Bhabha first-order wave equations: I. C, P, and T*

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We discuss properties of Bhabha first-order wave equations for arbitrary spin, of which the Dirac and Duffin-Kemmer-Petiau (DKP) equations are special examples. The C, P, and T transformation matrices for the Dirac field are reviewed in various representations, and the C, P, and T transformation matrices for the DKP and general Bhabha cases are then derived. The Bhabha transformation matrices are polynomials of order $2\mathcal{S}$ in the algebra matrices, where $\mathcal{S}$ is the maximum spin of a particular Bhabha algebra. For the cases $\mathcal{S} = 1$ and $\frac{1}{2}$ they reduce to the DKP and Dirac transformation matrices. We also discuss C, P, and T for the Sakata-Taketani (ST) reduction of the DKP equation, and explicitly exhibit the "subsidiary component" ST Hamiltonian equation, as well as the known "particle component" ST equation. Throughout we emphasize that physical insight which can be gained from the use of the first-order Bhabha formalism, including a possible connection between meson nonconservation and CP violation.

I. INTRODUCTION

Since the early 1930's the Dirac[1] first-order wave equation

$$(\partial\cdot\gamma+m)\psi^D=0\ , \tag{1.1}$$

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}\ , \tag{1.2}$$

whose field (ψ^D) has dimensions $(\text{mass})^{3/2}$, has been accepted as the best choice for describing particles of spin $\frac{1}{2}$. [We will consistently use the metric of (1.2), take our algebra matrices to be self-adjoint, and denote $\partial\cdot\gamma\equiv\partial_\lambda\gamma_\lambda$.] At the end of the 1930's, Duffin,[2] Kemmer,[3] and Petiau[4] (DKP) derived a first-order wave equation

$$(\partial\cdot\beta+m)\psi^{\text{DKP}}=0\ , \tag{1.3}$$

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=\beta_\mu\delta_{\nu\lambda}+\beta_\lambda\delta_{\nu\mu}\ , \tag{1.4}$$

whose field (ψ^{DKP}) also has dimensions $(\text{mass})^{3/2}$, and which describes particles of both spin 0 and spin 1.

As one can see from Eqs. (1.1) and (1.3), the DKP equation is similar in structure to the Dirac equation. In fact, Bhabha[5] showed that this similarity is more than superficial when he derived a system of first-order relativistic equations for arbitrary spin which includes the Dirac and DKP equations as special cases, thus relating the two. (We note here and later that the Bhabha equations for higher spin do *not* need their subsidiary equations to be obtained by external operations, as does the Rarita-Schwinger[6] equation, for example.)

Despite the success of the Dirac equation and the clear relationship between the Dirac and DKP equations shown by Bhabha, the formulations generally accepted for describing spin-0 and spin-1 particles are the second-order Klein-Gordon[7] (KG) and Proca[8] equations, respectively, with field dimensions $(\text{mass})^1$. This is mainly due to detailed investigations[9] which appeared to show that KG and Proca give the same results as the DKP formulation, and so the use of the DKP spinors and matrices could be avoided. Partially as a result of this, some formal aspects of the DKP and Bhabha systems, such as the charge conjugation (C), parity (P), and time-reversal invariance (T) symmetries, were never thoroughly studied.

However, as has been recently pointed out,[10-12] although the KG-Proca and DKP formulations do yield identical results in the case of conserved currents as studied in Ref. 9, when there is symmetry breaking relating the fields of a meson of one mass to a meson of another mass (a concept not around when the articles in Ref. 9 were written), this no longer holds in general. Then the same dynamics used with the various formulations can yield different results, and these can then be compared with experiment.[10, 11] Intuitively one can understand this new observation by realizing that a functional (field) with dimensions $(\text{mass})^{3/2}$ when extrapolated from an initial mass to a different final mass need not in general give the same results as when done with a field with dimensions $(\text{mass})^1$. Technically the result comes about because the β matrices, when taken between initial and final spinors of particles with different mass, mix up the mass and 4-momentum quantities in the same way that, for example, the Dirac matrices in baryon semileptonic decays mix up the Σ and n quantities in the process $\Sigma^-\to ne\bar{\nu}$.

In any event, with this new observation,[10-12] it is of greater interest than before to investigate some of the more formal properties of the DKP and Bhabha equations. In particular, in this paper the C, P, and T transformations will be considered, our main object being to obtain the C, P, and T transformation matrices. It will be helpful for the reader to review standard CPT transformation properties,[13-16] especially for the Dirac case, since except for commutation relations the DKP and Bhabha formalisms are symbolically the same as the Dirac second-quantized case. For example, the DKP fields are described as

$$\psi^{\mathrm{DKP}} = \left(\frac{m}{p_0 V}\right)^{1/2} \sum_{\vec{p}} \left[a_p u(p)\, e^{ip\cdot x} + b_p v(p)\, e^{-ip\cdot x}\right] \ , \quad (1.5)$$

$$\overline{\psi}^{\mathrm{DKP}} = \left(\frac{m}{p_0 V}\right)^{1/2} \sum_{\vec{p}} \left[a_p^\dagger \overline{u}(p)\, e^{-ip\cdot x} + b_p^\dagger \overline{v}(p)\, e^{ip\cdot x}\right], \quad (1.6)$$

where u and v are the particle and antiparticle spinors, and the rest of the notation is given in Ref. 17 (which discusses second-quantized DKP and Bhabha quantum electrodynamics) and/or Sec. II A.

In Sec. II we will discuss properties of the Dirac, DKP, and Bhabha equations and algebras, such as the nature of the algebras, the adjoint operators, expectation values, and built-in subsidiary conditions, that are pertinent to understanding the physical implication of what follows. (This will include a discussion of the difference between the pseudoscalar operator and the CPT matrix operator which, by an algebraic accident, are both represented by γ_5 in the Dirac case.) In Sec. III we will first review the well-known Dirac CPT properties in all of the standard representations of the γ matrices. (This will be useful for making intuitive physical observations in comparison with our other results.) We will explicitly derive the C, P, and T operators for both the DKP case and the general Bhabha case, and then verify that for spin $\frac{1}{2}$, and 0 and 1, the Bhabha results reduce to the Dirac and DKP cases, respectively. In Sec. IV we will discuss CPT for the Sakata-Taketani (ST) system of equations[12,18] which is obtained from the DKP Hamiltonian equation by decoupling the "particle (and antiparticle) components" from the "subsidiary components," and explicitly exhibit the "subsidiary component" equation for the first time. (The "particle components" by themselves were derived in another manner from the KG formulation by Feshbach and Villars.[19]) We will conclude in Sec. V with a discussion of the physical and mathematical implications of our results. This discussion will include one observation about T (or CP) violation which is similar in origin to remarks that have been made by Primakoff and Sharp about the implications of possible lepton nonconservation for the Dirac case.

Lastly, we should mention that in future works[20,21] (II and III of this series), we will discuss the generalized Sakata-Taketani reductions and the Lie algebras of the Poincaré generators, respectively, for Bhabha first-order wave equations of arbitrary spin. It will turn out that the generalizations of the built-in consequent equations, and other properties that we will discuss in this paper, are crucial to our results in II and III.

II. PERTINENT PROPERTIES OF THE FIRST-ORDER WAVE EQUATIONS AND ALGEBRAS

A. DKP vs Dirac properties

The DKP equation is given by

$$(\partial\cdot\beta + m)\psi^{\mathrm{DKP}} = 0\ , \quad (2.1)$$

where the β's satisfy the algebra

$$\beta_\lambda\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\lambda = \beta_\lambda\delta_{\mu\nu} + \beta_\nu\delta_{\mu\lambda}\ . \quad (2.2)$$

Actually, to obtain the complete semisimple DKP ring R, the unity operator I must be added by hand to Eq. (2.2).[22] The ring R is then reducible into a 1×1 trivial representation, a 5×5 spin-0 representation, and a 10×10 spin-1 representation. The reader can consult elsewhere for the details of these representations[23] and the operators (*not* including $\beta\cdot\beta$) which project out particular representations.[24,25] We simply note that the operators of interest are combinations of I, $E = I - I^{(1)}$ and

$$\theta \equiv \beta\cdot\beta(5-\beta\cdot\beta) = \left(\tfrac{5}{2}+\tfrac{3}{2}\zeta\right)\left(\tfrac{5}{2}-\tfrac{3}{2}\zeta\right) = \begin{cases} 6\,I^{(10)} \\ 4\,I^{(5)} \\ 0\,I^{(1)} \end{cases} . \quad (2.3)$$

The DKP algebra matrices β_μ do *not* have an inverse. That is, they generate a ring but not a group. In Sec. III C we will see that this is a property of the integer-spin algebras. In contrast, the half-integer-spin algebras, such as the Dirac algebra, do have inverses.

In addition to the β's, auxiliary matrices can be defined by

$$\eta_\lambda \equiv 2\beta_\lambda{}^2 - I\ . \quad (2.4)$$

These matrices are needed to obtain the C, P, and T properties, as will be shown in the next section. Further, η_4 is necessary in defining the adjoint equation

$$\overline{\psi}^{\mathrm{DKP}}(-\beta\cdot\partial + m) = 0,\ \ \overline{\psi}^{\mathrm{DKP}} = (\psi^\dagger)^{\mathrm{DKP}}\eta_4\ . \quad (2.5)$$

Note that this is different from the Dirac case, where the adjoint operator is one of the γ matrices, i.e.,

$$\overline{\psi}^{D} = (\psi^\dagger)^D\gamma_4\ . \quad (2.6)$$

We will see below that the above properties hold because the Bhabha adjoint operator η_4 is in general a polynomial in powers of the fourth matrix of the algebra α_4 [or β_4 as in Eq. (2.4)]. It is only in the case of the Dirac algebra that the polynomial reduces to the simple result of the matrix itself, i.e., γ_4. This will also hold for the C, P, and T operators. The Dirac case is special in having its adjoint, C, P, and T matrix operators being single products of its algebra matrices.

