Feynman-Dyson rules in parametric space*

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Concise and practical formulas for Feynman-parametric integrals are assembled and presented in the form of Feynman-Dyson rules in parametric space. These rules enable us to write down S-matrix elements directly in terms of some parametric functions. They are particularly useful for construction of ultraviolet- and infrared-divergence-free Feynman-parametric integrals in a form suitable for numerical integration.

I. INTRODUCTION

Over the years a number of parametric formulas for Feynman integrals have been introduced to satisfy various needs.[1-10] These formulas can be expressed concisely in terms of some auxiliary functions rather than Feynman parameters themselves. For instance, a significant simplification has been achieved in Ref. 8 in which the integrand (in particular the numerator) is expressed in terms of the so-called Kirchhoff currents. These auxiliary functions themselves can be expressed in various forms (e.g., as determinants, sums over loops,[2] sums over cut-sets,[5,6] etc.), some of which are relatively simple, while others may look simple but become very lengthy if worked out explicitly. Such differences are irrelevant to the lowest-order calculations, but for higher-order calculations it is crucial to write down explicitly the integrals in as simple a form as possible.

The purpose of this paper is to assemble simple expressions for these functions in a systematic and coherent manner so that they can be used for practical calculations. For this purpose it is most convenient to present them as Feynman-Dyson rules in Feynman-parametric space which enable us to write down the S-matrix elements directly in terms of the parametric functions. Similar attempts have already been made,[5,10-12] but we believe that the version presented here is the most direct and practical one. We wish to emphasize that most of the formulas in this paper are not new and can be found in the literature, especially in Ref. 2. We feel, however, that some of the formulas have not been sufficiently emphasized.

Our formulation of parametric representation provides a systematic and economical way of evaluating a large number of complicated Feynman integrals. In particular, it minimizes the labor of setting up a computer program for numerical integration.[13] By stressing common features of related diagrams through parametric functions defined directly from the topology of diagrams, our formulation also reduces the redundancy of computation for separate diagrams and provides means for crosschecking related diagrams.

The utility of our formulation extends, however, beyond numerical applications. For example, we have applied it to derive a general parametric-space technique for extraction of ultraviolet-divergent parts (as a step of an intermediate-renormalization scheme) and for a systematic separation of infrared-divergent parts of Feynman integrals.[14] The present paper was originally intended to be a section of a paper on the sixth-order radiative corrections to the anomalous magnetic moment of an electron.[13] However, in view of the generality and broader applicability of our formulation, we have decided to present it as a separate article.

In Sec. II we summarize our results in the form of a set of rules for writing down the spinor electrodynamical amplitudes directly in the Feynman-parametric space. The generalization to other field theories is straightforward, and it is not included here for the sake of compactness. In Sec. III we present a derivation of these rules. Section IV contains supplementary formulas useful for particular applications of the parametric representation. In Sec. V we show by an example how a Feynman amplitude is constructed using the parametric rules of Sec. II. A derivation of formula (97) is given in the Appendix.

II. FEYNMAN-DYSON RULES FOR PARAMETRIC INTEGRALS

In this section we give a set of rules which enables us to write down an invariant amplitude M for a Feynman diagram G directly in terms of parametric functions without going through the usual Feynman-Dyson rules in momentum or coordinate variables. We follow the notation and conventions of Bjorken and Drell.[7] We consider only those G that cannot be separated into two disconnected parts by cutting an internal line.

The diagram G consists of N internal lines labeled $1,2,\ldots,N$, some of which are electron lines and others are photon lines. Appropriately directed arrows are assigned to both electron and photon lines. An example is shown in Fig. 1(a). Feynman parameters $z_1, z_2, \ldots, z_N$ satisfying $\sum_{i=1}^{N} z_i = 1$, $z_i \geq 0$ are assigned to the internal lines. Variable momenta are completely integrated out and do not appear in the parametric formula. However, each line j carries fixed momentum q_j which satisfies the momentum conservation law at each vertex and hence depends linearly on the external momenta.

In our formulation all parametric functions are constructed from the basic parametric functions B_{ij} which are determined completely by the topological structure of the diagram G. For the diagram of Fig. 1(a), for instance, the topological structure is expressed in terms of the "chain diagram" shown in Fig. 1(b), which is obtained from G by amputating all external lines and disregarding distinction between electron and photon lines. In any given order the number of topologically distinct diagrams is quite small. All chain diagrams necessary for calculations of processes containing up to three internal loops are shown in Fig. 2. For each chain diagram the number of topologically different B_{ij} is also very small. It is therefore easy to prepare beforehand a complete list of B_{ij} up to a given order. All B_{ij} needed for calculations of *any* Feynman diagram with up to three internal loops are shown in Fig. 3. In these

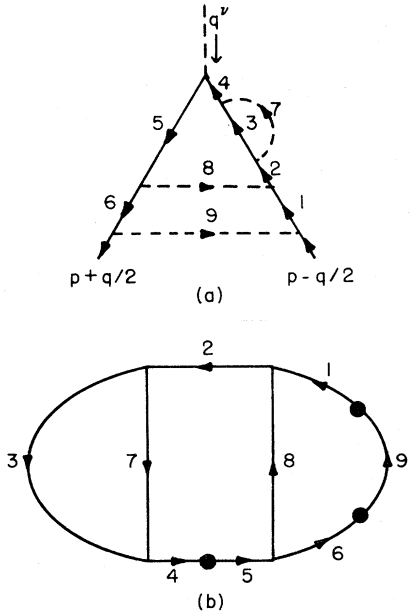

FIG. 1. (a) A sixth-order vertex diagram. (b) The chain diagram for the graph (a).

