fies F_1, F_2 expressions in the fashion indicated by Fig. 8. Further simplifications and projections can then be performed by REDUCE 2 (written by A. C. Hearn). For details, see P. Cvitanović, Cornell University Report No. CLNS-234, 1973 (unpublished), and *Proceedings of the Third Colloquium on Advanced Computing Methods in Theoretical Physics, Marseille, 1973*, edited by A. Visconti (Univ. of Marseille, Marseille, 1973). We have also written a program based on a somewhat different approach, where all the above steps are performed by SCHOONSCHIP (written by M. Veltman).

[25]W. Mayeda, *Graph Theory* (Wiley, New York, 1972), Sec. 7.6.

PHYSICAL REVIEW D VOLUME 10, NUMBER 12 15 DECEMBER 1974

New approach to the separation of ultraviolet and infrared divergences of Feynman-parametric integrals*

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(Received 28 March 1974)

A scheme for systematically separating ultraviolet divergences of Feynman amplitudes in parametric space is developed. It is summarized by an explicit formula which enables us to incorporate readily the ultraviolet-finite remainders thus constructed into the usual renormalization scheme. It is shown further that infrared divergences can be treated in a very similar way. Our method is particularly suitable for numerical integration.

I. INTRODUCTION

In order to evaluate Feynman integrals of higher orders numerically, it is necessary to locate and subtract the ultraviolet (UV) and infrared (IR) divergences beforehand. Since the removal of UV divergences is the essential aspect of the renormalization procedure, various prescriptions have been proposed in the literature for the extraction of UV-finite parts, although they vary in mathematical rigor and practicality depending on the purpose for which they have been formulated. On the other hand, the treatment of IR divergences has been relatively underdeveloped, particularly in the Feynman-parametric form. Thus we have found it necessary to develop some workable scheme.[1a] The purpose of this article is to present a general and systematic scheme for separating both UV and IR divergences of Feynman integrals, following the line first suggested in Ref. 1b. This method has been applied to the evaluation of sixth-order contributions to the electron magnetic moment.[2,3]

Our method is based on the parametric representation of Feynman integrals summarized in the preceding article,[4] hereafter referred to as I. It is particularly suited for numerical calculation because of the following properties:

(i) After the removal of divergences the integral is almost as simple as the original divergent integral.

(ii) The singularity is subtracted at each point of the domain of integration (rather than having cancellation of contributions from different parts of the domain).

(iii) Subtraction terms introduce no new singularities. (Note that the standard renormalization introduces infrared divergences.)

(iv) Subtraction terms are factorizable into lower-order expressions. Thus they are easier to evaluate analytically or numerically than the original integral.

(v) Our construction of UV and IR subtraction terms is also useful for crosschecking of trace calculation.

In Sec. II we review the UV power-counting rule for arbitrary Feynman integrals and propose a method for removing all leading UV singularities of parametric integrands. In Sec. III we apply it to QED and derive an expression for Dyson-Salam-renormalized amplitudes in terms of finite integrals. A power-counting rule for the degree of superficial IR divergence is developed in Sec. IV for arbitrary QED amplitudes by examining the properties of their denominators. In Sec. V it is extended to the whole integrand, taking account of the structure of numerator functions. A method for removing all IR divergences of QED

amplitudes is sketched. Application of these methods is given in the subsequent article.[3]

II. UV SINGULARITIES

A Feynman-parametric integral associated with a one-particle irreducible diagram G can be written in the dimensionally regularized form as

$$M_G = \frac{i^n(-i)^N}{(4\pi i)^{n\omega}} \int dz_G J_G \,, \tag{2.1}$$

$$J_G = \frac{1}{U^\omega}\left[F_0 + F_1/(iU) + \cdots + F_m/(iU)^m\right] e^{-iV(p)} \,, \tag{2.2}$$

where the notation follows that of I.[5] We drop the suffix G referring to the diagram G whenever no ambiguity arises. For example, in the above, $U = U_G$, $V(p) = V_G(p)$, etc. p in $V(p)$ stands for the set of all external momenta of the diagram G.

We define the Feynman *integrand* as the value of J_G for $\omega = 2$. To determine its singularities, note that all parametric functions that comprise J_G are homogeneous forms of Feynman parameters z_i. U, B_{ij}, Q_i^μ, and V are of degree n, $n-1$, 0, and 1, respectively, n being the number of independent integration loops in G.

The integral M_G may diverge when U vanishes (for some $z_i \to 0$) or $V(p)$ vanishes (and some $z_i \to \infty$). The second possibility occurs only if masses of some internal lines vanish, and is associated with the IR divergence, or, more generally, the mass singularity.[6] In this section we shall concentrate on the first case.

If all z_i vanish, U has a zero of order n. U also vanishes if all Feynman parameters of a loop or a set of loops are set equal to zero. Since $z_i \to 0$ corresponds to large momentum flowing through the line i, such divergences will be the parametric space versions of familiar UV divergences in momentum space. In general, let S be a connected subdiagram of G (including G itself) consisting of several closed loops. Then J_G will be singular in the domain boundary defined by the limit $\epsilon \to 0$, where

$$z_i = \begin{cases} O(\epsilon) \,, & i \in S \\ O(1) \,, & i \notin S \,. \end{cases} \tag{2.3}$$

Let us examine the behavior of J_G in this singular region.

A. Overall UV singularities

Suppose $f(z_i)$ is a function of Feynman parameters associated with the diagram G. We define its *UV limit* $[f(z_i)]^S_{\rm UV}$ as the leading term in the expansion in small ϵ defined in (2.3). Note that the word "UV limit" is used to indicate possible relevance to a UV divergence of the integral, but does not necessarily mean that these functions actually lead to divergent integrals.

The case $S = G$ will be referred to as the *overall* UV limit $[f(z_i)]^G_{\rm UV} \equiv [f(z_i)]_{\rm UV}$.

Let us now examine the overall UV limit of the integrand J_G. The exponent $V(p)$ does not affect the nature of this singularity since

$$[e^{-iV(p)}]_{\rm UV} = 1 \,. \tag{2.4}$$

Since B_{ij} is of degree $n-1$ in z_i, the most singular term of J_G is the one with most contractions (see I for definition). Thus we have

$$[J_G]_{\rm UV} = \frac{F_m}{U^2(iU)^m} \quad [= O(\epsilon^{-2n-m})] \,. \tag{2.5}$$

In the integral (2.1) this singularity is suppressed by the phase space dz_G which vanishes in the overall UV limit as

$$dz_G = O(\epsilon^N) \,. \tag{2.6}$$

It follows from (2.5) and (2.6) that the integral (2.1) is convergent in the domain (2.3) if

$$N_G - 2n_G - m_G > 0 \,, \tag{2.7}$$

where N_G, n_G, and m_G are the number of internal lines of G, the number of integration loops, and the maximum number of contractions, respectively. The Dyson-Nakanishi power-counting rule[7,8] follows from (2.7) if one rewrites it in terms of the number of external lines.

An obvious way to construct an integrand less singular than J_G in the limit (2.3) is to replace it by $J_G - [J_G]_{\rm UV}$. However, $[J_G]_{\rm UV}$ might have an IR divergence of its own since it is not damped exponentially for large z_i because of (2.4). It corresponds to renormalizing Feynman amplitudes at the point where all external momenta as well as masses of all internal lines vanish. Instead let us perform renormalization by defining the subtraction integrand $K_G J_G$ by

$$K_G J_G = [J_G]_{\rm UV} e^{-iV} \,, \tag{2.8}$$

$$V = V(p)|_{p \text{ on mass shell}} \,, \tag{2.9}$$

where K_G stands for the operation of constructing an "on-the-mass-shell" subtraction term.[9] For now we shall take (2.8) as the definition of the K operation. By (2.4) we have

$$[K_G J_G]_{\rm UV} = [J_G]_{\rm UV} \,. \tag{2.10}$$

Thus $J_G - K_G J_G$ is also less singular than J_G itself in the overall UV limit.

Unlike the usual Dyson-Salam construction of on-the-mass-shell subtraction terms, this one introduces no new IR divergence since the definition (2.8) keeps only the most-contracted term F_m in (2.2). In the case of QED, this is an im-

portant advantage over the standard method which, as is well known, introduces spurious IR divergences associated with the renormalization procedure. Moreover, since our subtraction is also defined on the mass shell, it will not be difficult to express the usual renormalized amplitude in terms of UV-finite integrals constructed by K operations. (See Sec. III.)

The numerator F_k depends in general on the external momenta p, so that one might wonder whether F_m in (2.8) is unambiguous. Since m is the maximum number of contractions, F_m has either all factors D contracted or one D remains uncontracted. In the first case there is no ambiguity since F_m does not contain external momentum. The only case of importance of the second possibility is the electron self-energy diagram. In this case we evaluate F_m in (2.8) for the same external momentum p as for the original integrand J_G.

B. Subdiagram UV singularities

From the definition (I.24) [Eq. (24) of paper I]

$$U=\det(U_{st}),\quad U_{st}=\sum_{i\in G}\eta_{is}\eta_{it}z_i\ ,$$

where the indices s and t run over a set of independent integration loops, it is clear that U vanishes if $z_i\to 0$ for all $i\in S$, S being a connected subdiagram of G consisting of several loops. More precisely, in the UV limit (2.3), U factors as[10]

$$[U]^S_{\rm UV}=U_S U_{G/S}\quad [=O(\epsilon^{n_S})]\ , \tag{2.11}$$

where n_S is the number of independent loops within S. By G/S we mean the reduced diagram obtained by shrinking the subdiagram S of G to a point. Whenever a parametric function has a subscript or superscript S, G/S, ..., it is to be understood as defined in terms of parameters z_i belonging to the diagrams S, G/S, ... alone.[11]

Let us now derive the UV limit of B_{ij} starting from the definition (I.51):

$$B_{ij}=\sum_c \eta_{ic}\eta_{jc}U_c\ ,$$

where U_c is a U function for the diagram obtained from G by shrinking a closed loop c to a point.