Writing the DKP spin-0 solutions in terms of the normalizations of Eqs. (1.5) and (1.6), the spinors explicitly are

$$u(p)=(2m^2)^{-1/2}\begin{bmatrix} ip_0 \\ ip_x \\ ip_y \\ ip_z \\ -m \end{bmatrix},$$

$$\bar{u}(p)=(2m^2)^{-1/2}[-ip_0, ip_x, ip_y, ip_z, -m] , \qquad (2.7a)$$

$$v(p)=(2m^2)^{-1/2}\begin{bmatrix} -ip_0 \\ -ip_x \\ -ip_y \\ -ip_z \\ -m \end{bmatrix},$$

$$\bar{v}(p)=(2m^2)^{-1/2}[ip_0, -ip_x, -ip_y, -ip_z, -m] . \qquad (2.7b)$$

The adjoint property of Eq. (2.5) gives the reason for the unusual Hermiticity properties of DKP expectation values. Recall that in the Dirac case, the expectation value of an operator Ω is given by

$$\langle\Omega\rangle^D=\int\bar{\psi}^D(\gamma_4\Omega)\psi^D\,d\tau , \qquad (2.8)$$

where γ_4 represents the fourth component of the current

$$(j_\lambda)^D=i\bar{\psi}^D\gamma_\lambda\psi^D . \qquad (2.9)$$

But because of the adjoint properties of the Dirac equation (2.6), Eq. (2.8) reduces to

$$\langle\Omega\rangle^D=\int(\psi^\dagger)^D\Omega\psi^D\,d\tau . \qquad (2.10)$$

This is not the case with the DKP equation, since

$$(j_\lambda)^{\text{DKP}}=i(\bar{\psi})^{\text{DKP}}\beta_\lambda\psi^{\text{DKP}},\ \ \eta_4\beta_4=\beta_4\neq I . \qquad (2.11)$$

Thus, it is necessary to define the expectation value of an operator Ω by the quantity

$$\langle\Omega\rangle^{\text{DKP}}=\int\bar{\psi}^{\text{DKP}}\langle\!\langle\beta_4\Omega\rangle\!\rangle\psi^{\text{DKP}}d\tau$$
$$=\int(\psi^\dagger)^{\text{DKP}}[\eta_4\langle\!\langle\beta_4\Omega\rangle\!\rangle]\psi^{\text{DKP}}d\tau , \qquad (2.12)$$

where Kemmer[3] suggested that $\langle\!\langle\beta_4\Omega\rangle\!\rangle$ is some combination [say $(\beta_4\Omega)$ itself or the symmetric combination $(\beta_4\Omega+\Omega\beta_4)$] such that the $\langle\!\langle\beta_4\Omega\rangle\!\rangle$ is Hermitian with respect to η_4. That is, we need

$$[\eta_4\langle\!\langle\beta_4\Omega\rangle\!\rangle]^\dagger=[\eta_4\langle\!\langle\beta_4\Omega\rangle\!\rangle] . \qquad (2.13)$$

We will return to this point more precisely in III.[21]

Lastly in this subsection we recall the point, originally made by Kemmer,[3] that if one multiplies the DKP equation (2.1) by $(\partial\cdot\beta)\beta_\lambda$ and then uses both the DKP algebra equation (2.2) and the original equation (2.1) to reduce the resultant products of one and three β's to products of zero and two β's, one obtains the equation

$$\partial_\lambda\psi^{\text{DKP}}=(\partial\cdot\beta)\beta_\lambda\psi^{\text{DKP}} . \qquad (2.14a)$$

Handling the adjoint DKP equation (2.5) similarly yields

$$\bar{\psi}^{\text{DKP}}\partial_\lambda=\bar{\psi}^{\text{DKP}}\beta_\lambda(\partial\cdot\beta) . \qquad (2.14b)$$

These are called the *consequent equations*, and their origin lies in the fact that the DKP representations for spin 0 and 1 are 5- and 10-dimensional instead of 2- and 6-dimensional [$(2S+1)\times 2$ for particle-antiparticle, since from Eq. (1.5) we have a charged equation]. (In the Bhabha formalism for higher spin discussed in Sec. II C it is shown that another manifestation of having more components than necessary is multiple mass solutions.) However, contrary to other high-spin formalisms, these constraints (and multiple solutions) are built into the system and do not have to be added externally. Also, the 2 and 6 dimensions for spin 0 and spin 1 are exactly the "particle components" of the Sakata-Taketani version of the DKP system discussed in Sec. IV.

It turns out that the "consequent equations" (2.14) are necessary for a complete covariant DKP formalism. The $\lambda=4$ equation is necessary to obtain the Hamiltonian. The $\lambda=1, 2, 3$ equations and/or the free wave equation are necessary to satisfy the Lie algebra of the Poincaré generators, as will be discussed in III.[21] Specifically, these equations are needed to satisfy the commutation relations $[K_i, K_j]=-i\epsilon_{ijk}J_k$ and $[K_i, H]=iP_i$.

B. The *CPT* operator as distinct from the pseudoscalar coupling operator

We wish to remind the reader that the *CPT* operator, which, for example, in the Dirac case is

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4 , \qquad (2.15)$$

and which in the DKP case will turn out to be

$$\eta_5=\eta_1\eta_2\eta_3\eta_4 , \qquad (2.16)$$

is *not* the same thing as the "pseudoscalar operator" (ps) used in pseudoscalar coupling. In the Dirac case the fact that both of these operators are γ_5 is an algebraic accident. The pseudoscalar operator (ps) as used, for instance, in the axial-vector part of the $V-A$ weak lepton current,

$$l_\lambda = \bar{u}^D \gamma_\lambda [1+(\mathrm{ps})] u^D , \tag{2.17}$$

is defined in the following manner:

$$(\mathrm{ps}) = N\epsilon_{\mu\nu\lambda\sigma}\, \alpha_\mu \alpha_\nu \alpha_\lambda \alpha_\sigma , \tag{2.18}$$

where N is a normalization constant, and the α's are proportional to the matrices of the algebra. For example,

$$\alpha_\lambda = \gamma_\lambda/2, \quad N=\tfrac{2}{3}, \quad \text{for Dirac} \tag{2.19a}$$

$$\alpha_\lambda = \beta_\lambda, \quad N=\tfrac{1}{4}, \quad \text{for DKP .} \tag{2.19b}$$

These results come from the simple calculations which show that for Dirac

$$\tfrac{2}{3}\, \epsilon_{\mu\nu\lambda\sigma}\, \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma / 2^4 = \gamma_1\gamma_2\gamma_3\gamma_4 = \gamma_5 , \tag{2.20}$$

whereas for DKP the (ps) operator does not reduce and one has

$$4\beta_5 \equiv \epsilon_{\mu\nu\lambda\sigma}\beta_\mu \beta_\nu \beta_\lambda \beta_\sigma = \epsilon_{\mu\nu\lambda\sigma}({}_{\mu\nu}R)(R_{\lambda\sigma}) \neq 4\eta_5 , \tag{2.21}$$

where the R's are spin-1 projection operators defined in Ref. 23.

The reason why we can use the spin-1 projection operators in Eq. (2.21) is that β_5 is identically zero in the spin-0 DKP representation. That is, one cannot couple two spin-0 particles of the same parity with a pseudoscalar coupling. For spin 1 an example of β_5 in DKP would be the β_5 pseudoscalar coupling of two vector mesons.

First, we give the spin-1 solution for the DKP equation written in terms of the massive-photon electromagnetic analogy (4-vector potential A_λ, electric and magnetic fields $\vec{E}$ and $\vec{B}$):

$$\psi^{\mathrm{DKP}} = e^{ip\cdot x}\left(\frac{m}{p_0 V}\right)^{1/2} u(p) , \tag{2.22a}$$

$$u(p) = (2m^2)^{-1/2} \begin{bmatrix} E_x \\ E_y \\ E_z \\ B_x \\ B_y \\ B_z \\ -mA_x \\ -mA_y \\ -mA_z \\ -mA_0 \end{bmatrix} = (2m^2)^{-1/2} \begin{bmatrix} -\partial_x A_0 - \partial_t A_x \\ -\partial_y A_0 - \partial_t A_y \\ -\partial_z A_0 - \partial_t A_z \\ \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \\ -mA_x \\ -mA_y \\ -mA_z \\ -mA_0 \end{bmatrix} . \tag{2.22b}$$

The spinor $\bar{u}(p)$ is given by

$$\bar{u}(p) = u^\dagger(p)\, \eta_4 , \tag{2.22c}$$

where in Kemmer's spin-1 representation[3]

$$\eta_4 = \text{diagonal}\, [111, -1-1-1, 111, -1] . \tag{2.22d}$$

The antiparticle spinors are

$$v(p) = u^*(p), \quad \bar{v}(p) = \bar{u}^*(p) . \tag{2.22e}$$

These actually turn out to be the same as the particle spinors in terms of the electromagnetic field quantities if these quantities are all real as in Eqs. (4.28) and (4.30) below.

In any event, one finds for the β_5 coupling inelastic case $p_\lambda \neq p'_\lambda$ (where the primed fields can be complex conjugated for complex fields)

$$i\bar{\psi}^{\mathrm{DKP}'}\, \beta_5 \psi^{\mathrm{DKP}} = \frac{e^{i(p-p')\cdot x}}{(mm'p_0 p'_0 V^2)^{1/2}} \left[\tfrac{1}{2}(\vec{E}'\cdot\vec{B} + \vec{B}'\cdot\vec{E})\right] . \tag{2.23}$$

This is as expected since the vector product of a vector ($\vec{E}$) and an axial vector ($\vec{B}$) is a pseudoscalar.

C. Bhabha's equations

Bhabha's system of first-order wave equations for arbitrary spin can be written as[5]

$$(\partial\cdot\alpha + \chi)\psi = 0 , \tag{2.24}$$

where χ is an integer or half-integer multiple of the mass and the α_μ's are the operators $J_{\mu 5}$ ($\mu = 1, 2, 3, 4$) of the Lie algebra so(5).

