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Are two-dimensional Yang-Mills gauge theories "color" dependent?*

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(Received 18 July 1974)

If the symmetry of the theory under global transformations generated by the charges is normal, the physical states of the system must be "color" singlets. (This is analogous to the physical states of two-dimensional quantum electrodynamics being neutral.) Consequently, the local color currents vanish in physical states. The (two-dimensional) inhomogeneous Lorentz invariance of the theory is also discussed.

I. INTRODUCTION

In the past year, the discovery of asymptotic freedom in non-Abelian gauge theories¹ has been accompanied by enormous enthusiasm over the tantalizing possibility that this class of theory might also provide a mechanism for confining quarks. The hopes that exist in this direction arise from the observation that such theories are very infrared singular.² Calculations employing renormalization-group techniques indicate the effective coupling constant grows at large distances, which suggests it may be energetically favorable

for the quanta of the theory to condense locally in regions of space. We have here a sort of Orwellian liberty, where one is free only as long as one does not wander off too far.

So far there are no firm calculations that actually support these hopes, or more ambitious speculations based upon them, in four-dimensional space-time. Of course entrapment might also occur in theories which are not of non-Abelian-gauge type, as indicated by several recent investigations.³ Nevertheless, the basic aesthetic reasoning underlying Yang-Mills theories⁴ is so appealing that it is urgent to explore further whether the behavior

suggested by the renormalization group is in fact realized.

In this paper we examine Yang-Mills theories based on arbitrary simple, connected, compact Lie groups in one time and one space dimension (TDYM theories). For simplicity, we shall call the physical meaning of the group "color," so that, for example, states that transform as singlets are color-free, etc. The basic result of our investigation is that

$$\langle \psi'_{\text{phys}} | J_{\mu}^a(x, t) | \psi_{\text{phys}} \rangle = 0, \quad (1.1)$$

where $J_{\mu}^a(x, t)$ is any component of the local conserved color current, and where $|\psi_{\text{phys}}\rangle$ are necessarily color-free states. This means that no physical state of the system can contain isolated observable color-dependent components. For this statement to make any sense, we require that the charges be "normal," i.e., that the symmetry be not realized in a Nambu-Goldstone manner.

Our result generalizes a recent discovery by 't Hooft that quarks are absent from the asymptotic states in $U(N)$ TDYM theories, when N tends to infinity, with (gN) fixed.⁵ As in 't Hooft's calculation, we observe the only way the growth of the Coulomb energy with spatial separation can be kept from diverging is for screening to develop, so the net sources of the "electric" fields add up to zero. This in turn obliterates the single-quark propagation. The non-Abelian character of the group leads to Eq. (1.1).

It is because the essential features of TDYM theories arise from the infrared behavior of the theory that we consider our investigation relevant to what can occur in four dimensions. At the very least, one might have considered the trapping arguments in four dimensions to be far shakier if a general result could not be found in two dimensions. Much more speculatively, if (in the broader sense of the renormalization-group approach⁶) large-momentum field components are integrated out⁷ of a four-dimensional gauge field theory, it may be that a two-dimensional structure remains to control the infrared behavior of the theory.⁸

The paper is organized as follows. In Sec. II we examine TDYM theories in the axial gauge. Elimination of the timelike gauge fields is carried through, so the theory involves only fermion degrees of freedom. Commutators of the components of the stress-energy tensor are evaluated to examine questions of inhomogeneous Lorentz invariance. It is discovered that the algebra of densities contains anomalous pieces, but these do not affect Lorentz invariance. In Sec. II we derive Eq. (1.1). It is noteworthy that detailed dynamical calculations are not necessary to establish this result. In

Sec. IV we discuss our result, with emphasis on the implicit assumptions that go into the derivation.

II. THE YANG-MILLS THEORY IN TWO DIMENSIONS

In the first part of this section, we review the canonical TDYM theory, and eliminate the gauge-field degrees of freedom by working in the gauge $A_1^a(x) = 0$. In the second part, we examine the general form of the one-dimensional Green's function, and discuss a prescription for dealing with integrations by parts and surface terms. Finally, we examine the inhomogeneous Lorentz-covariance properties of the theory, utilizing the approach of Schwinger, and resolve an apparent difficulty that arises.