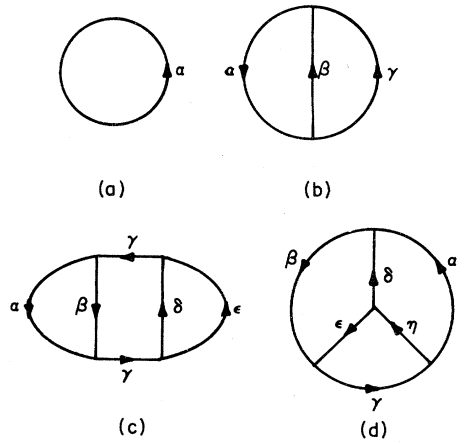

FIG. 2. Chain diagrams of (a) one-loop, (b) two-loop, (c) three-loop "pretzel," and (d) three-loop "Mercedes" types.

diagrams the name of a chain also denotes the Feynman parameter assigned to it. The Feynman parameter α for the chain α consisting of lines $i,j,\ldots,k$ is defined as $\alpha = z_i + z_j + \cdots + z_k$. The overall sign of $B_{\alpha\beta}$ changes if the direction of either line α or β changes. Recursive formulas for calculating B_{ij} of any order will be given at the end of this section.

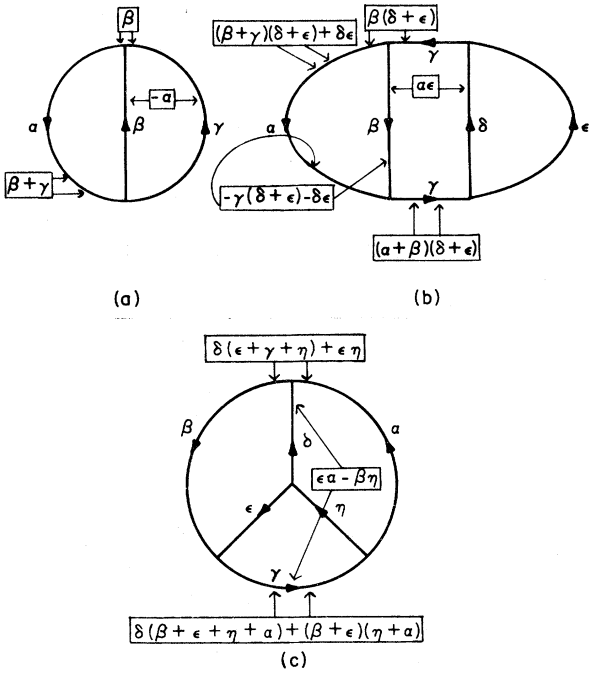

FIG. 3. $B_{\alpha\beta}$ functions for (a) two-loop, (b) three-loop "pretzel," and (c) three-loop "Mercedes" chain diagrams. For simplicity the names of chains $\alpha, \beta, \ldots$ are also used to represent the corresponding Feynman parameters $z_\alpha, z_\beta, \ldots$.

We are now ready to state the rules:

1. Find the parametric functions B_{ij} from the diagrams of Fig. 3.

2. Construct the parametric function U using

$$U=\sum_{j=1}^{N}\eta_{js}z_jB_{jk}\,,\quad \text{any } k\in s \tag{1}$$

where s is any directed loop of G and η_{js} is the projection $(\pm1,0)$ of line j along the loop s. Note that the number of terms actually contributing to U in (1) can be minimized by choosing the shortest loop in G.

3. Construct the parametric functions $Q_i'^{\mu}$ and V:

$$Q_i'^{\mu}=-\frac{1}{U}\sum_{j=1}^{N}q_j^{\mu}z_jB_{ij}'\,, \tag{2}$$

$$V=\sum_{j=1}^{N}z_j(m_j^2-q_j\cdot Q_j')\,, \tag{3}$$

where m_i is the mass of the ith line and

$$B_{ij}'=B_{ij}-\delta_{ij}U/z_i\,. \tag{4}$$

Again, the number of terms actually contributing to (2) and (3) can be cut down substantially by an appropriate routing of external momenta through G.

4. Construct the parametric integral

$$\left(\frac{i}{16\pi^2}\right)^n(-1)^N(N-2n-1)!\int\frac{dz_G\delta(1-z_G)}{U^2(V-i\epsilon)^{N-2n}}\,, \tag{5}$$

where the integration domain is given by

$$dz_G=\prod_{j=1}^{N}\int_0^{\infty}dz_j\,,\quad z_G=\sum_{j=1}^{N}z_j\,, \tag{6}$$

and n is the total number of independent loops.

5. Multiply (5) by factors associated with the remaining elements of the diagram G[15]:

a. for each external electron line entering the diagram a factor $\sqrt{Z_2}\,u(p,s)$ or $\sqrt{Z_2}\,v(p,s)$, depending on whether the line is in the initial or final state; likewise, for each electron line leaving the diagram a factor $\sqrt{Z_2}\,\bar{u}(p,s)$ or $\sqrt{Z_2}\,\bar{v}(p,s)$;

b. for each external photon line a factor $\sqrt{Z_3}\,\epsilon_{\mu}$;

c. for each internal electron line j a factor $i(\not{p}_j+m_j)$;

d. for each internal photon line j a factor $-ig_{\mu\nu}$;

e. for each vertex a factor $-ie_0\gamma_{\mu}$;

f. for each mass counterterm a factor $i\delta m$;

g. for each closed electron loop a factor -1.

6. Let us denote by F the product of γ^{μ} from vertices, $(\not{p}_j+m_j)$ from electron lines, and appropriate spinor factors. Then the action of F on the integral (5) is defined by

$$\mathsf{F}\,\frac{1}{V^m}=\frac{F_0}{V^m}+\frac{1}{(m-1)U}\,\frac{F_1}{V^{m-1}}+\frac{1}{(m-1)(m-2)U^2}\,\frac{F_2}{V^{m-2}}+\cdots\,, \tag{7}$$

where the subscript k of F_k stands for the number of contractions. By contraction we mean picking out a pair of $(\not{p}_i+m_i)$, $(\not{p}_j+m_j)$ from F, replacing them by $\gamma^{\mu},\gamma_{\mu}$, putting a factor $-\frac{1}{2}B_{ij}$ in front, and summing the result of this operation over all distinct pairs. Noncontracted $\not{p}_i$ are then replaced by $\not{Q}_i'$. With this step the Feynman integral in the parametric space has been completely defined and we can now carry out the integrations.