To be more explicit, the α_μ's satisfy the double commutation relations

$$[[\alpha_\mu, \alpha_\nu], \alpha_\lambda] = \alpha_\mu \delta_{\nu\lambda} - \alpha_\nu \delta_{\mu\lambda}, \quad \mu, \nu, \lambda = 1, 2, 3, 4 . \tag{2.25}$$

It can be shown[5, 22] that the self-adjoint operators

$$J_{ab} = -J_{ba}, \quad a, b = 1, 2, 3, 4, 5 \tag{2.26}$$

which satisfy the so(5) commutation relations

$$[J_{ab}, J_{cd}] = i(\delta_{ac} J_{bd} + \delta_{bd} J_{ac} - \delta_{bc} J_{ad} - \delta_{ad} J_{bc}) , \tag{2.27}$$

can be given by

$$\alpha_\mu = J_{\mu 5} = -J_{5\mu}, \quad J_{\mu\nu} = -i[\alpha_\mu, \alpha_\nu], \quad J_{55} = 0 . \tag{2.28}$$

(Note from the above that all the algebra matrices here and later are self-adjoint.) It can also be shown that each inequivalent irreducible representation (irrep) of the Lie algebra so(5) determines an inequivalent irrep of the α_μ operator algebra (2.25), and vice versa. The dimension $d_5(\mathcal{S}, S)$ of each irrep is labeled by two numbers, $\mathcal{S}$ and S, both integers or half-integers such that

$$\mathcal{S} \geq S \geq 0 , \tag{2.29}$$

and is given by

$$d_5(\mathcal{S}, S) = \tfrac{1}{6}(2\mathcal{S}+3)(2S+1)[(\mathcal{S}+1)(\mathcal{S}+2) - S(S+1)] \ . \tag{2.30}$$

It also follows, from the Cayley-Hamilton theorem for Hermitian matrices, that each α_μ satisfies the characteristic equation

$$\prod_{n=-\mathcal{S}}^{\mathcal{S}} (\alpha_\mu - nI) = 0 \ . \tag{2.31}$$

(One can quickly see this by observing that any of the J_{ab} operators can be rotated into the third component of angular momentum, $J_3 = J_{12}$.)

Given this, one can then take the special case $N=5$ of Theorem 2 in Ref. 22 and obtain that if $\mathcal{B}(\mathcal{S})$ is the algebra generated by I, α_1, α_2, α_3, and α_4 which satisfy Eqs. (2.25), (2.31), and (2.29), that is

$$[[\alpha_\mu, \alpha_\nu], \alpha_\lambda] = \alpha_\mu \delta_{\nu\lambda} - \alpha_\nu \delta_{\mu\lambda} \ , \tag{2.32a}$$

$$\prod_{n=-\mathcal{S}}^{\mathcal{S}} (\alpha_\mu - nI) = 0 \ , \tag{2.32b}$$

$$\mathcal{S} \geq S \geq 0, \text{ both integer or half-integer} \ , \tag{2.32c}$$

then we have the following:

(i) $\mathcal{B}(0)$, with $d_5(0,0)=1$, is the trivial (DKP) 1-dimensional algebra with elements 0 and I.

(ii) $\mathcal{B}(\frac{1}{2})$, with $d_5(\frac{1}{2},\frac{1}{2})=4$, is the Dirac algebra of Eq. (1.2). To put the Dirac Eq. (1.1) into the Bhabha form Eq. (2.24), take $\alpha_\mu = \gamma_\mu/2$ and $\chi = m/2$.

(iii) $\mathcal{B}(1)$ is the DKP algebra of Eq. (1.4). This algebra is reducible to the $d_5(1,0)=5$ spin-0 case and to the $d_5(1,1)=10$ spin-1 case.

(iv) $\mathcal{B}(\mathcal{S}>1)$ are the Bhabha algebras for higher-spin equations which we will now discuss.

From the above one sees that each algebra $\mathcal{B}(\mathcal{S})$ satisfies a different set of commutation relations, containing products of α's up to order $(2\mathcal{S}+1)$. Further, each algebra is reducible, containing representations of spin

$$S = \mathcal{S}, \mathcal{S}-1, \ldots, 0, \quad \text{integer } \mathcal{S} \tag{2.33a}$$

$$S = \mathcal{S}, \mathcal{S}-1, \ldots, \tfrac{1}{2}, \quad \text{half-integer } \mathcal{S} \ . \tag{2.33b}$$

Given the well-known Dirac and DKP cases, the α_μ commutation relations quickly become very complicated as $\mathcal{S}$ increases, and the only cases we know that have been explicitly calculated are $\mathcal{S}=\frac{3}{2}$ and $\mathcal{S}=2$, done by Madhavarao.[26]

By inserting Eq. (2.31) into the Bhabha Eq. (2.24) taken in the rest frame, one can see that for $\mathcal{S}>1$ the free Bhabha equation will no longer satisfy a single-mass-value Klein-Gordon equation, but rather will actually satisfy[27]

$$\mathcal{S} = \text{integer:} \quad 0 = (\partial\cdot\alpha)[\Box - \chi^2][4\Box - \chi^2]\cdots[(\mathcal{S}-1)^2\Box - \chi^2][\mathcal{S}^2\Box - \chi^2]\psi \ , \tag{2.34}$$

$$\mathcal{S} = \text{half integer:} \quad 0 = [\tfrac{1}{4}\Box - \chi^2][\tfrac{9}{4}\Box - \chi^2]\cdots[(\mathcal{S}-1)^2\Box - \chi^2][\mathcal{S}^2\Box - \chi^2]\psi \ . \tag{2.35}$$

The above results can be viewed as the reason the high-spin Bhabha equations do not need external "constraint equations" or "subsidiary conditions." The extra degrees of freedom usually eliminated with external subsidiary conditions are used up by the existence of solutions for more than one mass and the factor $(\partial\cdot\alpha)$ in the integer case. These eliminations come from Eq. (2.31) or (2.32b), which determines the order of a particular algebra and which basically is the origin of the derivation of the DKP consequent equations (2.14). One can see, for example, from Eqs. (3.4) and (3.5) that the multiple mass solutions for $\mathcal{S}=\frac{3}{2}$ and 2 are [the exponential in ψ for a particular mass state is[28] $\exp(ip_j\cdot x)$, where $p_j\cdot p_j = -\chi^2/j^2$]

$$\chi = \tfrac{3}{2}m, \tfrac{1}{2}m, \quad \mathcal{S} = \tfrac{3}{2} \tag{2.36a}$$

$$\chi = 2m, m, \quad \mathcal{S} = 2 \ . \tag{2.36b}$$

From Eq. (2.31) or (2.32b) one can understand why the half-integer-spin algebras have inverses and the integer-spin algebras do not.[28] In integer-spin algebras there is always a factor (α_μ) multiplying the rest of the product in Eq. (2.31) or (2.32b). This factor comes from the value $n=0$. Thus, the matrix α_μ is singular and does not have an inverse because it always has an eigenvalue of zero. However, for half-integer-spin algebras, Eq. (2.31) or (2.32b) can be written

$$0 = (\alpha_\mu{}^2 - \tfrac{1}{4})(\alpha_\mu{}^2 - \tfrac{9}{4})\cdots(\alpha_\mu{}^2 - \mathcal{S}^2) \ . \tag{2.37}$$

Equation (2.37) always allows a solution for the inverse of α_μ, $(\alpha_\mu)^{-1}$, given by

$$I = \alpha_\mu(\alpha_\mu)^{-1} = (\alpha_\mu)^{-1}\alpha_\mu \ . \tag{2.38}$$

From (2.37) this solution is

$$(\alpha_\mu)^{-1} = 4\alpha_\mu, \quad \mathcal{S} = \tfrac{1}{2} \ , \tag{2.39}$$

and for $\mathcal{S} \geq \frac{3}{2}$,

$$(\alpha_\mu)^{-1}=\left\{\frac{(-1)^{\mathcal{S}-1/2}\,2^{2\mathcal{S}+1}}{[(2\mathcal{S})!!]^2}\right\}(\alpha_\mu)\left\{(\alpha_\mu{}^2)^{\mathcal{S}-1/2}+\sum_{k=1}^{\mathcal{S}-1/2}(-1)^k(\alpha_\mu{}^2)^{\mathcal{S}-1/2-k}\left[\sum_{n_1>n_2>\cdots>n_k\geq 1}^{\mathcal{S}+1/2}c(n_1)\,c(n_2)\cdots c(n_k)\right]\right\}, \tag{2.40a}$$

$$c(n)=(n-\tfrac{1}{2})^2, \tag{2.40b}$$

$$\mathcal{S}\geq\tfrac{3}{2}. \tag{2.40c}$$

Specific examples of Eq. (2.40) are

$$\mathcal{S}=\tfrac{3}{2},\ (\alpha_\mu)^{-1}=-\tfrac{16}{9}\,\alpha_\mu(\alpha_\mu{}^2-\tfrac{5}{2}), \tag{2.41}$$

$$\mathcal{S}=\tfrac{5}{2},\ (\alpha_\mu)^{-1}=\tfrac{64}{225}\,\alpha_\mu(\alpha_\mu{}^4-\tfrac{35}{4}\,\alpha_\mu{}^2+\tfrac{259}{16}), \tag{2.42}$$

$$\mathcal{S}=\tfrac{7}{2},\ (\alpha_\mu)^{-1}=\tfrac{-256}{11025}\,\alpha_\mu(\alpha_\mu{}^6-21\alpha_\mu{}^4+\tfrac{987}{8}\,\alpha_\mu{}^2-\tfrac{3229}{16}). \tag{2.43}$$

Formulas for higher $\mathcal{S}$ are available upon request.

Note that the Bhabha system is internally combining spin-$\frac{1}{2}$ objects and then projecting out the possible total spin pieces. The easiest way to see this is to quote the example of the DKP algebra, which can be represented as the product of two Dirac spaces. That is, the algebra (2.2) is satisfied by

$$\beta_\lambda=\tfrac{1}{2}[I^{(1)}\gamma_\lambda^{(2)}+\gamma_\lambda^{(1)}I^{(2)}], \tag{2.44}$$

and this (16×16) representation is reducible to the aforementioned trivial (1×1), spin-0 (5×5), and spin-1 (10×10) dimensional representations. The Dirac spin-$\frac{1}{2}$ particle really is special in this system.