A. TDYM theory in axial gauge

The TDYM theory is based on the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + i\bar{\psi} \gamma^{\mu} (\partial_{\mu} + ig\frac{1}{2}t^a A_{\mu}^a) \psi, \quad (2.1)$$

where the gauge-covariant tensor

$$F_{\mu\nu}^a = \partial_{\nu} A_{\mu}^a - \partial_{\mu} A_{\nu}^a - gC^{abc} A_{\mu}^b A_{\nu}^c. \quad (2.2)$$

The Greek indices take values $(0, 1)$, with Minkowski metric $g^{00} = 1$, $g^{11} = -1$. Latin indices are the group indices, and C^{abc} are the real, totally antisymmetric structure constants of an arbitrary simple, connected, compact, Lie group.

We may choose a representation for the Dirac matrices γ^{μ} where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_3.$$

(Note: γ_1 is not Hermitian.) However, for our purposes this is totally irrelevant.

The Euler-Lagrange equations of the theory are

$$(\not{\partial} + ig\frac{1}{2}t^a A^a) \psi \equiv \gamma^{\mu} D_{\mu} \psi = 0, \quad (2.3)$$

$$\partial^{\nu} F_{\mu\nu}^a = g(j_{\mu}^a + C^{abc} F_{\nu\mu}^b A^{\nu c}), \quad (2.4)$$

where

$$j_{\mu}^a(x, t) = \bar{\psi}(x, t) \gamma_{\mu} \frac{1}{2} t^a \psi(x, t) \quad (2.5)$$

is the gauge-invariant fermionic contribution to the current.

We now exercise the freedom of gauge choice to set $A_1^a(x, t) = 0$. The essential simplification of the theory in this gauge manifests itself in the form taken by Eq. (2.4), zeroth component,

$$\partial_x^2 A_0^a(x, t) = -g j_0^a(x, t), \quad (2.6)$$

since $F_{01}^a(x, t) = \partial_x A_0^a(x, t)$. Equation (2.6) is non-dynamical (no time derivatives are involved), so A_0^a can be solved for

$$A_0^a(x, t) = -g \int dx' V(x, x') j_0^a(x', t), \tag{2.7}$$

where

$$\partial_x^2 V(x, x') = \delta(x - x'). \tag{2.8}$$

Since $(\partial_0 A_0^a)$ is absent from \mathcal{L} , A_0^a has no canonically conjugate momentum, and (2.7) may be used to eliminate A_0^a from the theory completely. Of course, it will be convenient to carry it along as a concise expression for the right-hand side of (2.7).

Quantization of the system is carried out by imposing the canonical anticommutation relations

$$\{\psi_\alpha^{\dagger a}(x, t), \psi_\beta^b(y, t)\} = \delta^{ab} \delta^{\alpha\beta} \delta(x - y), \tag{2.9}$$

where α and β label the Dirac components of ψ . The fermionic currents j_μ^a satisfy the current algebra

$$[j_\mu^a(x, t), j_\mu^b(y, t)] = iC^{abc} j_0^c(x, t) \delta(x - y) \quad (\text{no sum on } \mu), \tag{2.10a}$$

$$[j_0^a(x, t), j_1^b(y, t)] = iC^{abc} j_1^c(x, t) \delta(x - y) + i\sigma \delta^{ab} \partial_x \delta(x - y). \tag{2.10b}$$

This form of the current algebra is true under the assumption it is determined by the short-distance behavior of the free theory. Equivalently, it is determined by postulating a free-field Fourier decomposition for $\psi(x, t)$ at fixed time. The time evolution of the system is then determined by the Heisenberg equation of motion. The coefficient of the Schwinger term in (2.10b) may be calculated explicitly under these assumptions, and is a c number, $\sigma = (2\pi)^{-1}$. It is finite because we are working in two dimensions.⁹

Since $A_0^a(x, t)$ is not an independent dynamical variable, its commutation relations are entirely determined by (2.10). In particular,

$$[A_0^a(x, t), A_0^b(y, t)] = ig^2 C^{abc} \int du V(x, u) V(y, u) j_0^c(u). \tag{2.11}$$

The nontrivial structure of this commutator is, of course, a consequence of the nonlocality of the definition of A_0^a . The commutator of field strengths F_{01}^a will be seen to have a more transparent form once we specify $V(x, x')$ with greater precision.

