\phi)^2$, and $(\phi^*\phi A^2)$ vertices are changed. $G^{-1}[\Phi,\Psi]$ is still given by the same formal expression of Eq. (14). The new WT identity of $\Gamma[\Phi,\Psi]$ is

$$G_{\beta\alpha}\left[\partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i} - e\zeta_{ac}^\sigma\left(\delta_{\sigma\beta}\Psi_c - iP_{ce}G_{\sigma\xi}\frac{\delta G^{-1}{}_{\xi\beta}}{\delta\Psi_e}\right)\frac{\delta\Gamma}{\delta\Psi_a}\right]$$
$$= -\frac{1}{\alpha}f'_\alpha\left[\Phi + i\Delta\frac{\delta}{\delta\Phi}\,,\ \Psi + iP\frac{\delta}{\delta\Psi}\right]\cdot 1$$
$$= -\frac{1}{\alpha}[\partial_i^\alpha\Phi_i - \tfrac{1}{2}\xi_{ij}^\alpha(\Phi_i\Phi_j + i\Delta_{ij}) - \tfrac{1}{2}\eta_{bc}^\alpha(\Psi_b\Psi_c + iP_{bc})]$$
$$(\zeta_{12} = -\zeta_{21} = 1\,,\ \zeta_{11} = \zeta_{22} = 0)\,. \tag{85}$$

We define renormalized fields, parameters, and renormalization constants by

$$\Psi_a = Z_\psi{}^{1/2}\Psi_a^{(r)}\,,\quad \eta = \eta^{(r)}V\,,$$
$$e = e^{(r)}XZ_\psi{}^{-1}Z^{-1/2}\,,\quad \mu^2 = \mu_a{}^{2(r)} + \delta\mu_a{}^2\,, \tag{86}$$

in addition to those defined in Sec. II D.

Expressing the WT identity in terms of renormalized quantities [and dropping the suffix (r)], we have

$$G_{\beta\alpha}\left[\partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i} + e\frac{X}{Z_\psi}\zeta_{ac}^\sigma\left(\delta_{\sigma\beta}\Psi_c - iP_{ce}G_{\sigma\xi}\frac{\delta G^{-1}{}_{\xi\beta}}{\delta\Psi_e}\right)\frac{\delta\Gamma}{\delta\Psi_a}\right]$$
$$= -\frac{1}{\alpha}\mathfrak{F}'_\alpha[\Phi,\Psi]\,, \tag{87}$$

where

$$\mathfrak{F}'_\alpha[\Phi,\Psi] = \frac{W}{\tilde{Z}}\left[\partial_i^\alpha\Phi_i - \frac{Y}{2\tilde{Z}}\xi_{ij}^\alpha(\Phi_i\Phi_j + i\Delta_{ij}) - \tfrac{1}{2}V\eta_{bc}^\alpha(\Psi_b\Psi_c + iP_{bc})\right]. \tag{88}$$

We define

$$L_a^\beta = e\frac{X}{Z_\psi}\left(\zeta_{ab}^\beta\Psi_b - i\zeta_{ac}^\sigma P_{ce}G_{\sigma\xi}\frac{\delta G^{-1}{}_{\xi\beta}}{\delta\Psi_e}\right). \tag{89}$$

Then the WT identity reads

$$G_{\beta\alpha}\left[\partial_i^\beta\frac{\delta\Gamma}{\delta\Phi_i} + L_a^\beta\frac{\delta\Gamma}{\delta\Psi_a}\right] = -\frac{1}{\alpha}\mathfrak{F}'_\alpha[\Phi,\Psi]\,, \tag{90a}$$

i.e.,

$$\left[\partial_i^\beta\frac{\delta\Gamma}{\delta\Phi_i} + L_a^\beta\frac{\delta\Gamma}{\delta\Psi_a}\right] = -\frac{1}{\alpha}\mathfrak{F}'_\alpha[\Phi,\Psi]\,G^{-1}{}_{\alpha\beta}\,. \tag{90b}$$

The second term in L_a^β [Eq. (89)] is identical in form to $K[\Phi,\chi,\bar{\chi}]$ of Sec. V C and has the same diagrammatic representation (see Fig. 16).