Case $i,j\in S$. For $c\subset S$, we have

$$[U_c]^S_{\rm UV}=U^{G/S}U^S_c\quad [=O(\epsilon^{n_S-1})]$$

since the whole loop c within S is shrunk to a point. If c belongs to both S and G/S, we have $U_c=O(\epsilon^{n_S})$, and its contribution can be neglected. We thus find

$$[B_{ij}]^S_{\rm UV}=U^{G/S}B^S_{ij},\quad i,j\in S\ . \tag{2.12}$$

Case $i,j\in G/S$. Applying (2.11) to each U_c and noting that $U^S=\sum_{P(AB)}U^S_P$, where $P(AB)$ is any path entering and leaving S at the points A and B, we find

$$[B_{ij}]^S_{\rm UV}=B^{G/S}_{ij}U^S,\quad i,j\in G/S\ . \tag{2.13}$$

Case $m\in S, j\in G/S$. This will be of interest only when S is a self-energy subdiagram. In this case, according to I, Sec. IV F, we find

$$[B_{jm}]^S_{\rm UV}=B^{G/S}_{ji}A^S_m U^S,\quad j\in G/S,\ m\in S \tag{2.14}$$

where i is the line of G/S in which S is inserted.

The UV limit of the scalar current $A_i\equiv A_i^{(AB)}$ follows from (I.74):

$$A_i=-\frac{1}{U}\sum_{j\in G}\eta_{jP}z_jB'_{ji}\ ,$$

where $P=P(AB)$ is any path from point A to point B. We can always choose a path avoiding the line i so that $B'_{ij}=B_{ij}$. If $i\in G/S$, we have $[B_{ij}]^S_{\rm UV}=O(\epsilon^{n_S})$ according to (2.13) and (2.14). Thus we can drop all terms with $j\in S$ in the above summation and obtain

$$[A_i]^S_{\rm UV}=A^{G/S}_i,\quad i\in G/S\ . \tag{2.15}$$

If S is a self-energy insertion, we have according to (I.93)

$$[A_m]^S_{\rm UV}=A^{G/S}_iA^S_m,\quad m\in S\ . \tag{2.16}$$

Since scalar currents remain rational homogeneous functions of degree 0 in the UV limit, all z_i, $i\in S$, in the defining formula (I.3) of V drop out and we obtain

$$[V(p)]^S_{\rm UV}=\sum_{j\in G/S}z_j(m_j{}^2-q_j\cdot[Q'_j]^S_{\rm UV})\ .$$

Decomposing Q'^{μ}_j into scalar currents and using (2.15) we find

$$[V(p)]^S_{\rm UV}=V_{G/S}(p)\ . \tag{2.17}$$

We are now ready to consider the integrand $[J_G]^S_{\rm UV}$ as a whole. From (2.12) it is seen that only contractions within the subdiagram S maintain the most singular feature in the UV limit (2.3). The leading singularity is thus given by all terms in F_k that have the maximum number of contractions within S. If S is a vertex part, all D's [see (I.17) for a definition of D] in S can be contracted among themselves and the entire integrand factorizes as

$$[J_G]^S_{\rm UV}=J_{G/S}[J_S]^S_{\rm UV}\ , \tag{2.18}$$

where we have used (2.11) through (2.17) and (2.5). Thus the power-counting rule (2.7) applies not only to the diagram G as a whole but also to any subdiagram S. Again (2.18) itself is not suitable as a subtraction term since it might introduce new IR divergences. Instead, by generalizing (2.8) we choose

$$K_S J_G = [J_G]^S_{\rm UV}\, e^{-iV_S} \tag{2.19}$$

as the subtraction term.

If the number of D's in S is odd, the factorization of the subtraction integrand is slightly more complicated than (2.18). This case will be treated later in connection with the electron self-energy subdiagram.

C. Formal definition of K_S operation

Let us now restrict our attention to Feynman integrals having at most logarithmic UV divergences. Then the K subtraction defined above is sufficient to render such integrals UV-finite.[12] This operation may be cast in a form similar to the renormalization prescription of Refs. 13–15 as follows: Let us define a generalized integrand $J_G(\alpha_1, \alpha_2, \dots)$ by first scaling

$$z_j \to \alpha_S z_j, \quad j \in S \tag{2.20}$$

within J_G and then multiplying the scaled J_G with

$$\alpha_S{}^{N_S} e^{-i(1-\alpha_S)V_S} \tag{2.21}$$

for each subdiagram S, where N_S is the number of internal lines of S. The factor $\alpha_S{}^{N_S}$ takes care of the scaling of the phase space (2.6), and V_S is the function $V_S(p)$ for the diagram S evaluated with all external momenta p on the mass-shell.

The generalized integrand has the following properties:

$$J_G(1, 1, \dots) = J_G, \tag{2.22}$$

$$J_G(1, \alpha_S = 0, 1, \dots) = \begin{cases} 0 & \text{if } N_S - 2n_S - m_S > 0\,, \\ [J_G]^S_{\rm UV}\, e^{-iV_S} & \text{if } N_S - 2n_S - m_S = 0\,, \end{cases} \tag{2.23}$$

$$J_G(\alpha_G = 0, 1, \dots) = \begin{cases} 0 & \text{if } N_G - 2n_G - m_G > 0\,, \\ [J_G]_{\rm UV} e^{-iV} & \text{if } N_G - 2n_G - m_G = 0\,. \end{cases} \tag{2.24}$$

Thus the K-subtraction terms of (2.8) and (2.19) are obtained by setting appropriate α_S equal to zero. According to (2.23) and (2.24) the subtraction terms will be nonvanishing only for $S \in \mathcal{S}$, $\mathcal{S}$ being the set of all superficially divergent subdiagrams S of G (possibly including G itself) such that $N_S - 2n_S - m_S = 0$. In this formulation the operation $K_{S_i} K_{S_j} \cdots K_{S_m}$ is defined as setting $\alpha_{S_i} = 0$, $\alpha_{S_j} = 0, \dots, \alpha_{S_m} = 0$, and all remaining $\alpha_S = 1$. By construction an integrand of the form

$$\prod_{S_i \in \mathcal{S}} (1 - K_{S_i}) J_G(\alpha_1, \alpha_2, \dots)$$

leads to a UV-finite integral. We shall define this integral as the *K-finite part* of the Feynman amplitude M_G and denote it symbolically as

$$\Delta' M_G = \prod_{S \in \mathcal{S}} (1 - K_S) M_G\,. \tag{2.25}$$

In order to ensure that $\Delta' M_G$ is well defined, we must show that it is independent of the order of K_S factors. This is easy to confirm if $S_i \cap S_j = 0$ (no common point) or if $S_i \subset S_j$ or $S_j \subset S_i$. The only case where K_{S_i} and K_{S_j} do not commute is when S_i and S_j are overlapping diagrams. This happens, for example, when S_i and S_j are overlapping vertex parts within a self-energy diagram S_k such that $S_i \cap S_j \neq 0$ and $S_i \cup S_j = S_k$. Fortunately our definition (2.25) enables us to avoid this complication automatically.[14,15] To see this, note that each z_n that acquires a factor α_{S_k} by (2.20) will also have α_{S_i} or α_{S_j} or both factors. Since setting $\alpha_{S_i} = 0$ and $\alpha_{S_j} = 0$ simultaneously is the same as setting $\alpha_{S_k} = 0$, we find

$$(1 - K_{S_k}) K_{S_i} K_{S_j} = 0\,, \tag{2.26}$$

which may also be written as

$$(1 - K_{S_k})(1 - K_{S_i})(1 - K_{S_j}) = (1 - K_{S_k})(1 - K_{S_i} - K_{S_j})\,. \tag{2.27}$$

Thus the overlapping divergence $K_{S_i} K_{S_j} M_G$ never contributes to the formula (2.25), and the K-finite part $\Delta' M_G$ is uniquely defined independent of the order of K_S operations in (2.25).

III. RENORMALIZED QED AMPLITUDES

In this section we shall study how to express the renormalized QED amplitudes in terms of UV-finite integrals. We postpone the discussion of IR divergences to Secs. IV and V. For now we shall assume a finite photon mass λ.

As is well known, the only superficially divergent diagrams of QED are electron self-energy, photon self-energy, vertex and photon-photon scattering diagrams. Due to gauge invariance, photon-photon scattering integrals are actually convergent. (A method for rendering individual photon-photon scattering diagrams separately convergent is given in Ref. 16. Hence we shall not discuss it here.) We define the remaining three UV-divergent diagrams, i.e., proper electron self-energy part, proper photon self-energy part, and proper vertex part, using the conventions of Ref. 17. In the dimensionally regularized form they are

$$\Sigma_G = i \left[\frac{\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G}{U^\omega}\, \mathrm{F} e^{-iV(p)}\,, \tag{3.1}$$

$$\Pi_G^{\mu\nu} = i\left[\frac{\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G}{U^\omega}\ \mathsf{F}^{\mu\nu} e^{-iV(p)}\ , \tag{3.2}$$

$$\Gamma_G^{\nu} = \left[\frac{\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G}{U^\omega}\ \mathsf{F}^{\nu} e^{-iV(p,q)}\ . \tag{3.3}$$

Here n is the number of integration loops. Operators F, $\mathsf{F}^{\mu\nu}$, F^{ν} consist of $(\not{p}_i + m_i)$ factors from fermion lines, γ^α factors from vertices, and a factor (-1) for each closed fermion loop. When there is an ambiguity as to which amplitude they refer to, we shall write them as $\mathsf{F} = [\Sigma_G]$, $\mathsf{F}^{\mu\nu} = \mathsf{F}[\Pi_G^{\mu\nu}]$, etc. The maximum number of contractions is $m_G = n_G - 1$ for electron self-energy, and $m_G = n_G$ for photon self-energy and vertex. According to the power-counting rule (2.7) electron self-energy and vertex diagrams are (superficially) logarithmically divergent, whereas the photon self-energy is linearly divergent (remember that $z_i \sim 1/{p_i}^2$ "dimensionally" so that this corresponds to the familiar quadratic divergence in the momentum space).