Comment. When the number of closed loops is more than three, we need B_{ij} not shown in Fig. 3. In general we can construct B_{ij} using the formula

$$B_{ij}=\sum_c\eta_{ic}\eta_{jc}U_c\,, \tag{8}$$

where the sum goes over all (not necessarily independent) self-nonintersecting loops c that contain both lines i and j, and U_c is the U function for the reduced diagram obtained from G by shrinking the loop c to a point. [η_{ic} is defined in (1).] Since U_c contains one less loop, it can be calculated recursively by the formulas (1) and (8) until all reduced diagrams become simple loops. For such a loop consisting of lines $i,j,\ldots,k$ we have

$$U_c=z_i+z_j+\cdots+z_k\,. \tag{9}$$

III. DERIVATION

We consider a Feynman diagram G which cannot be separated into two disconnected parts by cutting an internal line. It consists of N internal lines $1,2,\ldots,N$, some of which are electron lines and some are photon lines. Both electron and photon lines are assigned (arbitrarily) directed arrows. A convenient choice of directions will be specified later when chain diagram is introduced. Given the diagram G, the usual Feynman-Dyson rules generate a Feynman integral of the form

$$M=\int\frac{F(p_i)\prod_{s=1}^{n}dr_s}{\prod_{i=1}^{N}(p_i^2-m_i^2)}\,, \tag{10}$$

where m_i is the mass of the line i, n is the number of independent integration loops, r_s is the integration momentum around the loop s, and $F(p_i)$ is a polynomial in the line momenta $p_1,p_2,\ldots,p_N$. To avoid unnecessary complication we shall assume in this section that (10) is free from divergences. Regularization of divergent integrals will be discussed in Sec. IV A.

In order to introduce various parametric functions, we shall begin by giving one of the conventional derivations[2] of parametric representation for the integral (10). The first step is to separate out the dependence of p_i on the integration momenta[16]

$$p_i=q_i+k_i\,, \tag{11}$$

$$k_i = \sum_{s=1}^{n} \eta_{is} r_s , \tag{12}$$

where η_{is} is the projection $(\pm 1, 0)$ of p_i along r_s, and is known as the "circuit matrix" in graph theory.[9] The conservation of momentum at any vertex v can be expressed in terms of the "incidence matrix" ϵ_{vj}:

$$\sum_{j=1}^{N} \epsilon_{vj} p_j + p_v = 0 , \tag{13}$$

where

$$\epsilon_{vj} = \begin{cases} 1 \text{ if the line } j \text{ enters the vertex } v , \\ -1 \text{ if the line } j \text{ leaves the vertex } v , \\ 0 \text{ otherwise} , \end{cases} \tag{14}$$

and p_v is the sum of all external momenta incident on the vertex v. The definition of k_i in (12) in terms of loop momenta r_s implies that k_i satisfy the conservation law of their own

$$\sum_{j=1}^{N} \epsilon_{vj} k_j = 0 . \tag{15}$$

This can be interpreted as the momentum conservation law for a diagram G_A obtained from G by amputating all external lines, i.e., by setting all external momenta equal to 0. All lines of G_A can be classified into sets $\alpha, \beta, \ldots$ according to whether they carry the same momentum k_i (within $\pm$ sign) or not. These sets will be called chains.[5] It is convenient to choose the direction of arrows so that $k_i = k_j$ for any pair of lines i, j belonging to the same chain. We shall stick to this convention hereafter. The chain diagram G_C is a diagram obtained from G by replacing all internal lines by corresponding chains. It represents the basic topological structure of the diagram G, and plays a very important role in our formulation.

It is seen from (11), (13), and (15) that the constant momenta q_j are conserved by themselves at each vertex

$$\sum_{j=1}^{N} \epsilon_{vj} q_j + p_v = 0 . \tag{16}$$

There are basically two ways to derive a parametric integral depending on how the terms linear in r_s in the combined denominator of (20) are eliminated. One is by shifting the origin of integration variables,[2] and the other is by imposing some restriction on q_j.[16] We shall adopt the first approach here because we want to treat q_j as completely independent. In fact we shall temporarily suspend even the conservation law (16), and reinstate it only after the operations in (37) are carried out. With this understanding we can replace p_i in the numerator function $F(p_i)$ by the operator[17]

$$D_i^{\mu} = \tfrac{1}{2} \int_{m_i^2}^{\infty} dm_i^2 \frac{\partial}{\partial q_{i\mu}} . \tag{17}$$

Now we can rewrite (10) in the form

$$M = F(D_i) \int \frac{\prod_{s=1}^{n} dr_s}{\prod_{i=1}^{N} (p_i^2 - m_i^2)} , \tag{18}$$

which enables us to concentrate initially on the structure of the denominator alone.

Let us introduce Feynman parameters $z_1, z_2, \ldots, z_N$ by Feynman formula

$$\prod_{i=1}^{m} \frac{(n_i - 1)!}{a_i^{n_i}} = (N-1)! \int \frac{\delta(1 - z_1 - z_2 \cdots - z_m) z_1^{n_1 - 1} dz_1 \cdots z_m^{n_m - 1} dz_m}{(z_1 a_1 + z_2 a_2 + \cdots + z_m a_m)^N} , \tag{19}$$

where $N = n_1 + n_2 + \cdots + n_m$, and transform (18) into

$$M = (N-1)! F(D_i) \int \delta\left(1 - \sum_{i=1}^{N} z_i\right) \prod_{j=1}^{N} dz_j \int \frac{\prod_{s=1}^{n} dr_s}{[\sum_{i=1}^{N} z_i (p_i^2 - m_i^2)]^N} . \tag{20}$$

To perform the r_s integration we expand $\sum z_i p_i^2$ as

$$\sum_{i=1}^{N} z_i q_i^2 + 2 \sum_{s=1}^{n} \sum_{i=1}^{N} z_i \eta_{is} (q_i \cdot r_s) + \sum_{s,t=1}^{n} U_{st} (r_s \cdot r_t) , \tag{21}$$

where

$$U_{st} = \sum_{j=1}^{N} z_j \eta_{js} \eta_{jt} . \tag{22}$$

The terms linear in r_s can be removed by shifting the origin of r_s:

$$r_s \to r_s - \sum_{j=1}^{N} \sum_{t=1}^{n} z_j q_j \eta_{jt} (\mathfrak{U}^{-1})_{st} . \tag{23}$$