We shall see that, like the general linear gauge (discussed in Ref. 2), the scalar field develops a vacuum expectation value even though $\mu_{\rm ren}{}^2 > 0$.

Consider the WT identity of Eq. (90a) for $\Phi_i = 0$, $\Psi_a = u_a$, where u_a are real constants. Let v_a be the vacuum expectation value of ϕ_a. In a vacuum,

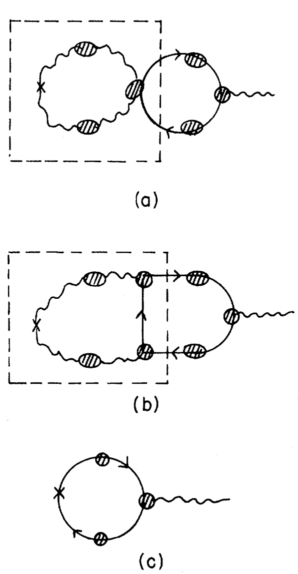

FIG. 21. Additional subtractions needed for $\Delta_\mu(p)$.

$$\left.\frac{\delta\Gamma}{\delta\Psi_a}\right|_{\text{vac}} \equiv 0\,, \quad \frac{\delta\Gamma}{\delta\Phi_i} \equiv 0\,. \tag{91}$$

Here the meaning of $\delta\Gamma/\delta\Phi_i|_{\text{vac}} \equiv 0$ should be carefully noted. For $\Phi = 0$ and $\Psi_a = u_a$, $\delta\Gamma/\delta\Phi_i$ has, in momentum space, the form

$$\text{F.T.}\left\{\left.\frac{\delta\Gamma}{\delta\Phi_i}\right|_{\Phi=0,\ \Psi_a=u_a}\right\} = p_\mu J(p^2, (u_1^2+u_2^2)^{1/2})|_{p\to 0}$$

and thus it is zero for any u_a. However, it is not true that $J(p^2, (u_1^2+u_2^2)^{1/2})|_{p\to 0}$ is zero for any u_a. In vacuum, $J(p^2, (v_1^2+v_2^2)^{1/2}) = 0$, so that

$$\begin{aligned}\text{F.T.}\left\{G_{\beta\alpha}\partial_i^\beta \frac{\delta\Gamma}{\delta\Phi_i}\right\}_{\Phi=0,\ \Psi_a=v_a} &\sim \frac{1}{p^2} p_\mu p^\mu J(p^2, (v_1^2+v_2^2)^{1/2})|_{p\to 0} \\ &= J(p^2, (v_1^2+v_2^2)^{1/2})|_{p\to 0}\end{aligned}$$

is zero too. Hence from Eq. (90a) we obtain that v_a satisfies

$$\mathfrak{F}'_\alpha[\Phi = 0, \Psi_a = v_a] = 0\,. \tag{92}$$

v_a is, in general, nonzero and ϵ-dependent (i.e., infinite in the limit $\epsilon \to 0$). v_a's are to be determined from solutions of

$$\left.\frac{\delta\Gamma}{\delta\Psi_a}\right|_{\Psi_a=v_a,\ \Phi=0} = 0\,. \tag{93}$$

We, then, define the shifted fields Ψ'_a by

$$\Psi_a = v_a + \Psi'_a$$

and make this substitution in the effective action. The proper vertices of the theory are obtained by expanding $\Gamma[\Phi_i, \Psi'_a]$ around $\Phi_i = 0$, $\Psi'_a = 0$.

We note that as a result of the substitution $\Psi_a = \Psi'_a + v_a$ in the effective Lagrangian, there are new vertices ($\phi' A_\mu$, $\phi' A^2$, ϕ'^3, ϕ') created in higher orders. All these vertices have dimensions three or lower.

We note that the presence of these vertices does not create any new renormalization subdiagrams in $G^{-1}[\Phi, \Psi']$, $L[\Phi, \Psi']$, and $\mathfrak{F}'_\alpha[\Phi, \Psi']$ of the kind that would need further internal subtractions (i.e., subtractions not taken care of by renormalization counterterms—see Secs. II E and V D). This follows because, as mentioned earlier, such renormalization parts have $D = 0$ at most, and the inclusion of any of the new vertices lowers D by one.