For the moment, it suffices to notice that in spite of (2.11), the vector current with components

$$J_0^a(x, t) = j_0^a(x, t), \tag{2.12a}$$

$$J_1^a(x, t) = j_1^a(x, t) + \frac{1}{2} C^{abc} \{F_{01}^b(x, t), A_0^c(x, t)\} \tag{2.12b}$$

is conserved. This is expected from the equation of motion (2.4). However, the symmetrization noted in J_1^a is required for actual conservation because of (2.11). It is also required, in any case, for Hermiticity.

B. The potential $V(x, x')$ and surface terms

So far, no form has been specified for the "potential" $V(x, x')$, Eq. (2.8). A general solution to that equation symmetric in x and x' is

$$V(x, x') = \frac{|x - x'|}{2} + Ax x' + B(x + x') + C, \tag{2.13}$$

where A (C) may be given dimensions of L^{-1} (L). We will refer to the novel terms in V as the A , B , and C terms for short. One expects the A and B terms to break translational invariance, and indeed the A term does so in a violent fashion. The reader may verify this for him (her) self in the expressions appearing in Sec. IIC. Carrying the A terms is cumbersome and in the end uninstruc-tive, so we shall set $A = 0$ for our presentation. The B term turns out to be more subtle, however, and will be retained for the present.

Physically, the A , B , and C terms affect the boundary values of V . Thus, we must preface further discussion by commenting on integrations by parts. Whenever integrations by parts are performed on operators involving the fields $\psi(x, t)$, we make the conventional prescription that wave-packet rather than plane-wave expansions be used, such that surface terms damp to zero.¹⁰ Not all surface terms will involve the field points at the spatial boundary explicitly, however, and in these cases special care must be taken. Typically these dangerous surface integrals involve $V(x, x')$, with one of its arguments at the boundary. These quantities can diverge linearly.

An illustration (mild because $\partial_x V$ enters) of the need for a prescription is provided by integrating the time component of the Maxwell equation over all space:

$$F_{01}^a(x, t) |_{\Sigma_x} = -gQ^a, \tag{2.14}$$

where the charge $Q^a = \int dx j_0^a$, and where Σ_x denotes the "surface" in the single spatial variable x . The validity of this identity is, of course, guaranteed by

$$\partial_x V(x, x') |_{\Sigma_x} = 1,$$

which follows directly from (2.8).

Explicitly, though, one is assigning a value to $\epsilon(z)$ for $z = \pm\infty$. A consistent way to do this is to work in the spatial interval $[L, -L]$, then take the $\lim L \rightarrow \infty$ at the end of the calculation. That is,

$$\lim_{L \rightarrow \infty} \int_{-L}^L dw \epsilon(w-z)f(w) \Big|_{z=\pm L} = \mp \int_{-\infty}^{\infty} dw f(w);$$

thus

$$\lim_{L \rightarrow \infty} F_{01}^a(\pm L) = \mp g Q^a / 2.$$

Note that (2.14) places no constraint on B or C .

A second example is provided by the requirement that charge be conserved. This holds if $J_1^a(x, t)$ vanish on the surface of x . Referring to (2.13b), the condition is that

$$C^{abc} \int \int du dv \{j_0^b(u), j_0^c(v)\} [V(x, u) \partial_x V(x, v)] \Big|_{\Sigma_x} = 0.$$

The surface term is evaluated for finite L , and

$$[F_{01}^a(x, t), F_{01}^b(x', t)] = -igC^{abc} [\partial_x V(x, x') F_{01}^c(x') + \partial_{x'} V(x, x') F_{01}^c(x) + gQ^c (B^2 - \frac{1}{4})]. \tag{2.16}$$

This commutator will be important for our consideration of inhomogeneous Lorentz invariance, to which we now turn.