A. Photon self-energy diagram

In practical calculations it is more economical to use the Källén-Lehmann spectral representation for renormalized photon propagators than to utilize the K_S subtraction method for divergences associated with photon self-energy diagrams. Nevertheless, for completeness, we shall include the K renormalization of photon propagators in our discussion. We know that the sum of a gauge-invariant set of photon self-energy diagrams has the form

$$\Pi^{\mu\nu} = (p^\mu p^\nu - g^{\mu\nu}p^2)\Pi(p^2)\ . \tag{3.4}$$

Thus the factor of $p^\mu p^\nu$ in (3.2) may be regarded as a contribution of the diagram G to $\Pi(p^2)$ and denoted as $\Pi_G(p^2)$. Thus, if we consistently replace everywhere as

$$\Pi_G^{\mu\nu}(p) \to -g^{\mu\nu}p^2\Pi_G(p^2) \tag{3.5}$$

(dropping $p^\mu p^\nu$), the sum over contributions from a gauge-invariant set of Feynman diagrams will remain unchanged by gauge invariance. Now, $\Pi_G(p^2)$ is only logarithmically UV-divergent, and K subtractions are sufficient to render finite all photon-self-energy contributions redefined by (3.5). In the Appendix we discuss the calculation of $\Pi_G(p^2)$ in the parametric form.

B. Renormalization constants

On-the-mass-shell renormalization constants are defined by

$$\delta m_G = \Sigma_G|_{\not{p}=m}\ , \tag{3.6}$$

$$B_G = \frac{p^\mu}{m}\frac{\partial \Sigma_G}{\partial p^\mu}\bigg|_{\not{p}=m} = (1 - Z_2^{-1})_G\ , \tag{3.7}$$

$$C_G = -{e_0}^2\Pi_G|_{p^2=0} = (1 - Z_3^{-1})_G\ , \tag{3.8}$$

$$L_G = \frac{p_\mu}{m}\Gamma_G^\mu|_{q=0,\not{p}=m} = (-1 + Z_1^{-1})_G \tag{3.9}$$

for the electron self-mass, wave-function renormalization, charge renormalization, and vertex renormalization, respectively. δm_G, C_G, and L_G are obtained by evaluating Σ_G, Π_G, and Γ_G^μ on the mass shell. B_G can be calculated from

$$\frac{\partial \Sigma_G}{\partial p_\mu} = i\left[\frac{\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G}{U^\omega}\ (\mathsf{E}^\mu + 2ip^\mu G\mathsf{F})e^{-iV(p)} \tag{3.10}$$

and

$$\mathsf{E}^\mu \equiv \frac{\partial \mathsf{F}}{\partial p_\mu} = \sum_{i,\ \text{electron only}} A_i\, \mathsf{F}_i^\mu\ , \tag{3.11}$$

where F_i^μ is obtained from F by the replacement $(\not{p}_i + m_i) \to \gamma^\mu$, and G is given by (I.36). Since G is linear in z_i, the second term in (3.10) is not overall UV-divergent. The maximum number of contractions in F_i^μ is still $m_G = n_G - 1$, so that B_G has a logarithmic overall UV divergence arising from the E_{m_G} term.

C. Overall UV divergences of renormalization constants

Let us now apply K_S operations to isolate the overall UV divergences of the renormalization constants. According to (2.8) the overall UV-divergent parts of δm_G, B_G, C_G, and L_G are

$$\delta\hat{m}_G \equiv K_G \delta m_G = -\left[\frac{-i\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G F_{n-1}[\delta m_G]}{U^{\omega+n-1}}\, e^{-iV}\ , \tag{3.12}$$

$$\hat{B}_G \equiv K_G B_G = -\left[\frac{-i\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G F_{n-1}[B_G]}{U^{\omega+n-1}}\, e^{-iV}\ , \tag{3.13}$$

$$\hat{C}_G \equiv K_G C_G = {e_0}^2\left[\frac{-i\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G F_{n-1}[\Pi_G]}{U^{\omega+n-1}}\, e^{-iV}\ , \tag{3.14}$$

$$\hat{L}_G \equiv K_G L_G = \left[\frac{-i\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G F_n[L_G]}{U^{\omega+n}}\, e^{-iV}\ . \tag{3.15}$$

$F_{n-1}[B_G]$ in (3.13) is defined by

$$E_{n-1}^\mu[\Sigma_G] \equiv \gamma^\mu F_{n-1}[B_G]\ , \tag{3.16}$$

noting that all fermion lines are contracted so that E_{n-1}^μ is proportional to γ^μ [see (3.11) for a defini-

tion of E]. $F_{n-1}[\Pi_G]$ in (3.14) is defined by

$$\left.\frac{\partial^2 F_{n-1}[\Pi_G^{\mu\nu}]}{\partial p_\alpha \partial p_\beta}\right|_{\alpha=\mu,\,\beta=\nu,\,\alpha\neq\beta} \equiv F_{n-1}[\Pi_G] \,. \tag{3.17}$$

Finally $F_n[L_G]$ in (3.15) is given by

$$F_n[\Gamma_G^\mu] \equiv \gamma^\mu F_n[L_G] \tag{3.18}$$

since all fermion lines are contracted. We shall not compute any of the UV-divergent parts explicitly. However, they will be useful in showing the equivalence of the Dyson-Salam approach and ours.

D. Subdiagram UV divergences

We have defined the subtraction integrand $K_S J_G$ by (2.19). We shall now examine how the integral over $K_S J_G$ is to be interpreted in QED.

For definiteness let G be a vertex diagram containing a vertex subdiagram S. Then the number of $(\not{p}_i + m_i)$ factors, $i \in S$, is even $(2n_S)$. According to (2.18) and (3.18) all dependence of the numerator on the parameters z_i of the subdiagram S is contained in the factor $F_{n_S}[L_S]$, and the subtraction integral takes the form

$$K_S\Gamma_G^\mu = \left[\frac{\alpha}{(4\pi i)^{\omega-1}}\right]^{n_{G/S}} \left[\frac{-i\alpha}{(4\pi i)^{\omega-1}}\right]^{n_S} \int \frac{dz_{G/S}\,dz_S}{U_{G/S}^{\ \omega} U_S^{\ \omega+n_S}} \, F_{n_S}[L_S]\, \mathrm{F}[\Gamma_{G/S}^\mu] e^{-i(V_{G/S}(p)+V_S)} \,, \tag{3.19}$$

which is clearly factorizable as

$$K_S\Gamma_G^\mu = \hat{L}_S \Gamma_{G/S}^\mu \,. \tag{3.20}$$

Actually G does not have to be a vertex diagram. Any QED diagram which contains a vertex subdiagram S will factor in the same fashion.

If S is an electron self-energy subdiagram, the numerator does not factor since not all $\not{D}_m$, $m \in S$, are contracted. For the uncontracted $\not{D}_m$, because of the factorizations (2.14) and (2.16), we can write symbolically

$$[\not{D}_m]_{\mathrm{UV}}^S = A_m^{\ S} \not{D}_{i''}^{G/S}, \quad m \in S \tag{3.21}$$

where i'' is a fictitious line for which we set $p_{i''} = p_{i'} = p_i$ after $\not{D}^{G/S}$ operations have been carried out.

For notational convenience let us set $T^* = G/S$ and denote as T the diagram $G/S, i'$ obtained from T^* by shrinking the line i' to a point. All parametric functions for T are obtained from those for T^* by the replacement $z_{ii'} \to z_i$. The Feynman diagram for T^* has one more propagator than T so that

$$N_{T^*} = N_T + 1 \,, \tag{3.22}$$

$$\int dz_{T^*} = \int dz_T dz_{i'} \,, \tag{3.23}$$

and the numerator of T^* contains an additional factor $i(\not{p}_{i'} + m_{i'})$.

Let us first suppose that S is a second-order electron self-energy diagram. Then the part of the numerator referring to S may be written as

$$\gamma^\mu (A_m^{\ S} \not{D}_{i''}^{T^*} + 1)\gamma_\mu = A_m^S \gamma^\mu \not{p} \gamma_\mu (\not{D}_{i''}^{T^*} - 1) + \gamma^\mu (A_m^S \not{p} + 1)\gamma_\mu$$

$$= F_{n_S-1}[B_S](\not{D}_{i''}^{T^*} - 1) + F_{n_S-1}[\delta m_S] \,, \tag{3.24}$$

where $\not{p}$ is used symbolically; set $\not{p} = 1$ after all contractions over $\gamma^\mu \cdots \gamma_\mu$ are performed. $F_{n-1}[B_S]$ and $F_{n-1}[\delta m_S]$ are defined by (3.14) and (3.12), with $n_S = 1$ in this case. Equation (3.24) is just the leading UV singularity of the expansion

$$\Sigma_S(p_i) = \delta m_S + B_S(\not{p}_i - m_i) + \cdots$$

of the electron self-energy insertion S, where $\not{p}_i$ corresponds to $\not{D}_{i''}^{T^*}$ in the parametric space. Such a decomposition works for any n_S giving

$$K_S\Gamma_G^\mu = \left[\frac{\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_S\,dz_T\,dz_{i'}}{U_S^{\ \omega+n_S-1} U_{T^*}^{\ \omega}} (-1)^{n_S-1} \{F_{n_S-1}[B_S]\, \mathrm{F}[\Gamma_T^\mu](D_{i'}^{T^*} \cdot D_{i''}^{T^*} - 1) + F_{n_S-1}[\delta m_S]\, \mathrm{F}[\Gamma_{T^*}^\mu]\} e^{-i(V_S + V_{T^*}(p))} \tag{3.25}$$

The first term is obtained by noting that we can set

$$(\not{D}_{i'}^{T^*} + 1)(\not{D}_{i''}^{T^*} - 1) \Rightarrow (D_{i'}^{T^*} \cdot D_{i''}^{T^*} - 1) \,, \tag{3.26}$$

where the right-hand side corresponds to the numerator factor $p_i^{\,2} - m_i^{\,2}$ in the unparametrized Feynman integral, which cancels one of the propagators $(p_i^{\,2} - m_i^{\,2})^{-1}$. In parametric space the same effect is achieved by the use of Nakanishi's identity (I.83)

$$-i\int \frac{dz_{T^*}}{U_{T^*}^{\omega}}\,(D_{i'}^{T^*}\cdot D_{i''}^{T^*}-m_i{}^2)e^{-iV_{T^*}(p)}=\int \frac{dz_T}{U_T^{\omega}}e^{-iV_T(p)}\,. \tag{3.27}$$