The presence of these vertices creates new renormalization parts in derivatives of $\Gamma[\Phi, \Psi']$ and $\mathfrak{F}'_\alpha[\Phi, \Psi']$; they are

$$\frac{\delta^2\Gamma}{\delta\Phi_i\delta\Psi'_a}\,, \quad \frac{\delta^3\Gamma}{\delta\Psi'_a\delta\Phi_i\delta\Phi_j}\,, \quad \frac{\delta^3\Gamma}{\delta\Psi'_a\delta\Psi'_b\delta\Psi'_c}\,, \quad \frac{\delta\mathfrak{F}'_\alpha}{\delta\Psi'_a}\,.$$

We need to show that these become finite with the others.

$$\left.\frac{\delta^2\mathfrak{F}'_\alpha}{\delta\Phi_i\delta\Psi'_a}\right|_{\Phi=0=\Psi'_a}$$

and

$$\left.\frac{\delta G^{-1}{}_{\alpha\beta}}{\delta\Psi'_a}\right|_{\Phi=0=\Psi'_a}$$

are not renormalization parts.

Taking these facts into account, we can carry out an analysis of divergences in $G^{-1}{}_{\alpha\beta}[\Phi, \Psi'_a]$, $L_a^\alpha[\Phi, \Psi'_a]$, and $\mathfrak{F}'_\alpha[\Phi, \Psi'_a]$ analogous to that in Secs. II E and V D. In the discussion for $\mathfrak{F}'_\alpha[\Phi, \Psi']$, we only need to remember the need for additional subtractions, which are shown in Fig. 21. Qualitatively the result is the same, namely, with appropriate choices of W, Y, and V in each loop approximation, $\delta\mathfrak{F}'_\alpha/\delta\Phi_i|_{\Phi=0=\Psi'_a}$, $\delta^2\mathfrak{F}'_\alpha/\delta\Psi'_i\Psi'_j|_{\Phi=0=\Psi'_a}$, and higher derivatives of $\mathfrak{F}'_\alpha$ can be made finite. $\delta^2\mathfrak{F}_\alpha/\delta\Phi_i\delta\Psi'_a|_{\Phi=0=\Psi'_a}$ also becomes finite, since it is not a renormalization part. This, however, does not apply to $\delta\mathfrak{F}'_\alpha/\delta\Psi'_\alpha|_{\Phi=0=\Psi'_\alpha}$. Also, derivatives of $L_a^\alpha[\Phi, \Psi']$ become finite in the n-loop approximation once the counterterms up to the $(n-1)$-loop approximation are chosen to make $\Gamma[\Phi, \Psi']$ and $G[\Phi, \Psi']$ finite. $L_a^\alpha[\Phi = 0 = \Psi'_a]$ may contain divergence, which in momentum space is independent of external momentum.

To prove renormalizability, let us assume that the counterterms chosen up to the $(n-1)$-loop approximation and the choice of v up to the $(n-1)$-loop approximation make derivatives of $\Gamma[\Phi, \Psi'_a]$, $G^{-1}{}_{\alpha\beta}[\Phi, \Psi'_a]$, $L_a^\alpha[\Phi, \Psi'_a]$, and $\mathfrak{F}'_\alpha[\Phi, \Psi'_a]$ around $\Phi = \Psi'_a = 0$ finite. Then we have to show that we can choose the counterterms to the n-loop approximation (and determine v to the n-loop approximation) which will make the derivatives of $\Gamma[\Phi, \Psi'_a]$, $G^{-1}{}_{\alpha\beta}[\Phi, \Psi'_a]$, $L_a^\alpha[\Phi, \Psi'_a]$, and $\mathfrak{F}'_\alpha[\Phi, \Psi'_a]$ finite in the n-loop approximation.

Let us choose $z_{(n)}, \tilde{z}_n$ so as to make the transverse part of the photon propagator and the ghost propagator finite in the n-loop approximation. Let us further choose divergent parts of $w_{(n)}$, $y_{(n)}$, and $v_{(n)}$ so as to make $\delta\mathfrak{F}'_\alpha/\delta\Phi_i|_{\Phi=0=\Psi'_a}$, $\delta^2\mathfrak{F}'_\alpha/\delta\Phi_i\delta\Phi_j|_{\Phi=0=\Psi'_a}$, and $\delta^2\mathfrak{F}'_\alpha/\delta\Psi'_a\delta\Psi'_b|_{\Phi=0=\Psi'_a}$ finite in the n-loop approximation.