In a different study, Bhabha[28] obtained the Lagrangian, current, and energy-momentum tensor densities for his system, and he also investigated the nonrelativistic limit. He made physical interpretations about the nature of these quantities with respect to the various mass, spin, and particle-antiparticle properties of his equations.

To be more explicit about some of these properties, the Bhabha adjoint equation is

$$\overline{\psi}(\partial\cdot\alpha-m)=0, \tag{2.45}$$

where the adjoint field turns out to be

$$\overline{\psi}=\psi^\dagger\eta_4, \tag{2.46}$$

with η_4 defined in general for arbitrary spin in Sec. III C.

There is a conserved current

$$\partial_\lambda i\overline{\psi}\alpha_\lambda\psi=0, \tag{2.47}$$

and the expectation value of an operator, like DKP, is

$$\langle\mathcal{O}\rangle=\int d\tau\,\psi^\dagger[\eta_4\langle(\alpha_4\mathcal{O})\rangle]\psi, \tag{2.48}$$

where $\langle(\alpha_4\mathcal{O})\rangle$ is such that

$$[\eta_4\langle(\alpha_4\mathcal{O})\rangle]^\dagger=[\eta_4\langle(\alpha_4\mathcal{O})\rangle]. \tag{2.49}$$

Again we mention that a precise statement of the Hermiticity properties will be given in III.[21]

It should also be mentioned that Harish-Chandra[24] attempted to modify the Bhabha system by having the α_μ's obey the relation

$$(\alpha_\mu)^{2\mathcal{S}+1}=(\alpha_\mu)^{2\mathcal{S}-1} \tag{2.50}$$

instead of Eq. (2.31) or (2.32b). This would eliminate the multiple mass solutions but still agree with the Dirac and DKP algebras since (2.50) is just a different version of

$$(\gamma\cdot p)^2=-m^2,\ \text{Dirac} \tag{2.51a}$$

$$(\beta\cdot p)^3=-m^2(\beta\cdot p),\ \text{DKP}. \tag{2.51b}$$

However, Harish-Chandra was not able to find finite algebras satisfying (2.50) for the higher spins, a point also discussed elsewhere.[29] Harish-Chandra concluded that if finite algebras satisfying (2.50) rigorously did not exist, then fundamental particles of higher spin are not allowed in nature.

Finally, we mention that high-spin field theories have been plagued by at least two general problems[30]: (i) not preserving the field commutation relations when minimal electromagnetic substitution is introduced, and (ii) having acausal solutions to the classical field equations when interactions are introduced. We will discuss how these problems apply to the Bhabha case in III.[21]

III. *C, P, T*

A. The Dirac case

The Dirac C, P, and T transformations are, of course, extremely well known. We will quickly review their properties for two reasons. First, this will facilitate comparison with our new results. Second, it will turn out that the form of the equations necessary to derive the C, P, and T transformation matrices will be symbolically the same as for the Dirac case, but with γ_μ becoming β_μ or α_μ and m becoming χ. Thus, since these equations have to be derived once, we will do it in the familiar Dirac notation as an aid to the reader.

We start with the case of charge conjugation, C. This is obtained by considering the first-order (Dirac) equation with the minimal electromagnetic substitution $\partial_\mu \to \partial_\mu - ieA_\mu$:

$$0 = [(\partial_\mu - ieA_\mu)\,\gamma_\mu + m]\,\psi^D \; . \qquad (3.1)$$

The equation which is charge conjugate to (3.1) will have the opposite charge and the charge-conjugate field $(\psi^D)^C$. $(\psi^D)^C$ is related to ψ by complex conjugation and the matrix $\mathcal{C}$, which is *dependent on the representation of the γ matrices:*

$$eA_\mu \to -eA_\mu, \;\; \psi^D \to (\psi^D)^C \; , \qquad (3.2)$$

$$(\psi^D)^C = \mathcal{C}\,[(\psi^D)]^* \; . \qquad (3.3)$$

[We will use the form (3.3) for the charge-conjugate solution although we note[31] that for covariant purposes it is often easier to work with

$$(\tilde{\bar{\psi}}^D) = \tilde{\gamma}_4(\psi^D)^* \; . \qquad (3.4)$$

If one uses the form (3.4) then all our results here and later are simply modified by multiplication of the transpose of the adjoint operator, $\tilde{\gamma}_4$ or $\tilde{\eta}_4$.]

Inserting (3.2) and (3.3) into (3.1) one obtains

$$0 = [(\partial_\mu + ieA_\mu)\,\gamma_\mu + m]\,(\mathcal{C})\,(\psi^D)^* \; . \qquad (3.5)$$

By taking the complex conjugate of (3.4), noting that

$$\vec{A}^* = \vec{A}, \;\; A_4^* = -A_4 \; , \qquad (3.6)$$

and multiplying on the left by $(\mathcal{C}^{-1})^*$ one obtains

$$0 = (\mathcal{C}^{-1})^*\,[(\vec{\partial} - ie\vec{A})\cdot\vec{\gamma}^* - (\partial_4 - ieA_4)\,\gamma_4^*]\,(\mathcal{C})^*\psi^D \; . \qquad (3.7)$$

Equation (3.7) is the same as (3.1) if we have

$$(\mathcal{C}^{-1})^*\,(\vec{\gamma}^*, \gamma_4^*)\,\mathcal{C}^* = (\vec{\gamma}, -\gamma_4) \qquad (3.8a)$$

or

$$\mathcal{C}^{-1}(\vec{\gamma}, \gamma_4)\,\mathcal{C} = (\vec{\gamma}^*, -\gamma_4^*) \; . \qquad (3.8b)$$

Here is where the representation is important. In the Dirac-Pauli (DP) representation one has

$$(\gamma_{2,4}^{\rm DP})^* = \gamma_{2,4}^{\rm DP}, \;\; (\gamma_{1,3}^{\rm DP})^* = -\gamma_{1,3}^{\rm DP} \; . \qquad (3.9)$$

Therefore, (3.8) becomes

$$\mathcal{C}^{-1}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)\,\mathcal{C} = (-\gamma_1, \gamma_2, -\gamma_3, -\gamma_4) \; , \qquad (3.10)$$

with the solution

$$\mathcal{C}^{\rm DP} = \phi_C \gamma_2 \; , \qquad (3.11)$$

where any ϕ here and later is an arbitrary phase factor which to us (but not always[15]) is unimportant. Also, the ϕ's are not necessarily the same from representation to representation.

Using the same standard method, parity P can be discussed. Consider the space inverted equation

$$0 = \{[-\vec{\partial} - ie\vec{A}(-\vec{x}, t)]\cdot\vec{\gamma} + [\partial_4 - ieA_4(-\vec{x}, t)]\,\gamma_4 + m\}\,\psi^D(-\vec{x}, t) \; . \qquad (3.12)$$

By using

$$[\psi^D(\vec{x}, t)]^P = \mathcal{P}\psi^D(-\vec{x}, t) \; , \qquad (3.13)$$

$$\vec{A}(\vec{x}, t) = -\vec{A}(-\vec{x}, t), \;\; A_4(\vec{x}, t) = A_4(-\vec{x}, t) \qquad (3.14)$$

and multiplying (3.12) on the left by $\mathcal{P}$, one gets equality with (3.1) if

$$\mathcal{P}(\vec{\gamma}, \gamma_4)\,\mathcal{P}^{-1} = (-\vec{\gamma}, \gamma_4) \qquad (3.15)$$

or

$$\mathcal{P} = \phi_P \gamma_4 \; . \qquad (3.16)$$

Time reversal T is the most complicated transformation since it involves the product of a matrix operator $\mathcal{T}$ with an operator $\mathcal{A}$ that is antiunitary[31] on the field of complex numbers:

$$\hat{\mathcal{T}} \equiv \mathcal{T}\mathcal{A} \; , \qquad (3.17a)$$

$$\mathcal{A}\lambda\mathcal{A}^{-1} = \lambda^* \; . \qquad (3.17b)$$

Considering the time-reversed equation

$$0 = \{[\vec{\partial} - ie\vec{A}(\vec{x}, -t)]\cdot\vec{\gamma} + [-\partial_4 - ieA_4(\vec{x}, -t)]\,\gamma_4 + m\} \times\psi^D(\vec{x}, -t) \; , \qquad (3.18)$$

using (3.5)

$$[\psi^D(\vec{x}, t)]^T = \hat{\mathcal{T}}\psi^D(\vec{x}, -t) \; , \qquad (3.19)$$

$$\vec{A}(\vec{x}, -t) = -\vec{A}(\vec{x}, t), \;\; A_4(\vec{x}, -t) = A_4(\vec{x}, t) \; , \qquad (3.20)$$

and multiplying (3.18) on the left by $\hat{\mathcal{T}}$, one gets

$$0 = [(\partial_\mu - ieA_\mu)\,\hat{\mathcal{T}}(\gamma_\mu)\,\hat{\mathcal{T}}^{-1} + m]\,\psi^D(\vec{x}, t) \; . \qquad (3.21)$$

Since $\mathcal{A}$ in $\hat{\mathcal{T}} = \mathcal{T}\mathcal{A}$ is an antiunitary operator, this means for (3.21) to equal (3.1) one would need

$$\mathcal{T}(\gamma_\mu^*)\,\mathcal{T}^{-1} = \gamma_\mu \; . \qquad (3.22)$$

Using (3.9) for γ_λ^* in the DP representation, one gets for $\mathcal{T}$ the solution

$$\mathcal{T} = \phi_T \gamma_1 \gamma_3 \; . \qquad (3.23)$$

Therefore,[32]

$$(\psi^D)^{CPT} = -\phi_C\phi_P\phi_T\gamma_5[\psi^D(-\vec{x}, -t)]^* \; . \qquad (3.24)$$

From the above discussion, one sees that C and T are representation-dependent since they involve complex conjugation. In particular, when giving the transformation matrices, it is important to denote whether one is working in the Dirac-Pauli representation (which is most useful in the nonrelativistic limit), the Weyl (W) representation (which is most useful in the extreme relativistic limit—as for neutrinos), or the Majorana (M) representation (which mixes up particle and antiparticle components of the field as Majorana[33] hoped would be physically possible). These various representations actually just interchange the particular explicit matrices γ_1, γ_2, γ_3, γ_4, and γ_5 among themselves. In particular,

$$\begin{aligned}
\gamma_1^{\mathrm{DP}} &= -(\gamma_1^{\mathrm{DP}})^* = -\gamma_1^W = \gamma_4^M ,\\
\gamma_2^{\mathrm{DP}} &= (\gamma_2^{\mathrm{DP}})^* = -\gamma_2^W = \gamma_2^M ,\\
\gamma_3^{\mathrm{DP}} &= -(\gamma_3^{\mathrm{DP}})^* = -\gamma_3^W = \gamma_5^M ,\\
\gamma_4^{\mathrm{DP}} &= (\gamma_4^{\mathrm{DP}})^* = -\gamma_5^W = \gamma_1^M ,\\
\gamma_5^{\mathrm{DP}} &= (\gamma_5^{\mathrm{DP}})^* = -\gamma_4^W = \gamma_3^M .
\end{aligned} \tag{3.25}$$

By combining the C, P, and T Eqs. (3.8), (3.15), and (3.22) with the properties (3.25) of the different γ matrix representations, one can solve for C, P, and T in any of the above representations. These results are listed in the first three columns of Table I. The differences among the explicit representations will be useful to us later.