Applying this to the first term in (3.25) one finds

$$K_S\Gamma_G^{\mu}=\hat{B}_S\Gamma_T^{\mu}+\delta\hat{m}_S\Gamma_{T^*}^{\mu}\,. \tag{3.28}$$

Thus our single subtraction term contains both mass and wave-function renormalization terms. This decomposition is valid for any QED diagram, not just Γ_G^{μ}. However, (3.27) must be generalized if M_G has parametric functions other than U and $V(p)$ in its definition (before $\not{p}_i$ operation is performed). In such a case we must use (I.86):

$$-i(D_{i'}^{T^*}\cdot D_{i''}^{T^*}-m_{i'}{}^2)\int \frac{dz_{T^*}H_{T^*}(z_i)}{U_{T^*}^{\omega}}\,e^{-iV_{T^*}(p)}$$

$$=\int \frac{dz_T}{U_T^{\omega}}\left[\left(1+z_i\frac{\partial}{\partial z_i}\right)H_T(z_i)\right]e^{-iV_T(p)}\,, \tag{3.29}$$

where $H_T(z_j)$ is any homogeneous function of z_j, $j\in T$. As an example, the reader may prove (3.28) for the Feynman integral (3.10) for $\partial\Sigma_G/\partial p^{\mu}$, where the parametric integral contains explicit factors of A_i and G. For this purpose it is useful to note

$$\partial G/\partial z_i=A_i{}^2,\quad \partial A_j/\partial z_i=-A_j B_{ij}/U, \tag{3.30}$$

which follows from (I.36) and (I.101). We shall not dwell on such generalizations, however, since we can just as well carry out the $K_S\Sigma_G$ operation before taking the $\partial/\partial p^{\mu}$ derivative in (3.10) and in that case there is no doubt that the rule (3.27) applies.

If S is a photon self-energy diagram, by our rule of imposing gauge invariance in the form (3.5), the numerator already contains an explicit factor $-D_i\cdot D_{i'}$ (i, i' being photon lines, $m_i{}^2=0$) so that we have

$$K_S\Gamma_G^{\mu}=\hat{C}_S\Gamma_{G/S,i'}^{\mu}\,. \tag{3.31}$$

In summary, if S is a divergent subdiagram of the diagram G, and M_G is a Feynman amplitude defined on G, then

$$\begin{aligned}
&K_SM_G=\hat{L}_SM_{G/S} && \text{if } S \text{ is a vertex},\\
&K_SM_G=\delta\hat{m}_SM_{G/S}+\hat{B}_SM_{G/S,i'} && \text{if } S \text{ is an electron self-energy diagram},\\
&K_SM_G=\hat{C}_SM_{G/S,i'} && \text{if } S \text{ is a photon self-energy diagram},
\end{aligned} \tag{3.32}$$

where the self-energy subdiagram lies between the lines i and i' of the reduced diagram G/S.

E. On-the-mass-shell renormalization

We shall first rephrase the usual Dyson definition of the on-the-mass-shell renormalized amplitude[7] as follows: Let M_G be a Feynman amplitude in QED. Then its finite renormalized part is given *formally* by

$$\tilde{M}_G=\prod_{S\in\$}(1-\mathcal{C}_S)M_G\,, \tag{3.33}$$

where $\mathcal{C}_S$ is an operator describing the extraction of renormalization constant associated with the subdiagram S according to the following rules:

$$\begin{aligned}
&\mathcal{C}_SM_G=L_SM_{G/S} && \text{if } S \text{ is a vertex},\\
&\mathcal{C}_SM_G=\delta m_SM_{G/S}+B_SM_{G/S,i'} && \text{if } S \text{ is an electron self-energy diagram},\\
&\mathcal{C}_SM_G=C_SM_{G/S,i'} && \text{if } S \text{ is a photon self-energy diagram}.
\end{aligned} \tag{3.34}$$

The result of usual analysis of overlapping divergences can be incorporated in (3.33) by imposing the condition[14,15]

$$(1-\mathcal{C}_i)\mathcal{C}_j\mathcal{C}_k=0, \tag{3.35}$$

where j and k are overlapping vertex subdiagrams of a self-energy part i. Because of (3.35) the operators $\mathcal{C}_S$ in (3.33) become effectively commutative, enabling us to define $\tilde{M}_G$ unambiguously.

Let us now examine the relation between the renormalized amplitude $\tilde{M}_G$ and the K-finite $\Delta' M_G$. For this purpose we note that

$$K_S\mathcal{C}_S=K_S\,, \tag{3.36}$$

which follows from the definitions (3.32) and (3.34). This may also be written as

$$(1-\mathcal{C}_S)=(1-K_S)(1-\mathcal{C}_S)\ . \tag{3.37}$$

Substituting this in (3.33) we obtain

$$\tilde{M}_G=\prod_{S\in\mathbb{S}}(1-K_S)(1-\mathcal{C}_S)M_G\ . \tag{3.38}$$

Now all operators in (3.38) can be freely commuted since the contribution of overlapping subdiagrams is excluded from the defining formula. We can therefore rewrite (3.38) as

$$\tilde{M}_G=\prod_{S_i\in\mathbb{S}}(1-K_{S_i})\prod_{S_j\in\mathbb{S}}(1-\mathcal{C}_{S_j})M_G\ . \tag{3.39}$$

Comparing this with the definition of $\Delta' M_G$ in (2.25), we arrive at our main result

$$\tilde{M}_G=\prod_{S\in\mathbb{S}}(1-\Delta'\mathcal{C}_S)\Delta' M_G\ , \tag{3.40}$$

where $\Delta'\mathcal{C}_S$ is an operator extracting the K-finite part of the renormalization constants associated with the subdiagram S.

The meaning of the operator $\Delta'\mathcal{C}_S$ may be seen most clearly by working out some examples. Let us start by considering the simple case where S is the only UV-divergent vertex subdiagram of G. Then we have

$$\begin{aligned}
\tilde{M}_G &= (1-\mathcal{C}_S)M_G && \text{by (3.33)}\\
&= (1-K_S)(1-\mathcal{C}_S)M_G && \text{by (3.37)}\\
&= (1-K_S)M_G-(1-K_S)L_S M_{G/S} && \text{by (3.34)}\\
&= \Delta' M_G-\Delta' L_S M_{G/S} && \text{by (2.25)}\\
&= \Delta' M_G-\Delta' L_S\Delta' M_{G/S} && (\text{since } M_{G/S}=\Delta' M_{G/S})\\
&= (1-\Delta'\mathcal{C}_S)\Delta' M_G && (\text{definition of } \Delta'\mathcal{C}_S)\ ,
\end{aligned} \tag{3.41}$$

in agreement with (3.40). Next consider the case with two divergent vertex subdiagrams S and T such that $S\supset T$. In this case we find

$$\begin{aligned}
\tilde{M}_G &= (1-\mathcal{C}_S)(1-\mathcal{C}_T)M_G\\
&= (1-\mathcal{C}_T)(1-\mathcal{C}_S)M_G\\
&= \Delta' M_G-\Delta' L_S\Delta' M_{G/S}-\Delta' L_T\Delta' M_{G/T}\\
&\quad +\Delta' L_T\Delta' L_{S/T}\Delta' M_{G/S}\ ,
\end{aligned} \tag{3.42}$$

following steps similar to those of (3.41). The last term is nothing but the expanded form of

$$(1-\Delta'\mathcal{C}_S)(1-\Delta'\mathcal{C}_T)\Delta' M_G\ , \tag{3.43}$$

showing how the operators $\Delta'\mathcal{C}_S$ and $\Delta'\mathcal{C}_T$ work.

As is obvious from these examples, $\tilde{M}_G$ can be expressed uniquely as a sum of products of K-renormalized quantities $\Delta' M_G$, $\Delta' M_{G/S}$, ..., which are all UV-finite. This result therefore constitutes another proof that the amplitude $\tilde{M}_G$ renormalized in the usual sense is in fact finite. To complete the renormalization program of QED we must of course carry out the remaining steps of absorbing all divergent terms into a few multiplicative factors in the usual way.

IV. INFRARED SINGULARITIES

The integral (2.1) may also diverge if J_G is not sufficiently damped as $z_i\to\infty$. The damping is usually provided by the exponent $iV(p)$ which grows linearly with z_i. However, $V(p)$ may vanish in some parts of parametric space. The most familiar case for which this happens is the infrared (IR) divergence of QED. This problem has been analyzed in great detail so that there appears to be hardly any room for another article.[6,18] Nevertheless we believe that our treatment of IR divergences, which emphasizes the formal similarity with that of UV divergences in the preceding sections, presents a fresh and interesting point of view. Furthermore our method is readily applicable to numerical treatment of IR divergences.[3]

For a systematic analysis of IR singularity it is more convenient to go back to the original parametric integral (I.5) defined over a compact domain

$$M_G=\left(\frac{i}{16\pi^2}\right)^n(-1)^N\int dz_G\,\delta(1-z_G)J_G\ , \tag{4.1a}$$

with

$$J_G=\frac{(N-2n-1)!F_0}{U^2V^{N-2n}}+\frac{(N-2n-2)!F_1}{U^3V^{N-2n-1}}+\cdots+\frac{(N-2n-m-1)!F_m}{U^{2+m}V^{N-2n-m}}\ , \tag{4.1b}$$

where we assume that UV divergences have already been taken care of by the K method, so that no regularization is required. Unlike the formula (2.1) where IR divergences are associated with $z_i\to\infty$, IR divergences arise in (4.1) from the $V\to 0$ singularities of the integrand J_G, which can be treated in analogy with the $U\to 0$ singularities studied in Secs. II and III. This is why we prefer the representation (4.1) over (2.1).