The $n \times n$ matrix $\mathfrak{U} = (U_{st})$ is nonsingular, as is shown later by explicit calculation of its determinant

$$U = \det(U_{st}) . \tag{24}$$

Substitution of the shifted r_s yields

$$\sum_{i=1}^{N} z_i (p_i^2 - m_i^2) = -V + \sum_{s,t=1}^{n} U_{st} (r_s \cdot r_t) , \tag{25}$$

where

$$V=\sum_{i=1}^{N} z_i m_i^2 - G\,,$$

$$G=\sum_{i=1}^{N} z_i q_i^2 - \frac{1}{U}\sum_{i,j=1}^{N} z_i z_j B_{ij}(q_i\cdot q_j)\,, \tag{26}$$

and

$$B_{ij}=U\sum_{s,t=1}^{n}\eta_{is}\eta_{jt}(\mathfrak{U}^{-1})_{st}\,. \tag{27}$$

Since U_{st} is symmetric by definition (22) it can be brought to a diagonal form by an orthogonal transformation $A_{t't}$:

$$\sum_{s',t'=1}^{n}(A^{-1})_{ss'}U_{s't'}A_{t't}=U_s\delta_{st}\,. \tag{28}$$

Noting that the Jacobian of the transformation $\bar{r}_s=\sum_{t=1}^{n}A_{st}r_t$ of the loop momenta is unity, we can write

$$\int\frac{\prod_{s=1}^{n}dr_s}{[\sum_{i=1}^{N}z_i(p_i^2-m_i^2)]^N}=\int\frac{\prod_{s=1}^{n}d\bar{r}_s}{[\sum_{s=1}^{n}U_s\bar{r}_s^2-V]^N} \tag{29}$$

in (20). We can now carry out $\bar{r}_s$ integrations by repeated application of the formula

$$\int\frac{d^4p}{(p^2-m^2+i\epsilon)^n}=\frac{i\pi^2}{(n-1)(n-2)}\frac{1}{(-m^2+i\epsilon)^{n-2}}\,, \tag{30}$$

which turns (20) into the Feynman-parametric formula

$$M=(-1)^N(\pi^2 i)^n(N-2n-1)!\,F(D_i)\int\frac{dz_G\,\delta(1-z_G)}{U^2(V-i\epsilon)^{N-2n}}\,, \tag{31}$$

where

$$dz_G=\prod_{j=1}^{N}\int_0^\infty dz_j\,,\quad z_G=\sum_{j=1}^{N}z_j\,. \tag{32}$$

The only remaining step is to carry out the $F(D_i)$ operation which involves successive differentiations with respect to q_i^μ. For this purpose we introduce the functions

$$Q_i'^{\mu}=-\frac{1}{2z_i}\frac{\partial V}{\partial q_{i\mu}}\,, \tag{33}$$

$$g_{\mu\nu}B'_{ij}=\frac{U}{2z_iz_j}\frac{\partial^2 V}{\partial q_i^\mu\partial q_j^\nu}\,. \tag{34}$$

Since V is quadratic in q_j, $Q_i'^{\mu}$ is linear in q_j and B'_{ij} is independent of q_j. It is useful to note that $Q_i'^{\mu}$ and G can be expressed in terms of the functions B'_{ij}:

$$Q_i'^{\mu}=-\frac{1}{U}\sum_{j=1}^{N}q_j^\mu z_j B'_{ij}\,, \tag{35}$$

$$G=\sum_{j=1}^{N}z_j(q_j\cdot Q_j')\,. \tag{36}$$

Since V has no higher derivatives, the result of D_i^μ operations can be expressed in terms of $Q_i'^{\mu}$ and B'_{ij} as

$$\begin{aligned}
&D_{i\mu}\frac{1}{V^n}=\frac{Q'_{i\mu}}{V^n}\,,\\
&D_{i\mu}D_{j\nu}\frac{1}{V^n}=\frac{Q'_{i\mu}Q'_{j\nu}}{V^n}-\frac{1}{2(n-1)}\frac{g_{\mu\nu}B'_{ij}}{UV^{n-1}}\,,\\
&D_{i\mu}D_{j\nu}D_{k\lambda}\frac{1}{V^n}=\frac{Q'_{i\mu}Q'_{j\nu}Q'_{k\lambda}}{V^n}-\frac{1}{2(n-1)}\frac{Q'_{i\mu}g_{\nu\lambda}B'_{jk}+Q'_{j\nu}g_{\mu\lambda}B'_{ik}+Q'_{k\lambda}g_{\mu\nu}B'_{ij}}{UV^{n-1}}\,,\\
&\cdots\,.
\end{aligned} \tag{37}$$

From (37) follows immediately the rule (7) describing the action of operation F on the parametric integral (5), except that B'_{ij} appears instead of B_{ij}. As is easily seen from (26) and (36), however, B'_{ij} and B_{ij} are related by

$$B'_{ij}=B_{ij}-\delta_{ij}U/z_i\,. \tag{38}$$

Thus $B'_{ij}=B_{ij}$ for $i\neq j$, which is always the case for $F(D_i)$ in spinor quantum electrodynamics.

We may now reimpose the conservation law (16) for q_j. What we have done thus far is to derive a parametric representation for the Feynman integral M and introduce various parametric functions. We see from (26), (35), and (38) that the functions V and $Q_i'^{\mu}$ are linear combinations of B_{ij}. We shall see later that U can be expressed similarly. In this sense B_{ij} may be regarded as the basic building block of all parametric functions.

Even if we decide to express all parametric functions in terms of B_{ij}, there still remains a great flexibility in the explicit form that V and $Q_i'^{\mu}$ may take, because the constant momenta q_j can be chosen freely subject only to the conservation law (16). Thus, judicious choice of q_j can lead to extremely compact expressions for V and $Q_i'^{\mu}$. (An example is shown in Sec. V.)

This flexibility implies of course that various $Q_i'^{\mu}$ and B'_{ij} functions are related. These relations can be cast in the form of Kirchhoff's first and second laws for electrical networks.[8,10,16] They

are consequences of the momentum conservation laws (15) and (16) as well as the invariance of V under shifting of origins of loop momenta r_s.