B. The DKP case

Putting minimal electromagnetic substitution in the DKP equation yields

$$0 = [(\partial_\mu - ieA_\mu)\beta_\mu + m]\psi^{\mathrm{DKP}} . \tag{3.26}$$

Comparing this with the Dirac case one sees that obtaining C, P, and T for the DKP field is at first symbolically the same as the Dirac case, with β_μ substituted everywhere for γ_μ. Thus, the DKP transformation equations for C, P, and T can be taken over directly from the Dirac Eqs. (3.8), (3.15), and (3.22):

$$(\mathcal{C}^{-1})^*(\vec{\beta}^*, \beta_4^*)\,\mathcal{C}^* = (\vec{\beta}, -\beta_4) , \tag{3.27}$$

$$\mathcal{P}(\vec{\beta}, \beta_4)\,\mathcal{P}^{-1} = (-\vec{\beta}, \beta_4) , \tag{3.28}$$

$$\mathcal{T}(\beta_\mu^*)\,\mathcal{T}^{-1} = \beta_\mu . \tag{3.29}$$

Now, the standard Kemmer representation of the β matrices has

$$\vec{\beta}^* = \vec{\beta}, \quad \beta_4^* = -\beta_4 . \tag{3.30}$$

(Note that this is like the Majorana representation of the Dirac matrices. Also, since there is no longer a direct analogy to $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, one cannot simply go over to Dirac-Pauli- or Weyl-type representations of the β matrices by simply interchanging five known matrices.)

Combining Eqs. (3.27)–(3.30) gives

$$\beta_\mu \mathcal{C} = \mathcal{C}\beta_\mu , \tag{3.31}$$

$$\mathcal{P}(\vec{\beta}, \beta_4) = (-\vec{\beta}, \beta_4)\,\mathcal{P} , \tag{3.32}$$

$$\mathcal{T}(\vec{\beta}, \beta_4) = (\vec{\beta}, -\beta_4)\,\mathcal{T} . \tag{3.33}$$

It is directly verified that solutions to Eqs. (3.31)–(3.33) are

$$\mathcal{C} = \phi_C I , \tag{3.34}$$

$$\mathcal{P} = \phi_P \eta_4 , \tag{3.35}$$

TABLE I. C, P, T, and CPT transformation properties of fields of first-order wave equations. The ϕ's are arbitrary phase factors which should not be thought of as being necessarily the same for the various representations.

Field	Dirac			Duffin-Kemmer-Petiau	Bhabha	Sakata-Taketani: Particle components	Sakata-Taketani: Subsidiary components
Algebra representation	Dirac-Pauli (useful in nonrelativistic limit)	Weyl (useful in relativistic limit)	Majorana (mixes particle and antiparticle)	Kemmer $\vec{\beta}=\vec{\beta}^*$, $\beta_4=-\beta_4^*$	Majorana-Kemmer-type $\vec{\alpha}=\vec{\alpha}^*$, $\alpha_4=-\alpha_4^*$	Pauli$\otimes$spin algebra	DKP algebra in Kemmer rep.
C	$\phi_C\gamma_2\psi^*$	$\phi_C\gamma_2\psi^*$	$\phi_C\psi^*$	$\phi_C\psi^*$	$\phi_C\psi^*$	$\phi_C\tau_x\psi^*$	$\phi_C\psi^*$
P	$\phi_P\gamma_4\psi(-\vec{x})$	$\phi_P\gamma_4\psi(-\vec{x})$	$\phi_P\gamma_4\psi(-\vec{x})$	$\phi_P\eta_4\psi(-\vec{x})$	$\phi_P\eta_4\psi(-\vec{x})$	$\phi_P\psi(-\vec{x})$	$\phi_P\eta_4\psi(-\vec{x})$
T	$\phi_T\gamma_1\gamma_3\psi(-t)$	$\phi_T\gamma_1\gamma_3\psi(-t)$	$\phi_T\gamma_1\gamma_2\gamma_3\psi(-t)$	$\phi_T\eta_1\eta_2\eta_3\psi(-t)$	$\phi_T\eta_1\eta_2\eta_3\psi(-t)$	$\phi_T\psi(-t)$	$\phi_T\eta_1\eta_2\eta_3\psi(-t)$
CPT	$-\phi_C\phi_P\phi_T$	$-\phi_C\phi_P\phi_T$	$-\phi_C\phi_P\phi_T$	$+\phi_C\phi_P\phi_T$	$(-1)^{2S}\phi_C\phi_P\phi_T$	$\phi_C\phi_P\phi_T$	$\phi_C\phi_P\phi_T$
	$\times\gamma_5\psi^*(-x_\mu)$	$\times\gamma_5\psi^*(-x_\mu)$	$\times\gamma_5\psi^*(-x_\mu)$	$\times\eta_5\psi^*(-x_\mu)$	$\times\eta_5\psi^*(-x_\mu)$	$\times\tau_x\psi^*(-x_\mu)$	$\times\eta_5\psi^*(-x_\mu)$

$$\mathcal{T} = \phi_T \eta_1 \eta_2 \eta_3 \ , \tag{3.36}$$

if one can find four matrices η_μ satisfying

$$\{\eta_\mu, \beta_\lambda\} = 0, \quad \mu \neq \lambda \tag{3.37a}$$

$$[\eta_\mu, \beta_\mu] = 0, \quad \text{no summation} \tag{3.37b}$$

$$\eta_\mu^{\ 2} = I \ . \tag{3.37c}$$

But Eqs. (3.37) are just the properties of the DKP η_λ matrices defined in Eq. (2.4):

$$\eta_\mu = 2\beta_\mu^{\ 2} - I \ . \tag{3.38}$$

Thus, one has the solutions (3.34)–(3.36). Further, since

$$[\eta_\lambda, \eta_\mu] = 0 \ , \tag{3.39}$$

one also has that

$$(\mathcal{C}\mathcal{P}\mathcal{T}) = \phi_C \phi_P \phi_T \eta_5 \equiv \phi_C \phi_P \phi_T (\eta_1 \eta_2 \eta_3 \eta_4) \ . \tag{3.40}$$

These results[34] are tabulated in the fourth column of Table I.

C. General Bhabha case

Putting minimal electromagnetic substitution into the Bhabha equations for arbitrary spin yields

$$0 = [(\partial_\mu - ieA_\mu)\,\alpha_\mu + \chi]\psi \ . \tag{3.41}$$

(Note that χ is a constant times the unity matrix,[28] so that it commutes with all the α_μ.)

By now using the Kemmer-Majorana representation

$$\vec{\alpha}^* = \vec{\alpha}, \quad \alpha_4^* = -\alpha_4 \ , \tag{3.42}$$

we can simply take over all the analysis of Sec. III B for the DKP equation and immediately say that the C, P, and T transformations are given by

$$\mathcal{C} = \phi_C I \ , \tag{3.43}$$

$$\mathcal{P} = \phi_P \eta_4 \ , \tag{3.44}$$

$$\mathcal{T} = \phi_T \eta_1 \eta_2 \eta_3 \ , \tag{3.45}$$

assuming that one can find matrices η_μ in all representations of the α_μ (i.e., all $\mathcal{S}$ and S) which satisfy

$$\{\eta_\mu, \alpha_\lambda\} = 0, \quad \mu \neq \lambda \tag{3.46a}$$

$$[\eta_\mu, \alpha_\mu] = 0, \quad \text{no summation} \tag{3.46b}$$

$$\eta_\mu^{\ 2} = I \ . \tag{3.46c}$$

Indeed it turns out that η_μ's satisfying Eqs. (3.46) exist. Using Eqs. (2.32) for the α_μ algebra and Lagrange's interpolation technique, Madhavarao, Thiruvenkatachar, and Venkatachaliengar,[35] in a little known paper, found an explicit general formula for the η_μ's. They did this out of a purely algebraic interest having nothing to do with C, P, and T transformations.[36] The explicit formula, derived in the Appendix, is

$$\eta_\mu(\mathcal{S}) = f(\alpha_\mu, \mathcal{S}) \ , \tag{3.47a}$$

$$f(x, \mathcal{S}) = \frac{(x-\mathcal{S})(x-\mathcal{S}+1)\cdots(x+\mathcal{S}-1)(x+\mathcal{S})}{(2\mathcal{S})!} \times \sum_{n=0}^{2\mathcal{S}} \binom{2\mathcal{S}}{n} \frac{1}{(x-\mathcal{S}+n)} \ . \tag{3.47b}$$

For $\mathcal{S}$ a half integer or integer, Eq. (3.47b) reduces to

$$f(x, \mathcal{S} = \text{half integer}) = \frac{2x(x^2-\frac{1}{4})(x^2-\frac{9}{4})\cdots(x^2-\mathcal{S}^2)}{(2\mathcal{S})!} \sum_{n=1}^{\mathcal{S}+1/2} \binom{2\mathcal{S}}{\mathcal{S}+\frac{1}{2}-n} \frac{1}{[x^2-(n-\frac{1}{2})^2]} \tag{3.47c}$$

and

$$f(x, \mathcal{S} = \text{integer}) = \frac{(x^2-1^2)(x^2-2^2)\cdots(x^2-\mathcal{S}^2)}{(\mathcal{S}!)^2} + \frac{2x^2(x^2-1^2)\cdots(x^2-\mathcal{S}^2)}{(2\mathcal{S})!} \sum_{n=1}^{\mathcal{S}} \binom{2\mathcal{S}}{\mathcal{S}-n} \frac{1}{(x^2-n^2)} \ . \tag{3.47d}$$

Specific examples are

$$\left.\begin{aligned} &\eta_\mu = 2\alpha_\mu, \quad \mathcal{S} = \tfrac{1}{2}\,(\text{Dirac}) \\ &\eta_\mu = 2\alpha_\mu^{\ 2} - 1, \quad \mathcal{S} = 1\,(\text{DKP}) \\ &\eta_\mu = \tfrac{1}{3}\alpha_\mu(4\alpha_\mu^{\ 2} - 7), \quad \mathcal{S} = \tfrac{3}{2} \\ &\eta_\mu = \tfrac{2}{3}\alpha_\mu^{\ 4} - \tfrac{8}{3}\alpha_\mu^{\ 2} + 1, \quad \mathcal{S} = 2. \end{aligned}\right. \tag{3.48}$$

Formulas for higher $\mathcal{S}$ are available upon request.