We shall restrict ourselves to electron QED. Our method applies to any IR divergence. For simplicity, however, we shall develop it for a class of Feynman diagrams with two electron legs on the mass shell and l ($=0, 1, 2, \ldots$) photon legs shown in Fig. 1. The string of electron lines connecting the two electron legs will be referred to as the path $P^e=P^e(AB)$. The crosshatched area represents the photon cloud which may contain electron loops. To avoid unnecessary complication we shall assume that all external photons are attached to P^e. (This restriction can be easily lifted.) Then we may choose the fixed momenta q_i as

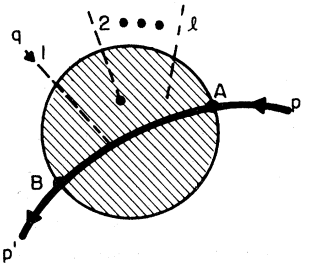

FIG. 1. Feynman diagram G with two electron legs, l photon legs, and a cloud of virtual photons and electrons.

$$q_i \neq 0 \text{ for } i \in P^e \ , \qquad q_i = 0 \text{ otherwise} \ , \tag{4.2}$$

and write $V(p)$ in the form

$$V(p) = \sum_{i \in P^e} z_i (1 - q_i \cdot Q_i') + \sum_{\substack{\text{electron} \\ \text{loops}}} z_i \ , \tag{4.3}$$

where we have set $m_e = 1$ and $\lambda = 0$. [The choice (4.2) is arbitrary, but it is sort of natural since IR divergences arise when virtual photons carry no momentum.]

A. Overall IR singularities

We shall first study the properties of $V(p)$ when all virtual photons are soft, i.e., when all external momenta are routed through the path P^e. For this purpose it is instructive to recall the electric-circuit analogy of z_i and A_i (or Q_i'): z_i corresponds to the resistance of the line i, and $A_i = A_i(AB)$ represents the fraction of the current flowing through the line i when it enters G at A and leaves at B. Thus, in order that the currents flow only through P^e, we must "short out" the path P^e (i.e., set $z_i = 0$ on P^e) while keeping the resistances of other lines finite. This observation leads us to the (tentative) definition of the overall IR limit $[f(z_i)]_{\rm IR}^G$ (where we shall drop the superscript G whenever no ambiguity arises) of an arbitrary parametric function $f(z_i)$ as the leading term in the δ expansion, where

$$z_i = \begin{cases} O(\delta), & \delta \ll 1 \text{ for } i \in P^e \ , \\ O(1) & \text{for } i \notin P^e \ . \end{cases} \tag{4.4}$$

If G has closed electron loops, we find $[V(p)]_{\rm IR} = O(1)$ because of the second term in (4.3). Thus the integral (4.1) is convergent (note also $[U]_{\rm IR} = O(1)$) and no overall IR divergence arises. From now on we shall therefore consider only diagrams with no electron loops. Then (4.3) reduces to the simple form

$$V(p) = \sum_{i \in P^e} z_i (1 - q_i \cdot Q_i') \ . \tag{4.5}$$

Since we now have $[V(p)]_{\rm IR} = O(\delta)$, the denominator $V(p)$ of the Feynman integral M_G vanishes in the limit (4.4). However, this does not lead to IR divergence of M_G immediately. To see this it is sufficient to examine the F_0 term of J_G, which is the most singular term in (4.1b). Noting that G for Fig. 1 has $n_e = 2n + l - 1$ electron lines on P^e and $n_p = n$ photon lines (hence $N = n_e + n_p = 3n + l - 1$) and that $U \neq 0$ in the limit (4.4), we obtain

$$\frac{dz_G}{U^2 V^{N-2n}} = O\left(\frac{\delta^{2n+l-1}}{\delta^{n+l-1}}\right) = O(\delta^n) \ . \tag{4.6}$$

This shows clearly that $V(p)$ must vanish more rapidly than δ in order that M_G develop an IR divergence.

To explore the IR property of Feynman integrals we must therefore study the structure of $V(p)$ more closely. It turns out that the short-circuit behavior of the current Q_i' mentioned above is crucial for this consideration. In terms of the scalar current A_i we may express it as

$$[A_i^{(AB)}]_{\rm IR} = \begin{cases} 1, & i \in P^e(AB) \ , \\ O(\delta), & i \notin P^e(AB) \ . \end{cases} \tag{4.7}$$

In order to prove this explicitly let us recall the formula (I.75):

$$A_i^{(AB)} = \frac{1}{U} \sum_{P(AB)} \eta_{iP} U_P \ , \tag{4.8}$$

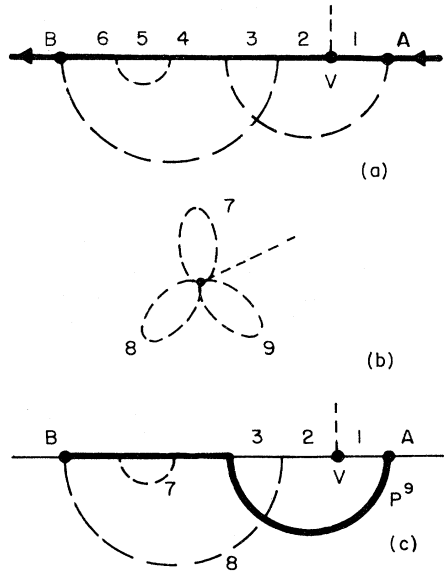

FIG. 2. (a) A diagram G with no internal photon loops. (b) The reduced diagram obtained by shrinking the electron path in G. (c) A path P^9 going through photon line 9.

where the summation is over all possible paths $P=P(AB)$. For concreteness let us consider the diagram in Fig. 2(a). Then, for U_{Pe}, obtained by shrinking the electron path P^e to a point [see Fig. 2(b)], one finds

$$[U_{Pe}]_{\rm IR}=U_{Pe}=\prod_{\substack{i\\ \text{photons}}} z_i \quad [=O(1)] \ . \tag{4.9}$$

Next consider the paths P^i which include one and only one photon line i, such as P^9 in Fig. 2(c). We find $[U_{P^9}]_{\rm IR}=z_{123}z_7z_8$. In general we have

$$[U_{P^i}]_{\rm IR}=z_i^e\prod_{\substack{j\neq i\\ \text{photons}}} z_j \quad [=O(\delta)] \ , \tag{4.10}$$

where z_i^e is the sum of Feynman parameters of all electron lines between the end points of the internal photon line i. All other paths contain two or more photon lines and are at most of order δ^2.

Thus

$$[U]_{\rm IR}=\Big[\sum_{P(AB)} U_P\Big]_{\rm IR}=U_{Pe} \ . \tag{4.11}$$

The formula (4.7) follows immediately from (4.11).

This result can be readily extended to the scalar current $A_i^{(CD)}$, C and D being any vertices on P^e:

$$[A_i^{(CD)}]_{\rm IR}=\Big[\frac{1}{U}\sum_{P(CD)}\eta_{iP}U_P\Big]_{\rm IR}=\begin{cases}1, & i\in P^e(CD)\\ O(\delta), & i\notin P^e(CD)\ .\end{cases} \tag{4.12}$$

We are now ready to evaluate $[V(p)]_{\rm IR}$. To avoid unnecessary complication let us consider the case $l=2$, where external photons of momenta k_1 and k_2 are attached to the vertices C and D on the electron path P^e. Then we have

$$Q_i'=A_i^{(AC)}p+A_i^{(CD)}(p+k_1)+A_i^{(DB)}p' \quad (p'=p+k_1+k_2) \tag{4.13}$$

and

$$\begin{aligned}\sum z_i q_i\cdot Q_i' &= \sum_{i\in P(AC)} z_i[A_i^{(AC)}p^2+A_i^{(CD)}p\cdot(p+k_1)+A_i^{(DB)}p\cdot p']\\ &+\sum_{i\in P(CD)} z_i[A_i^{(AC)}p\cdot(p+k_1)+A_i^{(CD)}(p+k_1)^2+A_i^{(DB)}(p+k_1)\cdot p']\\ &+\sum_{i\in P(DB)} z_i[A_i^{(AC)}p\cdot p'+A_i^{(CD)}(p+k_1)\cdot p'+A_i^{(DB)}p'^2] \ .\end{aligned} \tag{4.14}$$

Taking account of (4.12), we can write the IR limit of (4.14) as

$$\sum_{i\in P(AC)} z_iA_i^{(AC)}p^2+\sum_{i\in P(CD)} z_iA_i^{(CD)}(p+k_1)^2+\sum_{i\in P(DB)} z_iA_i^{(DB)}p'^2+O(\delta^2) \ , \tag{4.15}$$

which leads to

$$V(p)\to\sum_{i\in P(AC)} z_i(1-A_i^{(AC)})+\sum_{i\in P(CD)} z_i[1-A_i^{(CD)}(p+k_1)^2]+\sum_{i\in P(DB)} z_i(1-A_i^{(DB)})+O(\delta^2) \ . \tag{4.16}$$

Note that the first and third terms are of order δ^2 because of (4.12) and the mass-shell condition $p^2=p'^2=1$. On the other hand, the second term is of order δ unless $(p+k_1)^2=1$, which is not the case for general values of k_1. Obviously these features of (4.16) can be readily generalized to arbitrary l.

As was noted already for (4.6), the integral will not have an IR divergence unless $V(p)$ vanishes more rapidly than δ. Thus it would appear from (4.16) that the integral cannot be IR-divergent for $l\geq 2$. This is of course nonsense. The trouble is with (4.4) which is too simple and does not take advantage of the fact that some terms of (4.16) are already of order δ^2. One way to solve this problem,[19] which we adopt here, is to define the overall IR limit by a sharper condition

$$z_i=\begin{cases}O(\delta) & \text{if } i\in P_I^e=P^e(AC)+P^e(DB) \ ,\\ O(\delta^2) & \text{if } i\in P^e(CD) \text{ or } P^e/P_I^e \ ,\\ O(1) & \text{if } i\notin P^e=P^e(AB) \ ,\end{cases} \tag{4.17}$$

where C and D are external vertices nearest to A and B, respectively. Obviously $[V(p)]_{\rm IR}$ is now of order δ^2.