C. The Schwinger algebra

To study questions of Lorentz invariance of TDYM theories, we will calculate the Schwinger algebra of the components of the stress-energy tensor.¹¹ A superficially symmetric, conserved, gauge-invariant candidate for this tensor is

$$\theta^{\mu\nu} = -g^{\mu\nu} \mathcal{L} - F_a^{\mu\omega} F_{\omega\nu}^a + \frac{1}{4} i [\bar{\psi} \gamma^\mu \vec{D}^\nu \psi + (\mu \leftrightarrow \nu)]. \tag{2.17}$$

$$H = \int dx \theta^{00}(x, t) = -i \int dx \psi^\dagger \gamma^5 \partial_x \psi - \frac{1}{2} g^2 \int \int dx dx' V(x, x') j_0^a(x, t) j_0^a(x', t) + \frac{1}{2} g^2 Q^2 [C + 2L(B^2 + \frac{1}{4})]. \tag{2.19}$$

Here Q^2 is a convenient abbreviation for $\sum_a Q^a Q^a$. The term written proportional to L in H is an infinite operator quantity. However, it can be removed by adding an extra piece to the energy density,⁹

$$\theta^{00} \rightarrow \theta^{00} + \beta Q^2, \tag{2.20a}$$

where

$$\beta = -\frac{1}{2} g^2 (B^2 + \frac{1}{4}). \tag{2.20b}$$

Notice that even if $B=0$, $\beta \neq 0$ for nonvanishing coupling constant. It should also be noted that the C term in $V(x, x')$ exactly cancels the term $(\frac{1}{2} g^2 Q^2 C)$ exhibited in (2.19).

The spatial-displacement operator

antisymmetry of C^{abc} is used to show that the terms involved are identically zero for any value of L . Again, no constraint is required for B or C .

Once momentum conservation is established, it is possible to discuss surface terms in another fashion, by taking matrix elements between states of definite momentum:

$$\theta(x) \Big|_{\Sigma_x} \rightarrow \int dx \partial_x e^{i(k-k')x} \langle k | \theta(0) | k' \rangle, \tag{2.15}$$

where $\theta(x)$ is an arbitrary displacement-invariant operator. The value of the operator at spatial infinity does not appear explicitly. An example of this method will be given in Sec. III.

Finally, making use of the $[L, -L]$ prescription, we can now express, from (2.11),

In particular, in the gauge $A_1^a=0$, we have

$$\theta^{00} = \frac{1}{2} F_{01}^a F_{01}^a - \frac{1}{2} i \psi^\dagger \gamma^5 \vec{\partial}_x \psi, \tag{2.18a}$$

$$\theta^{11} = -\frac{1}{2} F_{01}^a F_{01}^a - \frac{1}{2} i \psi^\dagger \gamma^5 \vec{\partial}_x \psi, \tag{2.18b}$$

$$\theta^{01} = \frac{1}{2} i \psi^\dagger \vec{\partial}_x \psi. \tag{2.18c}$$

These expressions for the components of the stress-energy tensor are in agreement with the results of Schwinger, when his formulas are specialized to the two-dimensional Minkowski space. Making use of (2.18), we can systematically work out the desired algebraic relations:

1. The Hamiltonian is

$$P = \int dx \theta^{01} \tag{2.21}$$

then turns out to be time-dependent,

$$\partial_0 P = i[H, P] = -\frac{1}{2} B g^2 Q^2. \tag{2.22}$$

This is, of course, the explicit violation of displacement invariance that one naively expects from the B term in V . To conserve momentum, we need $B=0$. However, the next calculation will show that we have not simply been creating a straw man by keeping $B \neq 0$.