From (11), (15), and (17) we have for each vertex v

$$\sum_{j=1}^{N} \epsilon_{vj}(D_j^\mu - q_j^\mu) \int \frac{\prod_{s=1}^{n} dr_s}{\prod_{i=1}^{N}(p_i^2 - m_i^2)} = 0 , \tag{39}$$

where (16) is to be imposed after the D operations are carried out. This leads to Kirchhoff's first law for the "current" $Q_j'^\mu$

$$\sum_{j=1}^{N} \epsilon_{vj} Q_j'^\mu + p_v^\mu = 0 . \tag{40}$$

If we define the "internal current"

$$Q_j^\mu = Q_j'^\mu - q_j^\mu , \tag{41}$$

(40) reduces to the alternate form

$$\sum_{j=1}^{N} \epsilon_{vj} Q_j^\mu = 0 . \tag{42}$$

The description "internal" is motivated by the observation that $D_j - q_j$ in (39) corresponds to the internal momentum variable k_j of (11). The law (42) is an obvious analog of the conservation law (40) for the amputated diagram G_A.

As is easily seen from (35), (38), and (41), Q_i^μ can be expressed in terms of B_{ij} as

$$Q_i^\mu = -\frac{1}{U} \sum_{j=1}^{N} q_j^\mu z_j B_{ij} , \tag{43}$$

which is the internal analog of (35). Substituting this in (42), we obtain Kirchhoff's first law for the "internal" function B_{ij}:

$$\sum_{i=1}^{N} \epsilon_{vi} B_{ij} = 0 \text{ for any vertex } v \text{ and any internal line } j . \tag{44}$$

Applying this to adjacent lines j,k belonging to the same chain, we find that

$$B_{ij} = B_{ik} . \tag{45}$$

This is easily seen to hold for any j,k that belong to the same chain, so that B_{ij} are actually completely determined by the chain graph G_C, and do not depend on any further properties of the graph G. Thus they should be referred to as $B_{\alpha\beta}$, where $i \in \alpha$ and $j \in \beta$. For the notational convenience, however, we shall frequently use some B_{ij} as a representative of $B_{\alpha\beta}$ to avoid introducing chain indices $\alpha,\beta,\ldots$ explicitly.

A useful consequence of (45) is that once $B_{\alpha\beta}$ are calculated for one diagram they can be used for any other diagram with the same chain structure.

The second Kirchhoff law is obtained from the observation that V does not depend on the choice of the loop momenta as long as the momenta conservation laws (16) are satisfied.[8] Hence, if we increase all momenta along any loop s by the same amount q_s,

$$q_i' = q_i + \eta_{is} q_s , \tag{46}$$

V must remain unchanged:

$$\frac{dV}{dq_s^\mu} = \sum_{i=1}^{N} \frac{\partial q_{iv}'}{\partial q_s^\mu} \frac{\partial V}{\partial q_{iv}'} = \sum_{i=1}^{N} \eta_{is} \frac{\partial V}{\partial q_i'^\mu} = 0 . \tag{47}$$

Setting $q_s = 0$ in (47) and substituting (33) in (47) we obtain Kirchhoff's second law for $Q_i'^\mu$

$$\sum_{i=1}^{N} \eta_{is} z_i Q_i'^\mu = 0 . \tag{48}$$

Equating coefficients of q_j to 0 yields Kirchhoff's second law for B_{ij}'

$$\sum_{i=1}^{N} \eta_{is} z_i B_{ij}' = 0 . \tag{49}$$

Substituting (38) in this equation, we obtain

$$\eta_{js} U = \sum_{i=1}^{N} \eta_{is} z_i B_{ij} , \tag{50}$$

which enables us to express U in terms of B_{ij}. We see that U can be obtained by going around any loop s that contains the line j. By choosing the shortest possible loops one can express U very concisely. See Sec. V for examples. If we choose j which does not belong to the loop s, (50) gives useful linear relations among B_{ij}'s.

Thus far we have shown that all parametric functions can be expressed in terms of the chain-diagram functions B_{ij}. This is motivated by the desire to make contact with graph theory through the simplest and most general graph-theoretical functions available, B_{ij}. The defining formula (27) for B_{ij} is rather unwieldy for calculation of B_{ij}. However, various graph-theoretical formulas[2,9,18,19] can be used for this purpose. The best suited is the "circuit representation"[2,9] in which

$$B_{ij} = \sum_{c} \eta_{ic} \eta_{jc} U_c , \tag{51}$$

where U_c is the U function of the chain diagram that is obtained by shrinking the circuit c to a point. Because of the presence of the factor $\eta_{ic}\eta_{jc}$, the summation in (51) is restricted to all circuits c containing both lines i and j.

As an illustration we shall calculate $B_{\alpha\beta}$ for the "Mercedes" chain diagram of Fig. 2(d):

$$B_{\alpha\beta} = (+1)(+1) U_{\{\alpha,\beta,\gamma\}} + (+1)(+1) U_{\{\alpha,\beta,\epsilon,\eta\}} , \tag{52}$$

where the reduced chain diagrams obtained by shrinking the circuits $\{\alpha,\beta,\gamma\}$ and $\{\alpha,\beta,\epsilon,\eta\}$ to

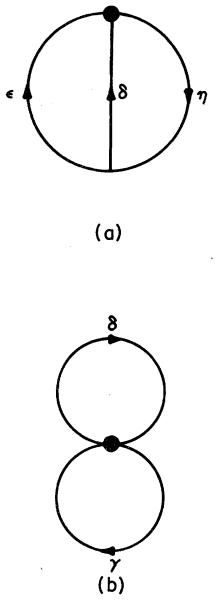

FIG. 4. Reduced chain diagrams of the chain diagram of Fig. 2(d) obtained by shrinking (a) the loop $\{\alpha, \beta, \gamma\}$, and (b) the loop $\{\alpha, \beta, \epsilon, \eta\}$.

points are shown in Figs. 4(a) and 4(b). They can in turn be calculated in terms of lower order $B_{\alpha\beta}$ by repeated use of (50) and (51) until all the U_c reduce to single loops like Fig. 2(a) for which by definitions (22) and (24)

$$U = z_\alpha \tag{53}$$

and by definition (27)

$$B_{\alpha\alpha} = 1\,, \tag{54}$$

where z_α is the Feynman parameter for the chain α and is given by

$$z_\alpha = z_i + z_j + \cdots + z_k \tag{55}$$

if the chain α consists of lines $i, j, \ldots, k$.