Differentiate Eq. (90b) with respect to Ψ'_b and set $\Phi_i = 0 = \Psi'_a$:

$$\partial_i^\beta \frac{\delta^2\Gamma}{\delta\Phi_i\delta\Psi'_b} + L_a^\beta \frac{\delta^2\Gamma}{\delta\Psi'_a\delta\Psi'_b} = -\frac{1}{\alpha} G^{-1}{}_{\alpha\beta} \frac{\delta\mathfrak{F}'_\alpha}{\delta\Psi'_b}\,. \tag{94}$$

In momentum space, the right-hand side and the first term on the left-hand side are proportional to p^2. Since $\delta^2\Gamma/\delta\Psi'_a\delta\Psi'_b$ does not have a zero in p^2 (for any a and b) it follows that $L_a^\beta(p^2)$ (for $a = 1, 2$) and in particular $[L_a^\beta(p^2)]_n^{\text{div}}$ are proportional to p^2. [This can be seen more easily if one performs a global U(1) transformation on ϕ_1, ϕ_2

such that only one of them has a vacuum expectation value of $(v_1^2+v_2^2)^{1/2}$.] But since $[L_a^\beta(p^2)]_n^{\rm div}$ must be a constant independent of p^2, it must be finite.

Differentiate Eq. (90b) with respect to Φ_j and set $\Phi=0=\Psi'_a$; then

$$\partial_i^\beta \frac{\delta^2\Gamma}{\delta\Phi_i\delta\Phi_j}+L_a^\beta\frac{\delta^2\Gamma}{\delta\Psi'_a\delta\Phi_j}=-\frac{1}{\alpha}\frac{\delta\mathfrak{F}'_\alpha}{\delta\Phi_j}G^{-1}{}_{\alpha\beta}. \tag{95}$$

We equate the n-loop divergence on both sides of Eq. (95) in momentum space; noting that

$$\left[\frac{\delta^2\Gamma}{\delta\Psi'_a\delta\Phi_j}\right]_0=0\,,$$

$[G^{-1}{}_{\alpha\beta}]_n$ is finite ,

$\left[\frac{\delta\mathfrak{F}'_\alpha}{\delta\Phi_j}\right]_n$ is finite ,

we obtain that

$p^\mu[\Gamma_{\mu\nu}(p)]_n^{\rm div}$ is finite .

Hence the longitudinal part of the photon propagator is also finite. Further, since $L_a^\beta(p^2)$ and $G^{-1}(p^2)$ are proportional to p^2, we find, from Eq. (95),

$$p^\mu\Gamma_{\mu\nu}\propto p^2p_\nu\,.$$

Hence, the photon mass is zero.

Henceforth, let us use a compact notation for derivatives of Γ, G, L, and $\mathfrak{F}'_\alpha$, for convenience. The letters $a,b,c,\ldots$ will be used for the scalar fields; the letters $i,j,\ldots$ will be used for the photon field. Thus

$$\Gamma_{aj}\equiv\frac{\delta^2\Gamma}{\delta\Phi_j\,\delta\Psi'_a}\bigg|_{\Phi=0=\Psi'_a}\,,\quad L^\alpha_{a,j}\equiv\frac{\delta L_a^\alpha}{\delta\Phi_j}\bigg|_{\Phi=0=\Psi'_a}\,,\ \text{etc.}$$

Differentiate the WT identity of Eq. (90b) with respect to Ψ'_b and Φ_j and set $\Psi'=0=\Phi$:

$$\partial_i^\beta\Gamma_{bij}+L_a^\beta\Gamma_{abj}+L^\beta_{a,j}\Gamma_{ab}+L^\beta_{a,b}\Gamma_{aj}$$
$$=-G^{-1}{}_{\alpha\beta}\frac{1}{\alpha}\mathfrak{F}'_{\alpha,bj}-\frac{1}{\alpha}\mathfrak{F}'_{\alpha,b}G^{-1}{}_{\alpha\beta,j}-\frac{1}{\alpha}\mathfrak{F}'_{\alpha,j}G^{-1}{}_{\alpha\beta,b}\,. \tag{96}$$

Now, $L^\beta_{a,j}$, $L^\beta_{a,b}$, $\mathfrak{F}'_{\alpha,bj}$, and $G^{-1}{}_{\alpha\beta,b}$ are not renormalization parts and $[L_a^\beta]_0=[L^\beta_{a,j}]_0=[\mathfrak{F}'_{\alpha,b}]_0=[\Gamma_{aj}]_0=0$.