2. Using the commutator (2.16) and the modified energy density given by (2.18) and (2.20), we find

$$[\theta^{00}(x, t), \theta^{00}(x', t)] = -i\partial_x \delta(x - x')[\theta^{01}(x, t) + \theta^{01}(x', t)] + ig^2(B^2 - \frac{1}{4})C^{abc}Q^c \{F_a^{01}(x, t), F_b^{01}(x', t)\}. \quad (2.23)$$

We have smeared on test functions to obtain the canonical terms of the algebra, and more singular c -number contributions have been neglected in this formula.

The extra noncanonical piece that is exhibited here can be traced to the $(B^2 - \frac{1}{4})C^{abc}Q^c$ term in the $[F^a, F^b]$ commutator, Eq. (2.16). This term survives in these calculations because it is not legitimate to disregard surface contributions of this type, as we have already explained.

Schwinger observed a similar phenomenon in the full four-dimensional Yang-Mills theory years ago.¹¹ He found that the commutator of spacelike-separated energy densities involved anomalous terms, which could only be removed by nontrivially modifying the definition of the energy density. However, the modifications needed to salvage the energy-density algebra required the existence of genuine transverse dynamical gauge-field degrees of freedom.¹² These are totally absent from the two-dimensional theory, so Schwinger's procedure cannot be carried through in this case.

We seem to be in the position, therefore, that either momentum conservation is violated (if $B^2 = \frac{1}{4}$), or the energy-density algebra fails (if $B=0$).

On the other hand, the Schwinger commutator conditions are sufficient, but not necessary, conditions for relativistic invariance if interactions with gravitation are not taken into account.¹³ It is reasonable to inquire whether the TDYM theory cares about the anomalous terms, since its inhomogeneous Lorentz group is much smaller than the ten-parameter Poincaré group of four-dimensional space-time.

3. To investigate the possibility that has just been raised, let us now set $B=0$ to maintain displacement invariance, and observe the following properties of

$$F_{01}^a(x, t) = -\frac{1}{2}g \int dy \epsilon(x-y) j_0^a(y, t). \quad (2.24)$$

First, with the $[L, -L]$ prescription, we have

$$\int dz \epsilon(z) = 0,$$

so that

$$\int dx F_{01}^a(x, t) = 0. \quad (2.25)$$

Second, one easily finds

$$\int dx x F_{01}^a(x, t) = -\frac{1}{2}g L^2 Q^a. \quad (2.26)$$

Now, condition (2.25) assures us that the extra piece in (2.23) does not contribute to $\partial_0 \theta^{00} = i[H, \theta^{00}]$. Thus, the energy-momentum density is indeed conserved, i.e., $\partial_\mu \theta^{0\mu} = 0$.

Next, to ensure that $\theta^{\mu\nu}$ transforms properly as a tensor, we must satisfy the commutation relation¹¹

$$i[\theta^{00}(x, t), K] = (x^0 \partial_x - x \partial^0) \theta^{00}(x) + 2\theta^{01}(x), \quad (2.27)$$

where the "boost" operator

$$K = x^0 P - \int dx x \theta^{00}(x, x^0). \quad (2.28)$$

However, Eq. (2.27) is violated due to (2.23), by a term proportional to

$$L^2 C^{abc} Q^c \{Q^a, F_{01}^b(x)\},$$

where (2.26) has been used in obtaining this result. Fortunately, one can use the antisymmetry of C^{abc} to prove this term vanishes identically.

But now we are finished, because there are no purely spatial rotations in the two-dimensional Minkowski space. Thus the inhomogeneous Lorentz algebra is perfectly acceptable even if $B=0$. (Actually, the full $[\theta^{00}(x), \theta^{01}(x')]$ equal-time algebra has not been displayed, but the reader can easily check that it holds, provided $\theta^{00} - \theta^{00} + \beta Q^2$ is accompanied by $\theta^{11} - \theta^{11} - \beta Q^2$.)