Shrinking a circuit may lead to two independent chain diagrams, as in Fig. 4(b). The corresponding U function is the product of U functions for each loop:

$$U_{\{\alpha,\beta,\epsilon,\eta\}} = z_\gamma z_\delta\,. \tag{56}$$

For $U_{\{\alpha,\beta,\gamma\}}$ we find, using (50) and (51),

$$\begin{aligned} U_{\{\alpha,\beta,\gamma\}} &= z_\delta B_{\delta\delta} + z_\eta B_{\delta\eta} \\ &= z_\delta(z_\epsilon + z_\eta) + z_\eta z_\epsilon\,. \end{aligned} \tag{57}$$

Thus we finally arrive at

$$B_{\alpha\beta} = z_\delta(z_\gamma + z_\epsilon + z_\eta) + z_\epsilon z_\eta\,. \tag{58}$$

This result is included in Fig. 3(c).

We note that, in the diagram of Fig. 2(d), there are only two topologically distinct functions $B_{\alpha\beta}$ for $\alpha \neq \beta$. Namely, they are $B_{\alpha\beta}$ and $B_{\gamma\delta}$. Functions of the type $B_{\alpha\alpha}$ are not independent and can be expressed, using Kirchhoff's first law, in terms of others. For example,

$$B_{\alpha\alpha} = B_{\alpha\beta} - B_{\alpha\delta} \tag{59}$$

for Fig. 2(d). If U is known already, $B_{\alpha\alpha}$ may also be obtained from the formula

$$B_{\alpha\alpha} = \frac{\partial U}{\partial z_\alpha}\,, \tag{60}$$

which is valid for any diagram. This formula is obtained by differentiating both sides of (50) with respect to z_α and noting that $B_{\alpha\beta}$ does not contain z_α as is seen from (51).

For the "pretzel" diagram Fig. 2(c) there are three topologically distinct functions $B_{\alpha\beta}$, $B_{\alpha\gamma}$, and $B_{\alpha\delta}$. Thus the formula (51) has to be used only five times to calculate all $B_{\alpha\beta}$ for all three-loop diagrams.

All $B_{\alpha\beta}$ functions for diagrams of up to three internal loops are shown in Fig. 3. If either α or β direction is reversed, $B_{\alpha\beta}$ changes its sign.

IV. FURTHER FORMULAS

This section consists of various extensions of the parametric method which are not essential to the basic derivation, but are necessary or useful in applying the rules of Sec. II to actual calculation or general study of parametric integrals.

A. Schwinger-Nambu representation[1]

If we use the identity

$$\frac{\Gamma(r)}{(V - i\epsilon)^r} = i^r \int_0^\infty dt\, t^{r-1} e^{-it(V - i\epsilon)} \tag{61}$$

and rescale $z_i \to z_i/t$ in (5) we obtain an alternative form of the parametric integral[12]

$$\frac{(-i)^N}{(16\pi^2 i)^n} \int \frac{dz_G}{U^2} e^{-iV}\,, \tag{62}$$

where each m_i^2 in V is understood to have an infinitesimal negative imaginary part. While this form is not suitable for numerical calculation, it is very convenient for renormalization arguments because the integrand of (62) factorizes trivially in the domain of the parametric space where the ultraviolet divergence takes place. All the rules of Sec. II apply if we replace (5) by (62), and (7) by

$$\mathrm{F}\, e^{-iV} = \left[F_0 + \frac{1}{iU} F_1 + \left(\frac{1}{iU}\right)^2 F_2 + \cdots\right] e^{-iV}\,. \tag{63}$$

B. Regularization

In general formula (10) for the Feynman amplitude is not well defined because of ultraviolet di-

vergences and must be regularized to be meaningful. We find it most convenient to use the dimensional regularization,[20] implemented by the analytic continuation of the momentum integration formula (30) into 2ω complex dimensions

$$\int \frac{d^{2\omega}p}{(p^2-m^2+i\epsilon)^n} = i(-\pi)^{\omega}\frac{\Gamma(n-\omega)}{\Gamma(n)}\frac{1}{(-m^2+i\epsilon)^{n-\omega}}. \tag{64}$$

The dimensionally regularized version of the parametric integral (5) is

$$\left(\frac{i}{(4\pi)^{\omega}}\right)^n (-1)^N \Gamma(N-\omega n)\int \frac{dz_G\delta(1-z_G)}{U^{\omega}(V-i\epsilon)^{N-\omega n}}, \tag{65}$$

and the Schwinger-Nambu form corresponding to (62) is

$$\left(\frac{i}{(4\pi i)^{\omega}}\right)^n (-i)^N \int \frac{dz_G}{U^{\omega}} e^{-iV}. \tag{66}$$

All the rules of Sec. II remain unchanged, except that formula (7) is replaced by a straightforward generalization

$$\mathsf{F}\frac{1}{V^m} = \frac{F_0}{V^m} + \frac{\Gamma(m-1)}{\Gamma(m)U}\frac{F_1}{V^{m-1}} + \frac{\Gamma(m-2)}{\Gamma(m)U^2}\frac{F_2}{V^{m-2}} + \cdots, \tag{67}$$

where m is a complex number, and the spinor traces and contractions appearing in F_i have to be evaluated by generalized rules.[20]

While the dimensional regularization is adequate for QED calculations, in practice it is simpler to avoid treating divergences of vacuum-polarization electron loops by using the Källén-Lehmann spectral representation for renormalized photon propagators. In case of second-order electron loops, this amounts to replacing the ith photon propagator $(p_i^2-\lambda^2+i\epsilon)^{-1}$ by[21,22]

$$\frac{\alpha}{\pi}\int_0^1 dt \frac{t^2(1-\frac{1}{3}t^2)}{1-t^2}\frac{1}{p_i^2-[4m^2/(1-t^2)]+i\epsilon}. \tag{68}$$

Fourth-order vacuum-polarization loops can be handled in the same manner.[22] It is easy to incorporate these formulas in our parametric scheme.