Hence, equating the n-loop divergence on both sides of Eq. (96), we get

$$\partial_i^\beta[\Gamma_{bij}]_n^{\rm div}+[L^\beta_{a,b}]_0[\Gamma_{aj}]_n^{\rm div}=-\frac{1}{\alpha}[\mathfrak{F}'_{\alpha,b}]_n^{\rm div}[G^{-1}{}_{\alpha\beta,j}]_0\,. \tag{97}$$

We define

$$\text{F.T.}\{\Gamma_{bij}\}=-i\Gamma^b_{\mu\nu}(p,q,r)\,,$$
$$\text{F.T.}\{\Gamma_{aj}\}=r_\nu\Gamma^{(a)}(r)\,.$$

Then, using $[L^\beta_{ab}]_0=e\zeta_{ab}$, Eq. (97) yields

$$p^\mu[\Gamma^b_{\mu\nu}(p,q,r)]_n^{\rm div}+e\zeta_{ab}r_\nu[\Gamma^a(r)]_n^{\rm div}=-\frac{\xi}{\alpha}r'_\nu[\mathfrak{F}'_{\alpha b}]_n^{\rm div}\,. \tag{98}$$

Since

$$[\Gamma^b_{\mu\nu}(p,q,r)]_n^{\rm div}=g_{\mu\nu}\times(\text{divergent constant})\,,$$

it follows from Eq. (98) that

$[\Gamma^b_{\mu\nu}(p,q,r)]_n^{\rm div}$ is finite

and

$$e\zeta_{ab}[\Gamma^a(r)]_n^{\rm div}=-\frac{1}{\alpha}\xi[\mathfrak{F}'_{\alpha,b}]_n^{\rm div}+\text{finite terms}\,, \tag{99}$$

while Eq. (94) gives

$$r^2[\Gamma^b(r)]_n^{\rm div}=-\frac{1}{\alpha}\xi[\mathfrak{F}'_{\alpha,b}]_n^{\rm div}r^2+\text{finite terms}\,. \tag{100}$$

Equations (99) and (100) imply that

$$[\Gamma^b(r)]_n^{\rm div}\ \text{is finite}\,,\qquad [\mathfrak{F}'_{\alpha,b}]_n^{\rm div}\ \text{is finite}\,. \tag{101}$$

Differentiate Eq. (90b) with respect to Ψ'_b, Ψ'_c and set $\Phi=0=\Psi'$:

$$\partial_i^\beta\Gamma_{bci}+L_a^\beta\Gamma_{abc}+L^\beta_{a,b}\Gamma_{ac}+L^\beta_{a,c}\Gamma_{ab}$$
$$=-\frac{1}{\alpha}G^{-1}{}_{\alpha\beta}\mathfrak{F}'_{\alpha,bc}-\frac{1}{\alpha}G^{-1}{}_{\alpha\beta,b}\mathfrak{F}'_{\alpha,c}-\frac{1}{\alpha}\mathfrak{F}'_{\alpha,b}G^{-1}{}_{\alpha\beta,c}\,. \tag{102}$$

Equating the n-loop divergence on both sides of Eq. (102), we find that

$$\partial_i^\beta[\Gamma_{bci}]_n^{\rm div}+e\zeta_{ab}[\Gamma_{ac}]_n^{\rm div}+e\zeta_{ac}[\Gamma_{ab}]_n^{\rm div}+e\left[\frac{X}{Z_\psi}\right]_n\{\zeta_{ab}[\Gamma_{ac}]_0+\zeta_{ac}[\Gamma_{ab}]_0\}\ \text{is finite}\,. \tag{103}$$