To summarize, we have seen that the TDYM theory is Lorentz-invariant in the gauge $A_1^a=0$, with

$$V(x, x') = \frac{|x-x'|}{2} + C, \quad (2.29)$$

giving

$$H = -i \int dx \psi^\dagger \gamma^5 \partial_x \psi - \frac{1}{4}g^2 \int \int dx dx' |x-x'| j_0^a(x) j_0^a(x'). \quad (2.30)$$

This Hamiltonian agrees with the canonical Hamiltonian (also modified to eliminate LQ^2 terms)

$$H = \int dx [-\mathcal{L} + \psi^\dagger \partial_0 \psi + \beta Q^2]$$

only if $C=0$. We shall set $C=0$ from now on.

Thus, we have eliminated the possible ambiguity in V by checking displacement invariance, showing the anomaly in (2.23) is irrelevant for Lorentz invariance, and requiring the Hamiltonian to be the canonical expression. As noted earlier, we could have kept the A term in V as well, but this leads to catastrophic terms in the Schwinger algebra.

III. ABSENCE OF COLOR-DEPENDENT STATES IN TWO-DIMENSIONAL YANG-MILLS THEORIES

Having established the formal properties of TDYM theories in the preceding section, we now proceed to prove the absence of nonsinglet states in the theory. The derivation exactly parallels the Brown-Zumino¹⁴ argument in two-dimensional QED (TDQED), but because of the non-Abelian character of the group it will be possible to prove an even stronger statement than that the charge vanishes. First, however, let us demonstrate

that this much is true.

The outline of the argument is as follows. The equation of current conservation, $\partial_\mu J_d^\mu = 0$, is supplemented by an equation of the form

$$i[H, J_1^a] - \partial_1 J_0^a = A^a. \quad (3.1)$$

The Lorentz pseudoscalar operator A^a is the "axial-vector current anomaly" if we identify $\epsilon^{\mu\nu} J_\nu^a$ as an axial-vector current in the theory. The precise form of this operator may be ascertained by performing the commutation of H , Eq. (2.30), with J_1^a , Eq. (2.12b). It is

$$A^a(x, t) = g\sigma F_{01}^a(x, t) - \frac{1}{2}gC^{abc}\{A_0^b(x, t), j_1^c(x, t)\} + \frac{1}{4}gC^{abc} \int du [\epsilon(x-u)\{A_0^b(x, t), \partial_u J_1^c(u, t)\} - |x-u|\{F_{01}^b(x, t), \partial_u J_1^c(u, t)\}]. \quad (3.2)$$

From current conservation and (3.1), it follows that

$$\square J_0^a(x, t) = \partial_x A^a(x, t). \quad (3.3)$$

Consequently, a condition of consistency on the theory may be deduced,¹⁵

$$\int dx \partial_x A^a(x, t) = 0. \quad (3.4)$$

In detail, this condition requires

$$g^2\sigma Q^a + \frac{1}{2}gC^{abc}\{A_0^b, j_1^c\}|_{\Sigma_x} + \frac{1}{8}gC^{abc} \int \int du dv \{j_0^b(v), \partial_u J_1^c(u)\} [\epsilon(x-u)|x-v| - \epsilon(x-v)|x-u|]|_{\Sigma_x} = 0. \quad (3.5)$$

Working consistently with our prescription for handling the surface terms, we have

$$\{A_0^b(x, t), j_1^c(x, t)\}|_{\Sigma_x} = 0, \quad (3.6a)$$

$$\lim_{L \rightarrow \infty} [\epsilon(x-u)|x-v| - \epsilon(x-v)|x-u|]|_{x=-L}^{x=L} = 0. \quad (3.6b)$$

This shows that all the charges $Q^a = 0$ for $g \neq 0$.

Since we have seen momentum is conserved in the theory, Eq. (3.6a) can be discussed in a different fashion that never involves evaluating $j_1^a(x, t)$ on the boundary, by using Eq. (2.15). The expression $\int dx \partial_x (A_0^b j_1^c)$ may be sandwiched between states of definite momentum to obtain $(k - k')\delta(k - k')\langle k | A_0^b(0) j_1^c(0) | k' \rangle$. This is zero provided the matrix element is well behaved. However, a potential difficulty arises because the Fourier transform of $A_0^b(0)$ behaves like $\int dq \tilde{j}_0^b(q) q^{-2}$. The problem can be resolved by taking greater care in observing the ($i\epsilon$) prescriptions inherent in the definition of this Fourier transform, for q in the neighborhood of zero. When this is done, Eq. (3.6a) is verified to be true.