C. Examples of Feynman integrals

We shall list the three simplest types of Feynman integrals for illustration of the rules of Sec. II.

Electron self-energy:

$$\Sigma^{(2n)} = -\left(\frac{-\alpha}{4\pi}\right)^n (n-2)!\,\mathsf{F}\int \frac{dz_G\delta(1-z_G)}{U^2V^{n-1}}. \tag{69}$$

Vertex:

$$\Gamma_{\nu}^{(2n)} = \left(\frac{-\alpha}{4\pi}\right)^n (n-1)!\,\mathsf{F}_{\nu}\int \frac{dz_G\delta(1-z_G)}{U^2V^n}. \tag{70}$$

Photon self-energy:

$$\Pi_{\mu\nu}^{(2n)} = -\left(\frac{-\alpha}{4\pi}\right)^n (n-2)!\,\mathsf{F}_{\mu\nu}\int \frac{dz_G\delta(1-z_G)}{U^2V^{n-1}}. \tag{71}$$

Our definitions of Σ, Γ, and Π are identical with those of Ref. 7. The suffix $2n$ is the order of the contribution and F, F_{ν}, and $\mathsf{F}_{\mu\nu}$ are the numerator expressions consisting of γ^{μ} from vertices, $(\not{p}_j+m_j)$ from electron lines, and appropriate spinor factors. If the corresponding diagrams contain fermion loops, additional factors of -1 for each loop should be included.

D. Scalar currents

In many applications it is convenient to replace the vectors $Q_i'^{\mu}$ by some scalar functions. Suppose an external momentum p^{μ} enters the graph G at point A and leaves it at point B. Then we may write

$$Q_i'^{\mu} = A_i^{(AB)}p^{\mu} + \text{linear combinations of other independent external momenta}. \tag{72}$$

The coefficient $A_i^{(AB)}$, which is completely determined by the Kirchhoff laws and represents a fraction of the "current" p^{μ} flowing through the line i, will be called scalar current. In order to find an explicit formula for $A_i^{(AB)}$, let us note that one choice of constant momenta q_j consistent with momentum conservation is

$$q_j^{\mu} = \eta_{jP}p^{\mu} + \text{other terms}, \tag{73}$$

where $P=P(AB)$ is any self-nonintersecting path starting at A and ending at B, and $\eta_{jP}=(1,-1,0)$ according as the line j lies (along, against, outside of) the path P. Substituting (72) and (73) in (35), we obtain

$$A_i^{(AB)} = -\frac{1}{U}\sum_{j=1}^{N}\eta_{jP}z_jB'_{ji}, \quad P=P(AB). \tag{74}$$

By choosing $P(AB)$ to be the shortest possible path, we can obtain a very compact expression for $A_i^{(AB)}$. We shall drop the superscript (AB) in $A_i^{(AB)}$ if it does not cause confusion.

While the formula (74) is efficient for numerical computations, a direct topological formula[2]

$$A_i^{(AB)} = \frac{1}{U}\sum_{P(AB)}\eta_{iP}U_P \tag{75}$$

is more powerful for other purposes. Here the summation goes over all distinct paths $P(AB)$ connecting the vertices A and B, and U_P is ob-

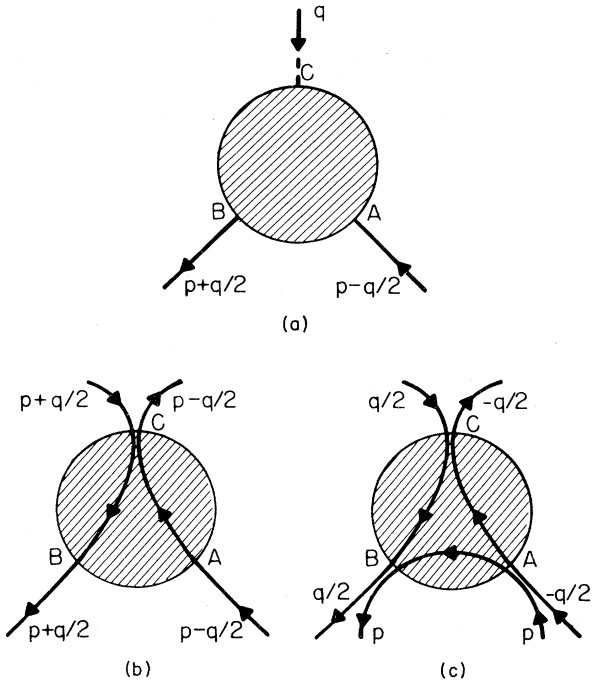

FIG. 5. Vertex diagram with given external momenta is shown in (a). Some possible paths of external momenta are indicated in (b) and (c).

tained from U by shrinking the path P to a point.

To illustrate the usefulness of scalar currents, we shall compute $Q_i'^{\mu}$ and G of (35) and (36) for a vertex diagram of Fig. 5(a). Alternative definitions of external momenta and their possible paths through the diagram are indicated in Figs. 5(b) and 5(c). The most efficient choice will depend on the particular diagram. If we follow the paths in Fig. 5(b), the constant momenta are given by

$$\begin{aligned} q_i &= p - \tfrac{1}{2}q\,, \quad i \in P(AC) \\ q_i &= p + \tfrac{1}{2}q\,, \quad i \in P(CB) \end{aligned} \tag{76}$$

and zero otherwise. "Kirchhoff's currents" $Q_i'^{\mu}$ then become

$$Q_i'^{\mu} = A_i^{(AC)}(p - \tfrac{1}{2}q)^{\mu} + A_i^{(CB)}(p + \tfrac{1}{2}q)^{\mu}\,. \tag{77}$$

If we choose the paths of Fig. 5(c), we obtain instead

$$Q_i'^{\mu} = A_i^{(AB)}p^{\mu} - A_i^{(AC)}\tfrac{1}{2}q^{\mu} + A_i^{(CB)}\tfrac{1}{2}q^{\mu}\,. \tag{78}$$

Comparing (77) and (78) we find

$$A_i^{(AB)} = A_i^{(AC)} + A_i^{(CB)}\,, \tag{79}$$

which is also obvious from the definition (74). Substituting (77) in (36) we obtain

$$G = m^2 \sum_{i \in P(AB)} z_i A_i^{(AB)} - \frac{1}{2}q^2 \left(\sum_{i \in P(AC)} z_i A_i^{(CB)} + \sum_{i \in P(CB)} z_i A_i^{(AC)} \right), \tag{80}$$

where we have assumed for simplicity that the external lines at A and B are on the mass shell $(p \pm \frac{1}{2}q)^2 = m^2$.