Choose $b=1$, $c=2$ $[\zeta_{12}=-\zeta_{21}=1]$. In this case, we express Eq. (103) in momentum space using

$$i\Gamma^{bc}_\mu(p;q,r)\equiv\text{F.T.}\{\Gamma_{bci}\}=+i[A(q+r)_\mu+B(q-r)_\mu]\,,$$
$$\text{F.T.}\{\Gamma_{ac}\}_n^{\rm div}=r^2(c_2+z_{\psi(n)})+(D_2-\delta\mu^2{}_{(n)})\,,$$
$$\text{F.T.}\{\Gamma_{ab}\}_n^{\rm div}=q^2(c_1+z_{\psi(n)})+(D_1-\delta\mu^2{}_{(n)})\,,$$
$$\text{F.T.}\{\Gamma_{ab}\}_0=(q^2-\mu^2)\,,$$
$$\text{F.T.}\{\Gamma_{ac}\}_0=r^2-\mu^2\,.$$

Then we obtain that

$$A(q+r)^2+B(q^2-r^2)+e[q^2(c_1+z_{\psi(n)})-r^2(c_2+z_{\psi(n)})]+e\left[\frac{X}{Z_\psi}\right]_n(q^2-r^2)+(D_2-D_1) \text{ is finite}. \quad (104)$$

Hence, it follows that

A is finite ,

$$D_1=D_2+\text{finite terms}, \quad (105)$$

$$C_1=C_2+\text{finite terms}. \quad (106)$$

From Eqs. (105) and (106), it follows that a mass renormalization term and a wave-function renormalization term of the forms $-\delta\mu^2{}_n(\phi_1{}^2+\phi_2{}^2)$ and $+z_{\psi(n)}(\partial_\mu\phi_1\partial^\mu\phi_1+\partial_\mu\phi_2\partial^\mu\phi_2)$, respectively, will remove divergences in the propagators for ϕ_1' and ϕ_2' fields with the choices

$$\delta\mu^2{}_{(n)}=D_1+\text{finite terms},$$

$$c_1=-z_n+\text{finite terms}.$$

Once this is done, Eq. (104) yields

$$B=e[X/Z_\psi]_n+\text{finite terms}.$$

Hence the choice of x_n such that $[X/Z_\psi]_n$ is finite will make $[\Gamma_\mu^{cb}(p,q,r)]_n$ finite. In particular, if we have chosen $x_r=z_{\psi(r)}$ $(0\le r\le n-1)$, then we may choose $x_n=z_{\psi(n)}$.

Now, choose $b=c=1$ in Eq. (103). We write

$$[\Gamma_\mu^{bc}(p,q,r)]_n^{\text{div}}=E(q+r)_\mu+F(q-r)_\mu,$$

$$\text{F.T.}[\Gamma_{ac}]_n^{\text{div}}=r^2G+H[a=2,\ c=1], \text{ etc.}$$

Then, we obtain

$$E(q+r)^2+F(q^2-r^2)=eG(r^2+q^2)+2He + \text{finite terms}.$$

Therefore, it is clear that E, F, G, and H are finite.

Finally, we differentiate Eq. (90b) with respect to Ψ_b', Ψ_c', and Ψ_d' and set $\Phi=0=\Psi'$. We equate the n-loop divergence on both sides. We obtain that

$$\zeta_{ab}[\Gamma_{acd}]_n^{\text{div}}+\text{(permutations) is finite}. \quad (107)$$

Choosing $b=c=d=1$ in Eq. (107), we obtain that

$$[\Gamma_{211}]_n^{\text{div}} \text{ is finite}. \quad (108)$$

Choosing $b=c=d=2$ in Eq. (107), we obtain that

$$[\Gamma_{122}]_n^{\text{div}} \text{ is finite}. \quad (109)$$

Choosing $b=2$, $c=d=1$ in Eq. (107) and using Eqs. (108) and (109), we find that

$$[\Gamma_{111}]_n^{\text{div}} \text{ is finite}.$$

Choosing $b=1$, $c=d=2$, we get that

$$[\Gamma_{222}]_n^{\text{div}} \text{ is finite}.$$

Thus we have shown that symmetric mass and wave-function renormalization counterterms remove divergences in the propagators of ϕ_1' and ϕ_2'. We have also shown that all the newly introduced renormalization parts become finite in the n-loop approximation. The rest of the proof (4-point functions, etc.) is trivial and hence will not be given here.

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