The charges Q^a cannot be identically zero as operators, however, without radically altering the

nature of the whole theory.¹⁴ Thus the conditions $Q^a = 0$ must be imposed as conditions on states. There is only one set of states on which all Q^a can vanish simultaneously, and these are the singlet states of the group. Thus all physical states in the theory, that is, those states for which the theory is self-consistent, must be color-free:

$$Q^a | \psi_{\text{phys}} \rangle = 0. \quad (3.7)$$

These conditions clearly rule out the possibility of observing a single color-dependent quark as a physical state. The Wigner-Eckart theorem assures us an even stronger assertion is true, however, namely that the color density itself vanishes locally between physical states. That is, from the group transformation property of the density, we have

$$[Q^a, j_0^b(x, t)] = iC^{abc} j_0^c(x, t); \quad (3.8)$$

therefore,

$$\langle \psi'_{\text{phys}} | j_0^a(x, t) | \psi_{\text{phys}} \rangle = 0. \quad (3.9)$$

In addition, making use of the Jacobi identity valid for Lie groups of the type we are considering,

$$C^{bcd}C^{ade} + C^{cad}C^{bde} + C^{abd}C^{cde} = 0, \quad (3.10)$$

we find that

$$[Q^a, J_1^b(x, t)] = iC^{abc}J_1^c(x, t). \quad (3.11)$$

Thus, Eq. (3.9) generalizes to

$$\langle \psi'_{\text{phys}} | J_\mu^a(x, t) | \psi_{\text{phys}} \rangle = 0, \quad (3.12)$$

as required for Lorentz covariance. (Actually, of course, a gauge change is required when a Lorentz transformation is made, to restore the axial gauge condition in the new frame of reference.)

The vanishing of the local color density in physical states means we cannot construct a physical singlet state consisting of quarks that are spatially well separated. Equation (3.12) then says this is true in any frame of reference. Whether this is a satisfactory general definition of "containment" we leave to the discretion of the reader.

Returning now to the anomaly in the Schwinger commutator, Eq. (2.22), we see that between physical states, the anomalous term is absent. Nevertheless, algebraic relations must be worked out *before* imposing the $Q^a = 0$ conditions. It is satisfying that the Lorentz algebra, and conservation of the stress-energy tensor, can be demonstrated as operator identities before taking matrix elements. In this vein, note that the algebra of color densities, (2.10a), is not a $0 = 0$ relation between physical states, since the left-hand side can receive nontrivial contributions from nonsinglet intermediate states. We cannot rule out the presence of such unphysical color-dependent states in completeness sums, but the operators in the theory that link physical to unphysical states are not members of a complete set of simultaneously commuting observables.

IV. DISCUSSION

In this section we will summarize the results of this paper, comment on the assumptions that have gone into obtaining those results, and attempt to shed some light onto the physical mechanisms at work.

The TDYM theory has been examined in axial gauge in order that the dependent longitudinal gauge field can be eliminated from the outset, leaving only genuine dynamical degrees of freedom to work with. An *a priori* ambiguity in the specification of the one-dimensional Green's function has been maintained, and the inhomogeneous Lorentz group studied using Schwinger's methods in order to remove the ambiguity. It was found that an anomalous nonlocal term in the commutator of spacelike separated energy densities persists, but in fact does not affect the Lorentz algebra, which

involves integrated quantities.

Next, an internal consistency condition of the theory was exploited to demonstrate that physical states are color-free. By use of the Wigner-Eckart theorem, it followed that the matrix elements of the full local color currents vanish in physical states. Consequently, isolated localized colored states are unphysical. Of course, at a given time, singlet states consisting of more than one fermion are *not* ruled out by the consistency conditions. This is true in any frame of reference. Consequently, in a Fock basis, fermionic degrees of freedom must be present in the physical sector.