Equations (3.47) show that η_μ is a polynomial in α_μ of even or odd order, depending on whether $\mathcal{S}$ is an integer or half integer. This fact, combined with Eq. (3.46a), means that

$$\left.\begin{aligned} &\eta_\mu\eta_\lambda - \eta_\lambda\eta_\mu = 0, \quad \mathcal{S}\text{ an integer} \\ &\eta_\mu\eta_\lambda + \eta_\lambda\eta_\mu = 0, \quad \mathcal{S}\text{ a half integer} \end{aligned}\right\} \mu \neq \lambda \ . \tag{3.49}$$

That is, η_μ and η_λ commute or anticommute depending on whether the spin is integer or half integer.

The significance of this fact for us is that there is an additional phase in the CPT operator. Specifically,

$$\begin{aligned} (\mathcal{C}\mathcal{P}\mathcal{T}) &= (-1)^{2\mathcal{S}}\,\phi_C\phi_P\phi_T\eta_1\eta_2\eta_3\eta_4 \\ &\equiv (-1)^{2\mathcal{S}}\,\phi_C\phi_P\phi_T\eta_5 \ . \end{aligned} \tag{3.50}$$

If one considers CTP or TCP this extra phase is avoided.

The results of this subsection are located in the fifth column of Table I. As claimed, for the special values $\mathcal{S}=(\frac{1}{2}$ and $1)$ the results reduce to the Dirac and DKP cases, respectively.

IV. *CPT* AND THE SAKATA-TAKETANI EQUATIONS

A. Properties of the ST equations

As mentioned in the Introduction, a physically useful formulation of the DKP equation was derived by Sakata and Taketani.[18] By use of what is known as a Peirce decomposition,[37] Sakata and Taketani[12,18] were able to separate out the $(2S+1)\times 2$ (for particle-antiparticle) components of the DKP formulation into one distinct Hamiltonian equation. The remaining components (essentially the built-in subsidiary conditions) are in a distinct equation that has to be satisfied simultaneously for a covariant description.

The method takes advantage of the fact that from Eq. (2.31) β_4^2 has eigenvalues of 0 and 1. In particular, there are $(2S+1)\times 2$ eigenvalues of 1, the rest being 0. Therefore, since β_4^2 and $(1-\beta_4^2)$ satisfy

$$\mathcal{G}\equiv\beta_4^2=\mathcal{G}^2 , \tag{4.1}$$

$$(1-\mathcal{G})=(1-\mathcal{G})^2 , \tag{4.2}$$

$$\mathcal{G}(1-\mathcal{G})=(1-\mathcal{G})\mathcal{G}=0,\quad I=\mathcal{G}+(1-\mathcal{G}) , \tag{4.3}$$

one can write any eigenvalue equation (and in particular the Hamiltonian equation) in the form

$$H\psi^{\mathrm{DKP}}=i\frac{\partial}{\partial t}\psi^{\mathrm{DKP}}=E\psi^{\mathrm{DKP}} , \tag{4.4}$$

$$H=\mathcal{G}H\mathcal{G}+\mathcal{G}H(1-\mathcal{G})+(1-\mathcal{G})H\mathcal{G}+(1-\mathcal{G})H(1-\mathcal{G}) , \tag{4.5}$$

$$E=E\mathcal{G}+E(1-\mathcal{G}) , \tag{4.6}$$

or

$$E[\mathcal{G}\psi^{\mathrm{DKP}}]=[\mathcal{G}H\mathcal{G}+\mathcal{G}H(1-\mathcal{G})]\psi^{\mathrm{DKP}} , \tag{4.7}$$

$$E[(1-\mathcal{G})\psi^{\mathrm{DKP}}]=[(1-\mathcal{G})H\mathcal{G}+(1-\mathcal{G})H(1-\mathcal{G})]\psi^{\mathrm{DKP}} . \tag{4.8}$$

In Eqs. (4.7) and (4.8) H is the DKP Hamiltonian with minimal electromagnetic substitution

$$H\psi^{\mathrm{DKP}}=i\frac{\partial}{\partial t}\psi^{\mathrm{DKP}}=-\frac{\partial}{\partial x_4}\psi^{\mathrm{DKP}}=E\psi^{\mathrm{DKP}} , \tag{4.9}$$

$$H=+\frac{1}{i}\partial_k^-\left(\frac{\beta_k\beta_4-\beta_4\beta_k}{i}\right)+m\beta_4 +\frac{ie}{2m}F_{\nu\rho}(\beta_\rho\beta_4\beta_\nu-\delta_{\rho 4}\beta_\nu)+eA_0 , \tag{4.10}$$

$$\partial_\mu^\mp=\partial_\mu\mp ieA_\mu . \tag{4.11}$$

What one wants to do with Eqs. (4.7) and (4.8) is to first write the second term on the right in (4.7) not as $\mathcal{G}H(1-\mathcal{G})$, but as something of the form $\mathcal{G}\mathcal{O}\mathcal{G}$. Similarly the first term on the right in (4.8) should be of the form $(1-\mathcal{G})\mathcal{O}(1-\mathcal{G})$. This would then decouple the two equations into particle-antiparticle components

$$\mathcal{G}\psi^{\mathrm{DKP}}\equiv\psi_P^{\mathrm{ST}} \tag{4.12}$$

and subsidiary components

$$(1-\mathcal{G})\psi^{\mathrm{DKP}}\equiv\psi_S^{\mathrm{ST}} . \tag{4.13}$$

This decoupling can easily be accomplished for (4.7) by simply multiplying Eq. (2.1) by $(1-\mathcal{G})$ to obtain the first "decoupling equation" in the form

$$-m(1-\mathcal{G})\psi^{\mathrm{DKP}}=(1-\mathcal{G})\vec{\partial}^-\cdot\vec{\beta}\psi^{\mathrm{DKP}}=\vec{\partial}^-\cdot\vec{\beta}\mathcal{G}\psi^{\mathrm{DKP}} . \tag{4.14}$$

Using (4.14), one finds the result obtained by Sakata and Taketani,[38]

$$E\psi_P^{\mathrm{ST}}=\mathcal{H}_P\psi_P^{\mathrm{ST}} , \tag{4.15}$$

$$\mathcal{H}_P=\mathcal{G}[H-m^{-1}H\vec{\beta}\cdot\vec{\partial}^-]\mathcal{G} \tag{4.16}$$

$$=m\beta_4+eA_0\mathcal{G}-m^{-1}\beta_4\beta_k\beta_i\partial_k^-\partial_i^- \tag{4.17}$$

$$\equiv m\beta_4+eA_0\mathcal{G}-\beta_4\left(\frac{1+\eta}{2}\right)m^{-1}\vec{\partial}^-\cdot\vec{\partial}^- +\beta_4\eta m^{-1}(\vec{S}\cdot\vec{\partial}^-)^2-\beta_4\left(\frac{1+\eta}{2}\right)\frac{e}{m}(\vec{S}\cdot\vec{B}) , \tag{4.18}$$

where $\vec{B}$ is the magnetic field, and the spin $\vec{S}$ and η are

$$iS_i\equiv\epsilon_{ijk}\beta_j\beta_k , \tag{4.19}$$

$$\eta\equiv\eta_1\eta_2\eta_3 . \tag{4.20}$$

By then observing that among themselves the surrounded operators

$$\mathcal{G}(\mathcal{G})\mathcal{G}\sim I,\quad \mathcal{G}(-i\beta_4)\mathcal{G}\mathcal{G}(\eta)\mathcal{G}\sim\tau_y , \quad \mathcal{G}(\eta)\mathcal{G}\sim\tau_x,\quad \mathcal{G}(\beta_4)\mathcal{G}\sim\tau_z , \tag{4.21}$$

form a Pauli algebra, and that further this Pauli algebra commutes with the surrounded spin algebra, the particle components of the ST system can be written in the final form

$$E\psi_P^{\mathrm{ST}}=\mathcal{H}_P\psi_P^{\mathrm{ST}} , \tag{4.22a}$$

$$\mathcal{H}_P=m\tau_z+eA_0-(\tau_z+i\tau_y)(\vec{\partial}^-\cdot\vec{\partial}^-+e\vec{S}\cdot\vec{B})(2m)^{-1} +i\tau_y(\vec{S}\cdot\vec{\partial}^-)^2m^{-1} . \tag{4.22b}$$

The subsidiary component ST equation can be obtained in a similar but more complicated manner. To our knowledge it has never been derived before and we will discuss the solution in more detail in II.[20] For now we simply note that by multiplying the DKP equation by $(\partial_4^- \beta_4 - m)\beta_4^2$ one has the second "decoupling equation" in the form

$$[(\partial_4^-)^2 - m^2]\mathcal{G}\psi^{\text{DKP}} = -(\partial_4^-\beta_4 - m)(\vec{\partial}^-\cdot\vec{\beta})[(1-\mathcal{G})\psi^{\text{DKP}}] \ . \tag{4.23}$$