We are now ready to examine how the integral M_G behaves in the limit (4.17). Suppose first that soft virtual photons are all attached to P_I^e and at least one photon connects $P^e(AC)$ and $P^e(DB)$. Then we find

$$\frac{dz_G}{U^2V^{N-2n}}=O\left(\frac{\delta^{2n}\delta^{2(l-1)}}{\delta^{2(n+l-1)}}\right)=O(1) \quad \text{for } l\geq 1 \tag{4.18a}$$

(for nonexceptional external photon momenta), which reveals the presence of IR divergence in M_G. The leading term of this logarithmic singularity is of the form $(\ln\lambda)^a$, where the value of a depends on the details of the soft photon structure of G. If some soft virtual photons are attached to $P^e(CD)$, on the other hand, the numerator will acquire extra powers of δ and the integral will be IR-finite.

For $l=0$ we obtain

$$\frac{dz_G}{U^2V^{N-2n}} = O(\delta^2) \tag{4.18b}$$

instead of (4.18a). Thus the electron self-mass δm_G is not overall IR-divergent. However, the wave-function renormalization constant B_G has an overall IR divergence, as is required by the Ward identity.

The requirement in (4.17) that z_i, $i \in P^e(CD)$, tend to zero faster than other means that these lines can be ignored insofar as the IR structure of M_G is concerned. In other words, the IR divergence is determined completely by the mass-shell property of electron lines on P_I^e. This is a generalization of the result found previously.[6]

Thus far we have considered only diagrams with two electron legs. For diagrams with 4, 6, ... electron legs some photons must be hard *in general* so that they do not fit exactly in the overall IR limit (4.17). However, by selecting a minimum number of photons which are required to be hard by kinematics and treating the remaining photons as soft, we can repeat the above analysis without trouble. Of course, it must be kept in mind that for some exceptional momenta the integral M_G may have IR singularities stronger than the general logarithmic behavior.

Before discussing the subdiagram IR singularities in the following subsection, we shall extend the above results to generalized diagrams that contain not only the electron-electron-photon vertices of the standard QED but also generalized vertices with any number of photon legs and any even number of electron legs.

Let us first consider a diagram containing a generalized vertex with $2k$ photon legs and no electron leg (any of these photons may be external) as shown in Fig. 3. Since it has $n_e = 2n + l - 1 - 2(k-1)$ electrons and $n_p = n+1$ photons, we obtain

$$\frac{dz_G}{U^2V^{N-2n}} = \begin{cases} O(\delta^{2(k-1)}) & \text{for } l=0\,, \\ O(\delta^{2(k-2)}) & \text{for } l \geq 1\,, \end{cases} \tag{4.19}$$

Thus the integral appears to be IR-divergent for $k=1$, $l \geq 0$ and $k=2$, $l \geq 1$. Actually no IR divergence occurs in these cases because of gauge invariance. (Strictly speaking, gauge invariance is outside the scope of this section since it requires consideration of numerators of the integral M_G. However, we only need some of its consequences.)

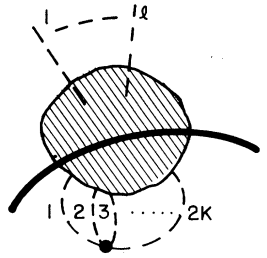

FIG. 3. A diagram with a generalized $2k$-photon vertex.

In the case $k=1$, which is obtained by shrinking a photon self-energy insertion S to a point, we may replace all photon lines with self-energy insertions by the Källén-Lehmann spectral representations for renormalized photon propagators. This leads to parametric integrals with "massive" internal photons. Diagrams with such massive photons will have no overall IR divergence since $V(p)$ does not vanish in the limit (4.17).

To dispose the case $k=2$, $l \geq 1$, which is obtained by replacing a photon-photon scattering subdiagram S by a four-photon vertex, let us rewrite the photon-photon scattering polarization tensor in a manifestly UV-finite form[16]

$$\Pi_{\kappa\rho\sigma\mu}(p_1,p_2,p_3,p) = -p^\nu \frac{\partial}{\partial p^\mu} \Pi_{\kappa\rho\sigma\nu}(p_1,p_2,p_3,p)\,, \tag{4.20}$$

which follows trivially from the gauge invariance of $\Pi_{\kappa\rho\sigma\mu}$. If p^ν is the momentum of an internal photon i, then (4.20) introduces an explicit D^ν factor in the numerator of the integral M_G. In the limit (4.17) we have $D_i^\nu \to A_i\, p^\nu = O(\delta)$ since $i \notin P^e$, so that the $k=2$ divergence of (4.19) is suppressed by an extra factor of δ in the numerator, rendering M_G finite in the overall IR limit (4.17).

Next consider a diagram containing a vertex with two electron legs and $m(=0, 1, 2, \ldots)$ photon legs (see Fig. 4). It has $n_e = 2n - 1 + l - (m-1)$ electron lines and $n_p = n$ photon lines. Thus in the limit (4.17) we have

$$\frac{dz_G}{U^2V^{N-2n}} = \begin{cases} O(\delta^m) & \text{for } l=0\,, \\ O(\delta^{m-1}) & \text{for } l \geq 1 \text{ vertex on } P_I^e\,, \\ O(\delta^m) & \text{for } l \geq 1 \text{ vertex on } P(CD)\,. \end{cases} \tag{4.21}$$

For $m \geq 2$ the integral M_G is convergent. The case $m=1$ is an ordinary vertex already covered by

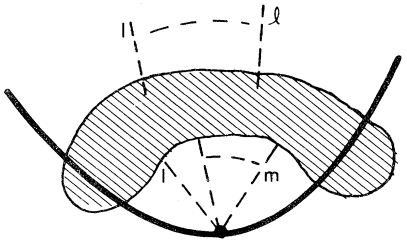

FIG. 4. A diagram with a generalized vertex with two electron and m photon lines.

(4.18). The case $m=0$ corresponds to mass counterterm insertion. Such a diagram can have worse IR divergence than a diagram without an insertion for $l \geq 1$. However, this can be reduced to the case (4.18) by carrying out the mass renormalization of electron lines.

Other conceivable generalized vertices have four or more electron legs. By the same kind of argument as above it can be shown that no new type of overall IR divergence arises in these cases.

B. Subdiagram IR singularities

In order that $V(p)$ behave as δ^2 it is not really necessary that the external momenta p, p' flow exclusively through the electron path P^e. What is crucial is the existence of nonempty paths $P^e(AC)$ and $P^e(DB)$ such that

$$1 - A_i^{(AC)} = O(\delta) , \qquad 1 - A_i^{(DB)} = O(\delta) . \tag{4.22}$$

As was noted already the behavior of $A_i^{(CD)}$ is irrelevant. This means in particular that parts of p and p' may be diverted to some photon lines. Thus IR divergences may appear even if some photons become hard insofar as other photons enable us to satisfy (4.22). When the photons i become hard, the corresponding z_i tend to zero in the IR limit and generate closed paths along which all z_i vanish. This means that U vanishes, too. In fact this is what is needed to maintain the degree of IR divergence in spite of the suppression of IR singularities coming from the phase space.

In order to generalize the IR limit (4.17) to subdiagrams, it is therefore necessary to combine (4.17) with the UV limit (2.3). Suppose we wish to examine the IR singularity arising from the reduced diagram G/S where the subdiagram S consists of a set of closed loops. Noting that $V(p)$ must be of order δ^2 for IR divergence to occur, let us define the IR limit $[f(z_i)]_{\mathrm{IR}}^{G/S}$ of an arbitrary parametric function $f(z_i)$ as the leading term in the double expansion in δ, ϵ, where

$$z_i = \begin{Bmatrix} O(\delta) \text{ for } i \in P_I^e \\ O(\delta^2) \text{ for } i \in P^e/P_I^e \\ O(1) \text{ for } i \notin P^e \end{Bmatrix} , \quad i \in G/S \tag{4.23a}$$

$$z_i = O(\epsilon), \ \epsilon \simeq \delta^2 \text{ for } i \in S , \tag{4.23b}$$

P_I^e being the subset defined in (4.17).

In Sec. II B we have already seen that (4.23b) leads to the separation of parametric functions into functions defined on S alone and ones defined on G/S alone. Then (4.23a) reduces the latter to an overall IR limit on G/S.

Let us consider a general subdiagram S of the diagram G shown in Fig. 5. The path $P^e(CD)$ may be empty, i.e., we allow a subdiagram S which does not contain any electron line from P^e. (In that case m is even by Furry's theorem.) Photons entering S can be either internal or external to the diagram G as a whole. By P^e/S we mean the reduced electron path obtained by shrinking $P^e(CD)$, the part of P^e contained in S, to a point. If there are any massive particle lines other than those of P^e, we have to consider only the case where they belong to S since V would be finite in the limit (4.23) otherwise.

The IR limit of various parametric functions now follows directly from the results of Sec. II B. First of all we have

$$[U]_{\mathrm{IR}}^{G/S} = U^S [U^{G/S}]_{\mathrm{IR}} \quad [= O(\epsilon^{n_S})] \tag{4.24}$$

$$[B_{ij}]_{\mathrm{IR}}^{G/S} = B_{ij}^S [U^{G/S}]_{\mathrm{IR}} \quad [= O(\epsilon^{n_S - 1})], \quad i, j \in S . \tag{4.25}$$

For other pairs of i and j, $[B_{ij}]_{\mathrm{IR}}^{G/S}$ is of order ϵ^{n_S} and can be ignored compared with (4.25).