To arrive at these conclusions, we have used certain assumptions that go beyond assuming that the equations of motion and canonical commutation relations are sensible as operator equations. One such extra assumption is that the Schwinger term in the equal-time commutator of time-space components of the fermionic current is a c number. As mentioned earlier, this is basically the assumption that at fixed time the fields $\psi(x, t)$ can be expanded in terms of Fourier coefficients satisfying Fermi Dirac statistics, and that the time evolution of operators is then determined by Heisenberg equations of motion.

A second assumption was that suitably smeared fields can be used, such that integrations by parts can be freely performed. This is a rather conventional assumption, and in the single crucial instance where some doubt may persist [Eq. (3.6a)], an alternate argument is available. Implicit in the alternate argument, however, is a third assumption used throughout, namely, that the charges exist as sensible operators, and that the vacuum is unique.

All of the above assumptions are present in operator solutions to TDQED as well,¹⁶ and, without further apology, we believe them to be reasonable. This does not mean, however, that alternative sets of mutually consistent assumptions may not exist. These would define a different theory.

Examining Eqs. (3.3)–(3.6), we see the mechanism at work here is the same as that in TDQED, namely that the current acquires a mass. We can rewrite (3.3) as

$$(\square + \mu^2)J_0^a = \partial_x \Phi^a,$$

where $\mu^2 = g^2 \sigma$. Unlike the case of TDQED, the color density obeys an "interacting" equation of motion rather than a free equation of motion. Equation (3.6) is the statement that the source current $\partial_x \Phi^a$ does not contribute to the total charge. The possibility that J_0^a acquires a discrete mass for any value of the coupling constant appears to be a peculiarity of two dimensions.

Indeed, the two-dimensional peculiarities of the

theory can be exhibited more clearly if the explicit dependence of the Hamiltonian on the Fermi fields is eliminated. This may be accomplished by using Sommerfield's identity¹⁷

$$\partial_x \psi = \frac{1}{2} i\pi \{ \gamma^5 j^0 + j^1, \psi \}$$

applied to each fermionic species separately. This identity is true in two dimensions, and can be demonstrated as an operator identity using fixed-time expansions for $\psi(x, t)$.¹⁸ Using this identity, we can rewrite

$$\begin{aligned} H_0 &= -i \int dx \psi^\dagger \gamma^5 \partial_x \psi \\ &= \frac{\pi}{2} \left[a \int dx [J_0^a]^2 + J_1^a{}^2 + b \int dx [j_0^a j_0^a + j_1^a j_1^a] \right], \end{aligned}$$

where J_μ is the group singlet current $\bar{\psi} \gamma_\mu \psi$, and the relative weights a and b depend upon the group under consideration. The entire Hamiltonian is now expressed in terms of current components. Together with the algebra (2.10), this specifies the quantum-dynamical problem completely.¹⁹ However, solution in terms of canonical bosons¹⁸ does not seem possible.

Finally, although the $Q^a = 0$ conditions of the theory should be treated in the sense of superselection

rules, we have not really proved that the color-free states form a complete basis for all simultaneous observables in the theory. There remains, therefore, a question of unitarity for the solutions in the physical sector. It is clear that more work on this problem is required.

Notes added in proof

(1) We have worked with the fermion mass equal to zero. The principal results of this paper continue to hold true even if the fermions have nonzero mass.

(2) It has been noted that if V , Eq. (2.13), contains a term $B'(x - x')$ instead of $B(x + x')$, then momentum is conserved; and $(B')^2 = \frac{1}{4}$ resolves the difficulty in Eq. (2.23). However, such a term leads to an explicit violation of charge conservation. This may be verified using the arguments between Eq. (2.14) and (2.15).

ACKNOWLEDGMENTS

We thank our colleagues at SLAC for stimulating discussions, especially T.-M. Yan for several exchanges of ideas. In addition, one of us (J.W.) is indebted to J. Arafune and Y. Nambu for a valuable suggestion.

*Work supported by the U. S. Atomic Energy Commission.

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