Using this to change the term $(1-\mathcal{G})H\mathcal{G}\psi^{\text{DKP}}$ in (4.8) to the form $(1-\mathcal{G})\Theta(1-\mathcal{G})\psi^{\text{DKP}}$ leads to $\{[(\partial_4^-)^2 - m^2]\psi^{\text{DKP}} \neq 0\}$,

$$\mathcal{H}_S\psi_S^{\text{ST}} = (1-\mathcal{G})\mathcal{H}[1 - ((\partial_4^-)^2 - m^2)^{-1}(\partial_4^-\beta_4 - m)(\vec{\partial}^-\cdot\vec{\beta})] \times(1-\mathcal{G})\psi_S^{\text{ST}} \ . \tag{4.24}$$

The right-hand side of (4.24) is not free of time derivatives, so the subsidiary components Hamiltonian is not a Hamiltonian in the ordinary sense. Its solution is an identity in terms of the particle components solution. This can most easily be seen in the free-particle case, where

$$\begin{aligned}\mathcal{H}_S\psi_S^{\text{ST}} &= -E(\vec{p}\cdot\vec{p})^{-1}(\vec{\partial}\cdot\vec{\beta})(\vec{\partial}\cdot\vec{\beta})(1-\beta_4^2)\psi_S^{\text{ST}} \\ &= E\psi_S^{\text{ST}} \ .\end{aligned} \tag{4.25}$$

Thus, although one can derive the C, P, and T transformations for ψ_S^{ST}, as we do in the next subsection, and one can obtain the Poincaré generators and show that they satisfy the proper commutation relations as we will do in III,[21] it is an operational exercise, the physics having been transferred elsewhere. We will return to this point in more detail in II.[20]

Finally we mention that as in the standard DKP formulation, the particle component expectation value of ST has a different adjoint expectation value than would be naively expected. It is

$$\begin{aligned}\langle\Omega\rangle^{\text{ST}} &= \int \overline{\psi}_P^{\text{ST}}(\tau_z\Omega)\psi_P^{\text{ST}}\,d\tau \\ &= \int(\psi_P^{\text{ST}})^\dagger(\tau_z\Omega)\psi_P^{\text{ST}}\,d\tau \ ,\end{aligned} \tag{4.26}$$

which, from Eq. (4.21) for β_4, is clearly a reflection that in the entire DKP formulation the expectation value is

$$\begin{aligned}\langle\Omega\rangle^{\text{DKP}} &= \int\overline{\psi}^{\text{DKP}}(\beta_4\Omega)\psi^{\text{DKP}}\,d\tau \\ &= \int(\psi^{\text{DKP}})^\dagger\eta_4(\beta_4\Omega)\psi^{\text{DKP}}\,d\tau \ .\end{aligned} \tag{4.27}$$

B. C, P, and T for Sakata-Taketani equations

Before doing C, P, and T for the particle components of ST, we have to quickly review the properties of $\vec{B}$, $\vec{E}$, and $\vec{S}$, since we no longer have a manifestly covariant formulation. From the properties of the four-vector potential A_μ given in Eqs. (3.6) and (3.14)

$$\begin{aligned}&(\vec{A}, A_4)^* = (\vec{A}, -A_4) \ , \\ &(\vec{A}(-\vec{x}), A_4(-\vec{x})) = (-\vec{A}(\vec{x}), A_4(\vec{x})) \ , \\ &(\vec{A}(-t), A_4(-t)) = (-\vec{A}(t), A_4(t)) \ ,\end{aligned} \tag{4.28}$$

combined with the definitions

$$\vec{B} = \vec{\nabla}\times\vec{A}, \ \ \vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t \ , \tag{4.29}$$

one has

$$\begin{aligned}&\vec{B}^* = \vec{B}, \ \ \vec{E}^* = \vec{E} \ , \\ &\vec{B}(-\vec{x}) = \vec{B}(\vec{x}), \ \ \vec{E}(-\vec{x}) = -\vec{E}(\vec{x}) \ , \\ &\vec{B}(-t) = -\vec{B}(t), \ \ \vec{E}(-t) = \vec{E}(t) \ .\end{aligned} \tag{4.30}$$

Further, from thinking about the orbital part of angular momentum being defined as $\vec{L} = \vec{r}\times(-i\vec{\partial})$, or that $\vec{S}$ is given by Eq. (4.19) with the $\vec{\beta}$ real, one can quickly realize that

$$\vec{S}^* = -\vec{S}, \ \ \vec{S}(-\vec{x}) = \vec{S}(-t) = \vec{S}(\vec{x}, t) \ . \tag{4.31}$$

One can then consider the C, P, and T transformations of Eq. (4.22) for ψ_P^{ST}. The only nontrivial transformation matrix is obtained for C. By the same method as in Sec. III, one can show that the C transformed equation of (4.22) is

$$0 = \mathcal{C}^{*-1}[-E - m\tau_z + eA_0 + (\tau_z + i\tau_y)(\vec{\partial}^-\cdot\vec{\partial}^- + e\vec{S}\cdot\vec{B})(2m)^{-1} - i\tau_y(\vec{S}\cdot\vec{\partial}^-)^2 m^{-1}]\,\mathcal{C}^*\psi_P^{\text{ST}} \ . \tag{4.32}$$

Equation (4.32) is the same as (4.22) if

$$\begin{aligned}&\mathcal{C}^{*-1}\tau_z\mathcal{C}^* = -\tau_z \ , \\ &\mathcal{C}^{*-1}\tau_y\mathcal{C}^* = -\tau_y \ .\end{aligned} \tag{4.33}$$

The solution is

$$\mathcal{C}^{\text{ST}} = \phi_C\tau_x \ . \tag{4.34}$$

The P and T transformed equations turn out to be identical to the original, so that

$$\mathcal{P}^{\text{ST}} = \phi_P I, \ \ \mathcal{T}^{\text{ST}} = \phi_T I \ , \tag{4.35}$$

and

$$(\mathcal{C}\mathcal{P}\mathcal{T})^{\text{ST}} = \phi_C\phi_P\phi_T\tau_x \ , \tag{4.36}$$

these results being in the sixth column of Table I.

With regard to the subsidiary components of the ST system, since all the η_μ commute with the surrounding operator $(1-\mathcal{G})$, and since Eq. (4.24) is still written in the DKP β algebra, one might quickly guess that the C, P, and T transformation matrices would be the same as for the DKP ordinary case. This guess turns out to be correct

Direct substitution of the C, P, and T transformations into Eq. (4.24) leads, after a fair amount of algebra, to the original Eq. (4.24) if the transformation matrices are the same as for the standard DKP case of Sec. III B. More precisely, the subsidiary component Sakata-Taketani C, P, and T transformation matrices must satisfy the same Eqs. (3.31)–(3.33) as the DKP matrices. These results are listed in the last column of Table I.

V. DISCUSSION

From the above we have seen how the DKP and general Bhabha C, P, and T transformation matrices are direct generalizations of the Dirac case. Since C and T involve complex conjugation their transformation matrices depend on the particular representation of the algebra matrices that are used. (The Dirac algebra transformation matrices in the Majorana, Dirac-Pauli, and Weyl representations are given in Table I.) The different representations yield different physical insights. One can choose the representation which best illuminates the area of interest.

For the DKP and Bhabha cases we used the "Kemmer representation," which corresponds to the Majorana representation of the Dirac matrices and which mixes up particle and antiparticle states. To see this, note that the DKP spinors in Eqs. (1.7) have first and fifth components proportional to E and m, not to $(E \pm m)$ as for the ordinary particle and antiparticle type solutions. For the Dirac case, one just changes around the explicit matrices for γ_1, γ_2, γ_3, γ_4, and γ_5 to go from one representation to the other, as shown in Eq. (3.25). Because this freedom is in general lost for higher-spin representations, one cannot just switch the matrices around. One has to perform rotations of the so(5) matrices.

Note, however, that the ST decomposition of the DKP equation leaves one with the "particle component" solutions exhibiting the particle-antiparticle nature of the DKP equation. The free solution spinors are proportional to mixtures of $(E \pm m)$. Specifically, for spin-0 they are $(V=1)$

$$\psi(p)=\frac{1}{(4mE)}\begin{bmatrix} E+m \\ -E+m\end{bmatrix} e^{ip\cdot x},$$

$$\overline{\psi}(p)=\psi(p)^\dagger \,, \tag{5.1}$$

$$\psi(p)^C=\frac{1}{(4mE)}\begin{bmatrix} -E+m \\ E+m\end{bmatrix} e^{-ip\cdot x},$$

$$\overline{\psi}(p)^C=[\psi(p)^C]^\dagger \,. \tag{5.2}$$

Furthermore, the Majorana-Kemmer representation can give us a physical insight into how CP or T violation might be set up in the K_L-K_S meson system. Remember that this representation is useful if *physical* particles and antiparticles can be mixed up. Because of conservation laws such as baryon or lepton conservation, one does not ordinarily allow for the possibility of this happening in the case of fermions. However, as was pointed out by Primakoff and Sharp,[39] if one allows lepton conservation to break down[40] (neutrinos are Majorana particles), this will lead to a prediction of CP nonconservation if an imaginary term is added to the $V-A$ current.

Mesons are not conserved. In other words, if one were to represent K_0 and $\overline{K}_0$ by the u and v spinors of Eq. (2.7), and then were to represent K_L and K_S by

$$u_+=\frac{-i}{\sqrt{2}}\,(u+v),\;\; u_-=\frac{-1}{\sqrt{2}}\,(u-v) \tag{5.3}$$

without the exponentials, then one would have the Majorana-type condition

$$(u_\pm)^*=u_\pm \,. \tag{5.4}$$

Thus, motivated by Primakoff and Sharp, adding an imaginary term to the weak hadronic meson current can involve CP violation, uniting these two nonconservation laws. (See Ref. 39 for more details.)

Intuitively one can think of this in the Feynman picture as mixing up the directions of time of particle and antiparticle when you allow them to combine.