From (2.15) the scalar currents within G/S satisfy

$$\begin{aligned} [A_i]_{\mathrm{IR}}^{G/S} &= [A_i^{G/S}]_{\mathrm{IR}} \\ &= 1 \quad \text{for } i \in P^e, \ i \in G/S \\ &= O(\delta) \text{ for } i \notin P^e, \ i \in G/S . \end{aligned} \tag{4.26}$$

For A_m, $m \in S$, we can determine the IR limit by

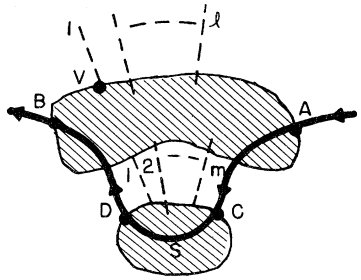

FIG. 5. A diagram G with a general subdiagram S.

first taking the UV limit of the defining formula (4.8)

$$[A_m]^S_{\rm UV} = \frac{1}{U^{G/S}U^S}\sum_{P(AB)}\eta_{mP}U_P^{G/S}U_P^S\,. \tag{4.27}$$

Now, in taking the limit (4.23a), we note that $[U_{P^e}^{G/S}]_{\rm IR} = U_{P^e}^{G/S}$ and $[U_P^{G/S}]_{\rm IR} = O(\delta)$ if $P \neq P^e$. Thus, only paths containing P^e/S will contribute to the above summation in the IR limit, yielding

$$[A_m]^{G/S}_{\rm IR} = A_m^S(CD) = A_m^S \text{ for } m\in S;\ C, D \text{ on } P^e$$
$$= O(\delta) \qquad \text{for } m\in S;\ C, D \text{ not on } P^e\,. \tag{4.28}$$

In terms of these results we can now evaluate $[V]^{G/S}_{\rm IR}$ as follows:

$$[V]^{G/S}_{\rm IR} = \left[\sum_{i\in P^e(CD)} z_m(1-q_m\cdot Q'_m) + \sum_{\substack{\text{electron}\\ \text{loops in } S}} z_i m_i{}^2\right]^{G/S}_{\rm IR} + \left[\sum_{i\in P^e/S} z_i(1-q_i\cdot Q'_i)\right]^{G/S}_{\rm IR}$$
$$= \left[\sum_{i\in P^e(CD)} z_m(1-q_m\cdot Q'^S_m) + \sum_{\substack{\text{electron}\\ \text{loops in } S}} z_i m_i{}^2\right] + \left[\sum_{i\in P^e/S} z_i(1-q_i\cdot Q'^{G/S}_i)\right]_{\rm IR}\,. \tag{4.29}$$

Thus in the IR limit the function V separates into the S and G/S parts of the same order

$$[V]^{G/S}_{\rm IR} = V_S + [V_{G/S}]_{\rm IR}\,. \tag{4.30}$$

Just as in Sec. III the separation of U and V into the S and G/S pieces enables us to factor the parametric integral over the IR limit integrand into a product of S and G/S integrals. This factorization is trivial in the Schwinger-Nambu representation (2.1). In the compact domain representation (4.1) it is implemented by inserting the identity

$$1 = \int \frac{ds}{s}\frac{dt}{t}\,\delta\left(1-\frac{z_S}{s}\right)\delta\left(1-\frac{z_{G/S}}{t}\right) \tag{4.31}$$

and rescaling

$$z_i \to s z_i\,, \quad i\in S$$
$$z_i \to t z_i\,, \quad i\in G/S \tag{4.32}$$

in the separated integral

$$\int \frac{dz_G\,\delta(1-z_G)}{U_S{}^2[U_{G/S}]_{\rm IR}{}^2(V_S+[V_{G/S}]_{\rm IR})^{N-2n}}\,.$$

The desired factorization is then achieved by the Feynman formula

$$\Gamma(k+l)\int \frac{\delta(1-s-t)s^{k-1}ds\,t^{l-1}dt}{(sA+tB)^{k+l}} = \frac{\Gamma(k)}{A^k}\frac{\Gamma(l)}{B^l}\,. \tag{4.33}$$

The reduced diagram G/S is one of the diagrams with generalized vertices discussed in Sec. IV A, and determining whether the diagram G has a divergence in the IR limit (4.23) reduces to determining all G/S that are overall IR-divergent.

Thus far we have concentrated on the no-contraction term in (4.1). In the overall IR limit this is sufficient for determining the leading singularity of the integrand since each contraction diminishes the power of V in the denominator. For the IR limit (4.23), however, it is possible that the higher contractions are as singular as the no-contraction term since the decrease in the singularity due to the vanishing of V may be compensated by the increased singularity coming from vanishing of U. Each contraction introduces a factor $B_{ij}V/U$. From (4.25) we note that if $m, n\in S$,

$$[B_{mn}V/U]^{G/S}_{\rm IR} = \frac{B^S_{mn}}{U^S}[V]^{G/S}_{\rm IR} = O(\delta^2\epsilon^{-1})\,. \tag{4.34}$$

Thus internal contractions within S will produce terms of the same order in the IR limit as the noncontracted term. All other contractions introduce factors of order δ^2 and do not contribute in this limit. Again IR divergences occur only if G/S has an overall IR divergence.

Let us summarize this section by giving a prescription for locating all IR singularities of a diagram G. Find all subdiagrams S, consisting of several closed loops, of G such that the reduced diagram G/S is free from any massive particle lines not belonging to the electron path P^e. Find the degree of overall IR divergence of the reduced diagram G/S in each case. Repeat this procedure for the reduced diagram G/S successively. Finally find the IR singularity of the diagram G itself. This procedure will exhaust all IR singularities associated with a QED diagram G.

V. SEPARATION OF IR DIVERGENCES

In Sec. IV we have derived rules for locating IR singularities in the parametric space and determining whether they give rise to IR divergences. For that purpose it was sufficient to consider zeros of the denominator $V(p)$. We have also

seen that, in the IR-singular region, parametric functions separate, suggesting the factorization of the integrand. We shall now extend our consideration to the whole integrand, determine the IR limits of all terms, and show that such a factorization indeed takes place. We shall then construct an IR subtraction scheme analogous to the K method for UV divergence.

To avoid unnecessary complication let us consider in this section only a vertex diagram G with the external photon momentum $q \neq 0$. The electron self-energy diagram can be treated as a special case of vertex diagram with $q=0$. Some extension of our method is necessary to deal with general diagrams.

A. Overall IR limit

The numerator F_0 of the no-contraction term consists of γ matrices from vertices and electron propagator factors $Q_i'+1$, where, according to (I.77),

$$Q_i'^{\mu} = A_i^{(AV)}(p - \tfrac{1}{2}q)^{\mu} + A_i^{(VB)}(p + \tfrac{1}{2}q)^{\mu} , \tag{5.1}$$

V being the position of the vertex γ^{μ} to which the external photon is attached. In the following we shall drop the superscript (AB) whenever no ambiguity arises, so that we denote $A_i = A_i^{(AB)}$, $V(p) = V^{(AB)}(p)$, etc. According to (4.12), in the overall IR limit, we have

$$[Q_i']_{\mathrm{IR}} = \begin{cases} p - \frac{1}{2}q = p' , & i \in P^e(AV) \\ p + \frac{1}{2}q = p'' , & i \in P^e(VB) \\ O(\delta) , & \text{otherwise} \end{cases} \tag{5.2}$$

i.e., all external momenta flow through the electron path P^e.

The general structure of the numerator F_0 is of the form

$$F_0 = \bar{u}(p'')F_0^{\nu}u(p') \rightarrow \bar{u}(p'')\gamma^{\alpha}(\not{p}''+1)\cdots\gamma^{\nu}\cdots(\not{p}'+1)\gamma_{\beta}u(p') , \tag{5.3}$$

where $\gamma^{\alpha}, \gamma^{\beta}, \ldots$ are contracted with $\gamma_{\alpha}, \gamma_{\beta}, \ldots$ somewhere in the above product. To simplify (5.3) we recall

$$(\not{p}'+1)\gamma_{\beta}u(p') = 2p'_{\beta}u(p') . \tag{5.4}$$

[In the following we shall not write the spinor factors $\bar{u}(p'')$, $u(p')$ explicitly. But it is to be understood that all F_k are evaluated sandwiched between spinors.] By repeated applications of (5.4) we can readily reduce the factors to the right (or left) of γ^{ν} in (5.3) to

$$\prod_{\beta}(2p'_{\beta}) \quad \left(\text{or} \prod_{\alpha}(2p'')^{\alpha}\right) .$$

Thus F_0 in (5.3) can be written, for $p'^2 = p''^2 = 1$, as

$$[F_0^{\nu}]_{\mathrm{IR}} = 4^n(p' \cdot p'')^m \gamma^{\nu} \equiv f_{\mathrm{IR}}^{G}\gamma^{\nu} \tag{5.5}$$

(if G has no fermion loop), where m is the number of photons crossing the external vertex V. In the overall IR limit the numerator therefore reduces to the bare vertex γ^{ν}, and hence the magnetic contribution is at most of order δ:

$$[F_0^{\nu}|_{\text{mag. mom.}}]_{\mathrm{IR}} = O(\delta) . \tag{5.6}$$

Since vertex diagrams are superficially logarithmically IR-divergent, (5.6) will suppress the singularity and make the contribution to the magnetic moment overall IR-finite. Charge form factors are of course overall IR-divergent. Let us define the overall IR-divergent part of the vertex diagram $\Gamma_{\nu}^{(2n)}$ of (I.70) by

$$\Gamma_{\mathrm{IR}}^{(2n)\nu} = \left(\frac{-\alpha}{4}\right)^n (n-1)!\,\gamma^{\nu} \int \frac{dz_G\,\delta(1-z_G)}{U^2 V^{N-2n}} f_{\mathrm{IR}}^{G} \equiv g_G \gamma^{\nu} . \tag{5.7}$$

Note that f_{IR}^{G} in (5.7) could be replaced by any parametric function that has the same IR limit as (5.5). For example, we could have defined $\Gamma_{\mathrm{IR}}^{(2n)\nu}$ using the entire F_0 for the numerator. Similarly U and V could have been replaced by some other functions, as long as they have the same IR limits as U and V.

B. Subdiagram IR limit

According to (4.26) and (4.28) we have

$$[Q_i']_{\mathrm{IR}}^{G/S} = \begin{cases} p', & i \in P^e(AV), & i \in G/S \\ p'', & i \in P^e(VB), & i \in G/S \\ O(\delta), & i \notin P^e, & i \in G/S \\ A_i^S p', & P^e(CD) \subset P^e(AV), & i \in S \\ A_i^S p'', & P^e(CD) \subset P^e(VB), & i \in S \\ Q_i'^S, & \text{if the vertex } V \text{ is in } S , & i \in S \end{cases} \tag{5.8}$$

where $A_i^S \equiv A_i^S(CD)$. The definition of the paths is clarified in Fig. 5.