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APPENDIX: DERIVATIONS OF THE OPERATORS η_μ

To derive the general operators η_μ, one starts with the basic Bhabha double commutation relations of Eq. (2.32a):

$$[[\alpha_\mu, \alpha_\nu], \alpha_\lambda]=\alpha_\mu\delta_{\nu\lambda}-\alpha_\nu\delta_{\mu\lambda} \,. \tag{A1}$$

Because one can rotate any of the generators J_{ab} into the third component of angular momentum, one can always find a representation where a particular α_μ is diagonal. We do this, as a calculational trick, but our results will be representation-independent. The matrix representation of α_μ is now

$$(\alpha_\mu)_{ij} = d_i \delta_{ij} \ , \tag{A2}$$

μ being definite and no sum involved in the i's or j's here or later. Take $\lambda = \mu$ and $\nu \neq \mu$ with a matrix representation

$$(\alpha_\nu)_{ij} = c_{ij}, \quad \nu \neq \mu \ . \tag{A3}$$

Then Eq. (A1) becomes

$$[[\alpha_\mu, \alpha_\nu], \alpha_\mu] = -\alpha_\nu \ , \tag{A4}$$

and combining (A2)–(A4) gives

$$c_{ik} = (d_i - d_k)^2 c_{ik} \ . \tag{A5}$$

Again i and j in (A5) denote a particular component of the matrix, and no sum is involved.

The first point to be made from (A5) is that

$$c_{ii} = 0 \ , \tag{A6}$$

that is, for α_μ diagonal, α_ν is off-diagonal. Further, (A5) shows that

$$c_{ik} = 0, \ \text{unless} \begin{cases} (d_i - d_k)^2 = 1 \ , \\ d_i \pm 1 = d_k \ . \end{cases} \tag{A7}$$

Because of the characteristic equation (2.32a) and the facts that

$$\eta_\mu \alpha_\nu + \alpha_\nu \eta_\mu = 0, \quad \mu \neq \nu \tag{A8}$$

$$\eta_\mu \alpha_\mu - \alpha_\mu \eta_\mu = 0, \quad \text{no sum} \tag{A9}$$

the η_λ are assumed to be expressible as a polynomial in α_λ of order $2\mathcal{S}$ for a particular algebra representation labeled by $\mathcal{S}$. Therefore,

$$\eta_\mu = f(\alpha_\mu, \mathcal{S}) \equiv \sum_{n=0}^{2\mathcal{S}} a_n (\alpha_\mu)^n \ . \tag{A10}$$

Since α_μ is diagonal, (A10) implies

$$(\eta_\mu)_{ij} = f(d_i, \mathcal{S})\, \delta_{ij} \ . \tag{A11}$$

Combining (A2), (A3), (A8), and (A11) gives

$$0 = c_{ik} [f(d_i, \mathcal{S}) + f(d_k, \mathcal{S})] \ . \tag{A12}$$

(A7) and (A12) together mean that

$$0 = f(d_i, \mathcal{S}) + f(d_k, \mathcal{S}), \ \text{if} \begin{cases} (d_i - d_k)^2 = 1 \\ d_i \pm 1 = d_k \end{cases} \tag{A13}$$

or

$$f(d_i, \mathcal{S}) = -f(d_i \pm 1, \mathcal{S}), \ \text{for all } i \ . \tag{A14}$$

But since η_μ is diagonal from (A11), and further it is unitary from

$$\eta_\lambda^2 = 1 \ , \tag{A15}$$

we then have that

$$(\eta_\mu^2)_{ij} = \delta_{ij} = f^2(d_i, \mathcal{S})\, \delta_{ij} \ , \tag{A16}$$

or

$$f^2(d_i, \mathcal{S}) = 1 \ \text{for all } d_i \ . \tag{A17}$$

(A14) and (A17) together give us that

$$f(\mathcal{S}, \mathcal{S}) = -f(\mathcal{S}-1, \mathcal{S}) = +f(\mathcal{S}-2, \mathcal{S}) = \cdots = (-1)^{2\mathcal{S}} f(-\mathcal{S}, \mathcal{S}) = +1 \ , \tag{A18}$$

or

$$f(\mathcal{S}-n, \mathcal{S}) = (-1)^n, \quad n = 0, 1, 2, \ldots, 2\mathcal{S} \ . \tag{A19}$$

[Choosing +1 on the right-hand side of (A18) is a convention that just as well could have been −1.]

But now since $f(x, \mathcal{S})$ is a polynomial of degree $2\mathcal{S}$ in x and Eqs. (A18) and (A19) give the value of $f(x, \mathcal{S})$ at $2\mathcal{S}+1$ points, $f(x, \mathcal{S})$ can be uniquely determined by using the Lagrange interpolation formula with zero remainder term. (That is, in this case the formula is exact instead of an interpolation.) For the polynomial $\mathcal{G}(x)$ of degree m determined at $m+1$ points x_i, the formula is

$$\mathcal{G}(x) = \sum_{i=0}^{m} f(x_i)\, l_i(x) \ , \tag{A20}$$

$$l_i(x) = \frac{\prod_{j=0}^{m}(x - x_j)}{[(x - x_i)\prod_{j=0,\, j\neq i}^{m}(x_i - x_j)]} \ . \tag{A21}$$

Therefore, the formula for $f(x, \mathcal{S})$ is, after translating the sum from (0 to $2\mathcal{S}$) to ($-\mathcal{S}$ to $\mathcal{S}$),

$$f(x, \mathcal{S}) = \sum_{n=-\mathcal{S}}^{\mathcal{S}} \frac{(-1)^{n-\mathcal{S}} \prod_{j=-\mathcal{S}}^{\mathcal{S}}(x - j)}{[(x-n)\prod_{j=-\mathcal{S},\, j\neq n}^{\mathcal{S}}(n - j)]} \tag{A22}$$

Finally, some simple algebraic manipulation transforms Eq. (A22) to the final result (3.47b) for arbitrary $\mathcal{S}$, and then if desired, the Eqs. (3.47c) and (3.47d) for half integer and integer spin, respectively.

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[31]See p. 117 of Ref. 16 for a discussion of Eq. (3.4), and pp. 81–84 on the antiunitarity of Eq. (3.17). An antiunitary operator can be thought of as a unitary operator times complex conjugation.

[32]As stated, we are using Wigner time reversal, T_W, instead of Schwinger time reversal, T_S. The connection between these and strong reflection (SR), which is usually used to discuss the *CPT* theorem, is carefully given in Ref. 16. See pp. 86, 142, and 143 for T_S vs T_W, and pp. 136–139, 146, and 147 for SR.: (SR) $= CPT_S$ $=$ (Hermitian conjugation) $\times (CPT_W)$.

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operator is the same as the P matrix, η_4. Also, in this paper Bhabha first used the trick of diagonalizing α_4 that was used to obtain the η_μ in Ref. 35.

[37]See p. 583 of Ref. 12.

[38]Corresponding to these equations the following misprints in Ref. 12 should be noted: Equation (3.13) should have a minus sign on the left, the second term on the right of (3.15) should have a minus sign, and in (3.19) the term $(\vec{S}\cdot\vec{\partial}^-)$ should be $(\vec{S}\cdot\vec{\partial}^-)^2$.

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Renormalization-group sum rules and the construction of massless field theories in 4 − ϵ dimensions

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We study massless ϕ^4 field theory and the Reggeon calculus with Pomeron intercept 1, in $4-\epsilon$ dimensions. We present sum rules which give the full propagator and the bare mass (or intercept) as integrals over the remaining (finite) renormalization constants of these theories. When an infrared-stable Gell-Mann–Low eigenvalue exists these sum rules can be used to extract the infrared behavior of the propagator. They can also be used to show that the perturbation series is an asymptotic expansion for small values of the coupling constant and large values of the momentum. The sum rules can be combined with the Schwinger-Dyson equations for each theory to give a perturbative construction of the Green's functions which is free of infrared divergences.

I. INTRODUCTION

Massless field theories in $4-\epsilon$ dimensions are of great interest in both solid-state and high-energy physics. In the study of critical-point phenomena the field theory of major interest is relativistic ϕ^4 theory (analytically continued to the Euclidean region).[1-3] In the high-energy Pomeranchuk problem the relevant field theory is the "nonrelativistic" ψ^3 theory, better known as the Reggeon calculus.[4-6] The unifying feature of these problems is that in both cases the development of long-range order leads to scaling laws for the correlation (Green's) functions in the infrared region. In both cases the critical exponents and scaling functions can be directly calculated using renormalization-group techniques. However, the construction of these theories in perturbation theory is (for finite rational ϵ) plagued with infrared divergences.[2,3,7] If these field theories are renormalizable at all in $4-\epsilon$ dimensions, then they are superrenormalizable. That is, infrared divergences can be related only to the mass renormalization. From dimensional analysis the bare mass m_0 in the ϕ^4 theory is related to the coupling constant g_0 by

$$m_0 = g_0^{1/\epsilon} f(\epsilon), \tag{1}$$

and so it gives rise to terms nonanalytic in g_0, and hence to divergences of perturbation theory. m_0 also contains an essential singularity at $\epsilon = 0$, and so in the usual ϵ expansion of the theory is taken to be zero. It is desirable, therefore, to have a method for constructing these theories which avoids the ϵ expansion, as well as the difficulties of perturbation theory.

In this paper we present sum rules, valid in both theories, which give both the bare mass and the full propagator as integrals over the finite renormalization constants of the theories. The integral representation for the propagator explicitly displays the anomalous-dimension infrared behavior when a stable Gell-Mann–Low eigenvalue is present (which in perturbation theory is the case in both theories, at least for small ϵ). It can also be used to show that the bare perturbation expansion is an asymptotic expansion valid for small values of the coupling constant or large values of the momentum. We further show that our sum rules can be combined with Schwinger-Dyson equations for each of the theories to give an iterative construction procedure which is free of infrared divergences. The question of the convergence of this iteration procedure goes beyond that of the convergence of the perturbation series for massive theories because of the nonanalyticity of m_0. However, at each step of the calculation the approximation for m_0 is systematically improved, as is explained in the text. As a result, one may be optimistic about the convergence of our procedure.