The contribution of G/S lines to the numerator F_k will reduce to the numerical factor $f_{\mathrm{IR}}^{G/S}$ while the lines from S will retain the γ-matrix structure. Because of (4.34) all contractions will occur within S, and the numerator factor arising from the (noncontracted) G/S lines will be the same for all F_k. We therefore expect factorization of the type $F \rightarrow [F_0^{G/S}]_{\mathrm{IR}} F^S$. Since there are only a few possible subdiagrams S for our vertex diagram G, let us establish it for each type of S separately.

(i) Electron self-energy. With the help of (5.8) we find

$$[F_0]_{\mathrm{IR}}^{G/S} = f_{\mathrm{IR}}^{G/S}\gamma^{\nu}F_0[\delta m_S] , \tag{5.9}$$

where $f_{\mathrm{IR}}^{G/S}$ is a generalization of (5.5) to the electron self-mass insertion diagram G/S. Repeating this for all F_k we obtain

$$[\mathsf{F}^{\nu}]_{\mathrm{IR}}^{G/S} = f_{\mathrm{IR}}^{G/S}\gamma^{\nu}\mathsf{F}[\delta m_S],$$

S = electron self-energy diagram. (5.10)

(ii) Internal vertex (excludes the vertex V). We have

$$[\mathsf{F}^{\nu}]_{\mathrm{IR}}^{G/S} = f_{\mathrm{IR}}^{G/S}\gamma^{\nu}\mathsf{F}[L_S] \ . \tag{5.11}$$

(iii) External vertex (includes the vertex V). We find

$$[\mathsf{F}^{\nu}]_{\mathrm{IR}}^{G/S} = f_{\mathrm{IR}}^{G/S}\mathsf{F}[\Gamma_S^{\nu}] \ . \tag{5.12}$$

Similar results are obtained if G is an electron self-energy diagram; G/S always gives a factor $f_{\mathrm{IR}}^{G/S}$. Thus we have established the factorization of parametric integrands in the IR limit.

C. Subtraction of IR singularities

The simplest way of constructing an integrand less singular than J_G in the IR limit (4.23) would be to replace it by $J_G - [J_G]_{\mathrm{IR}}^{G/S}$. However, $[U]_{\mathrm{IR}}^{G/S}$ is too singular in the $z_i \to 0$ limit, where i is a photon line in G/S. Thus such a subtraction would introduce a spurious UV divergence. We shall avoid this by defining the subtraction term using the $I_{G/S}$ operation defined by[20]

$$I_{G/S}J_G = f_{\mathrm{IR}}^{G/S}\mathsf{F}^{S}\frac{1}{U_S^{\,2}U_{G/S}^{\,2}(V_S+V_{G/S})^{N-2n}} \ , \tag{5.13}$$

where $f_{\mathrm{IR}}^{G/S}$ must be replaced by $\gamma^{\nu}f_{\mathrm{IR}}^{G/S}$ if G/S includes the external vertex V. By construction we have $[I_{G/S}J_G]_{\mathrm{IR}}^{G/S} = [J_G]_{\mathrm{IR}}^{G/S}$, so that $(1-I_{G/S})J_G$ is also less singular in the IR limit (4.23) than J_G. We choose to make the replacements $[U_{G/S}]_{\mathrm{IR}} \to U_{G/S}$, $[V_{G/S}]_{\mathrm{IR}} \to V_{G/S}$ in defining (5.13) so that our I operation parallels the K operation. We could also have replaced the numerator factor $f_{\mathrm{IR}}^{G/S}$ by any function $f^{G/S}$ such that $[f^{G/S}]_{\mathrm{IR}} = f_{\mathrm{IR}}^{G/S}$. (In the subsequent article[3] we shall find such a redefinition to be convenient.)

We are now ready to define the I- and K-finite part of the Feynman amplitude M_G by

$$\Delta M_G = \prod_{S_i}(1-I_{G/S_i})\Delta' M_G \ , \tag{5.14}$$

where $\Delta' M_G$ is given by (2.25) and the product goes over all electron self-energy and vertex subdiagrams S_i. Strictly speaking, (5.14) may still contain logarithmic IR divergences since the superficial IR divergence may be linear for the electron self-energy subdiagram S. However, such problems do not arise in our particular application.[3] We shall therefore not worry about this problem any further.

In terms of (4.33) and (5.10)–(5.12) we find that the subtraction terms factorize as

$$I_{G/S}\Gamma_G^{\nu} = \begin{cases} g_{G/S}\gamma^{\nu}\delta m_S & \text{if } S \text{ is an electron self-energy diagram,} \\ g_{G/S}\gamma^{\nu}L_S & \text{if } S \text{ is an internal vertex,} \\ g_{G/S}\Gamma_S^{\nu} & \text{if } S \text{ is an external vertex,} \end{cases} \tag{5.15}$$

where g_G is defined by (5.7). Physically the decomposition (5.15) means that an IR divergence occurs whenever the diagram G separates into a cloud of soft photons attached to the external electron lines (reduced diagram G/S) and an "inner" diagram S whose photons carry arbitrary momenta. $g_{G/S}$ may be a complicated integral. However, in calculating physical processes, these IR-divergent integrals will cancel each other identically and need not be evaluated explicitly.

Let us also note that, according to (5.15), only those diagrams containing an external vertex subdiagram will give IR-divergent contributions to the magnetic moment. All other magnetic moment contributions defined by (5.14) will be IR-finite.

ACKNOWLEDGMENTS

We should like to thank Dr. T.-M. Yan, Dr. G. Grammer, and Dr. D. R. Yennie for useful discussions. One of us (TK) wishes to thank the Japan Society for Promotion of Science and Dr. H. Miyazawa of the University of Tokyo for their kind hospitality.

APPENDIX: CALCULATION OF $\Pi_G(p^2)$

We shall describe how (3.5) can be consistently implemented in the parametrized Feynman integrals. Note that for $\mu \neq \nu$,

$$\frac{\partial}{\partial p^{\alpha}}\frac{\partial}{\partial p^{\beta}}\Pi_G^{\mu\nu} = (g^{\mu}{}_{\alpha}g^{\nu}{}_{\beta}+g^{\nu}{}_{\alpha}g^{\mu}{}_{\beta})\Pi_G + (p^{\mu},p^{\nu},\ \ldots\ \text{terms}) \ . \tag{A1}$$

All external momentum p dependence of $\Pi_G^{\mu\nu}$ is contained in $F_k^{\mu\nu}$ terms in the numerator and the exponent $V(p)$. One can symbolically write

$$\frac{\partial}{\partial p_{\alpha}}\frac{\partial}{\partial p_{\beta}}(\mathsf{F}^{\mu\nu}e^{-iV(p)}) = \frac{\partial^2\mathsf{F}^{\mu\nu}}{\partial p_{\alpha}\partial p_{\beta}}e^{-iV(p)} + (p^{\alpha},p^{\beta},\text{terms}), \quad \nu \neq \mu \tag{A2}$$

with

$$\frac{\partial^2\mathsf{F}^{\mu\nu}}{\partial p_{\alpha}\partial p_{\beta}} = \sum_{\substack{i,j \\ \text{electrons}}} A_i A_j \mathsf{F}_{ij}^{\mu\nu,\alpha\beta} \ , \tag{A3}$$

where (A3) is obtained by noting that all p depen-

dence of F_k is in the noncontracted factors $Q_i'+m_i$, $Q_i'=A_i\not{p}$. $F_{ij}^{\mu\nu,\alpha\beta}$ is obtained from $F^{\mu\nu}$ by replacing factors $\not{p}_i+m_i$ and $\not{p}_j+m_j$ by γ^α and γ^β. Then Π_G is obtained by

$$\Pi_G = i\left[\frac{\alpha}{(4\pi i)^{\omega-1}}\right]^n \int \frac{dz_G}{U^\omega}\,\frac{\partial^2 F^{\mu\nu}}{\partial p_\alpha \partial p_\beta}\,e^{-iV(p)}\,,$$
$$\alpha=\mu,\ \beta=\nu,\ \alpha\neq\beta\ . \quad \text{(A4)}$$

The redefinition (3.5) has to be consistently incorporated in all diagrams with photon self-energy subdiagram S. One way of doing this is by first parametrizing the subintegral $\Pi_S^{\mu\nu}$, substituting (3.5) and (A4), and then parametrizing the rest of the integral. It is not obvious that this two-step parametrization will lead to the same form of the parametric integral as our usual overall parametrization. However, following the arguments of Sec. III of I, it is easy to show that

$$U = U_S\, U_{G/S}(z_{ii'}\to z_{ii'}+G_S)\,, \quad \text{(A5)}$$

where S is inserted between the lines i and i' and G_S is given by

$$V_S = \sum_{j\in S} z_j m_j^2 - p_i^2 G_S\ . \quad \text{(A6)}$$

Thus the two-step parametrization is in fact the same as the overall parametrization. Making use of (I.95) we can also rewrite $A_m^S A_n^S$ appearing in (A3) in terms of overall parametric functions, noting that

$$A_m^S = A_m/A_i\,, \quad m\in S\ . \quad \text{(A7)}$$

Thus, if some diagram G contains a photon self-energy insertion S, we obtain

$$M_G = \frac{i^n(-i)^N}{(4\pi i)^{n\omega}}\int \frac{dz_G}{U^\omega}\,F(D_k)e^{-iV(p)}\,, \quad \text{(A8)}$$

with

$$F(D_k) = E^{\mu\nu}F_{\mu\nu}\,, \quad \text{(A9)}$$

where $E^{\mu\nu}$ contains numerator factors from $j\notin S$, and $F_{\mu\nu}$ is the numerator factor from the electron lines in S. Then (3.5) is implemented by the replacement

$$F(D_k)\to E_\delta^\delta(D_j)\,\frac{(-D_i\cdot D_{i'})}{A_i^2}\sum_{\substack{m,n\in S\\ \text{electron}}} A_m A_n F_{\mu\nu,mn}^{\alpha\beta}$$
$$\alpha=\mu,\ \beta=\nu;\ \alpha\neq\beta\ . \quad \text{(A10)}$$

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*Work supported in part by the U. S. Atomic Energy Commission and by the National Science Foundation.

†John Simon Guggenheim Foundation Fellow.

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