E. Nakanishi's identity

The Ward identity applied to Feynman diagrams in quantum electrodynamics relates each vertex diagram to a self-energy diagram obtained from it by removing the external vertex and one of the adjacent internal lines. In the momentum space the Ward identity reduces ultimately to a simple algebraic identity,

$$\frac{p_i \cdot p_j - m^2}{(p_i^2 - m^2)(p_j^2 - m^2)} = \frac{1}{(p_i^2 - m^2)} \quad \text{if } p_i = p_j\,. \tag{81}$$

The corresponding identity in the parametric representation is obtained by considering

$$(D_j \cdot D_{j'} - m_j^2) \frac{1}{(p_{j'}^2 - m_{j'}^2)\prod_{i=1}^{N}(p_i^2 - m_i^2)} = \frac{1}{\prod_{i=1}^{N}(p_i^2 - m_i^2)}, \tag{82}$$

where we set $p_j = p_{j'}$, $m_j = m_{j'}$ on the left-hand side only after $D_j^{\mu}, D_{j'}^{\nu}$ operations have been carried out. Let us refer to the diagram obtained from G by replacing the propagator $(p_j^2 - m_j^2)^{-1}$ by $(p_j^2 - m_j^2)^{-1}(p_{j'}^2 - m_{j'}^2)^{-1}$ as (mass insertion) diagram G^*. Parametrizing both sides as was done in Sec. III, formulas (18)–(31), we obtain Nakanishi's identity[2]

$$(-D_j \cdot D_{j'} + m_j^2) \int \frac{dz_{G^*}\delta(1 - z_{G^*})}{U_{G^*}^2 V_{G^*}^{N-2n+1}} = \frac{1}{N - 2n} \int \frac{dz_G \delta(1 - z_G)}{U_G^2 V_G^{N-2n}}\,. \tag{83}$$

Since the lines j and j' carry the same momentum and mass, V_{G^*} will depend only on the sum of their Feynman parameters $z_{jj'} = z_j + z_{j'}$. U_{G^*} also depends only on $z_{jj'}$, so that we can perform one z integration using the rule

$$\int_0^z dz_i \int_0^{z - z_i} dz_j F(z_i + z_j) = \int_0^z z_{ij}\, dz_{ij} F(z_{ij})\,. \tag{84}$$

If we now rename $z_{jj'}$ as z_j on the left-hand side of (83) we obtain

$$\int \frac{dz_G \delta(1 - z_G)}{U^2} \left[z_j(-D_j \cdot D_{j'} + m_j^2) \frac{N - 2n}{V^{N-2n+1}} - \frac{1}{V^{N-2n}} \right] = 0\,. \tag{85}$$

A more rigorous derivation of this identity was given by Nakanishi.[2] In his proof he makes essential use of the property that U and V are homogeneous functions of degree n and 1 in z, respectively. It is easy to generalize it to integrals involving B_{ij} and A_j, which are homogeneous functions of degree $n-1$ and 0. Suppose $H(z)$ is any homogeneous function of degree h in z. Then we have

$$\int \frac{dz_G \delta(1-z_G)}{U^2}\left[z_j(-D_j\cdot D_{j'}+m_j^2)\frac{(N-2n+h)H}{U^2V^{N-2n+h+1}}-\frac{(1+z_j\partial/\partial z_j)H}{U^2V^{N-2n+h}}\right]=0. \tag{86}$$

These rules, applied to $H=A_i$, G, etc., will be useful in the discussion of renormalization.

F. Some parametric formulas for self-energy insertions

We give some relations that will be needed for the consideration of ultraviolet and infrared divergences arising from divergent subdiagrams of an arbitrary Feynman diagram.

Consider a diagram G obtained by inserting a "self-energy" diagram S into the line i (thus splitting it into two lines i and i'; see Fig. 6) of the diagram T. The lines belonging to T and S will be denoted $i,j,k,\ldots$ and $m,n,\ldots$, respectively. In formula (51), whenever a circuit c contains lines i,i', U_c factors into a product of a function depending only on the parameters of S and a function depending only on the parameters of T. For example, for the circuit $c=i,m,i',j$ in Fig. 7(a), we have

$$U_c=z_{m'}z_k, \tag{87}$$

as is seen from Fig. 7(b). Generalizing these results we find from (51)

$$B_{ii'}=B_{i'i}=\left(\sum_{P(CD)}U_P^S\right)\left(\sum_c \eta_{ic}\eta_{i'c}U_c^T\right), \tag{88}$$

where the superscripts S,T indicate that the corresponding quantities are defined on the diagrams S,T alone, and the P summation is over all possible paths between vertices C and D of the self-energy diagram S. This factorization has occurred because all circuits c that give nonvanishing contributions to B_{ii} must go through lines i,i'. Making use of the formula[18,19]

$$U^S=\sum_{P(CD)}U_P^S \tag{89}$$

and (51), we can rewrite (88) as

$$B_{ii}=U^SB_{ii}^T, \quad i\in T. \tag{90}$$

Other B functions that factorize similarly are

$$\begin{aligned} &B_{ij}=U^SB_{ij}^T, \quad i,j\in T\\ &B_{im}=A_m^SU^SB_{ii}^T, \quad m\in S,\ i\in T\\ &B_{jm}=A_m^SU^SB_{ij}^T, \quad m\in S,\ i,j\in T \end{aligned} \tag{91}$$

where we have used (75).

In particular, when an external momentum p^μ enters the subdiagram T at vertex A and leaves T at vertex B, an interesting factorization of the scalar current $A_m, m\in S$, occurs. As a special case of (74) we have

$$A_m=-\frac{1}{U}\sum_{j\in G}\eta_{jP}z_jB'_{mj}, \tag{92}$$

where $P=P(AB)$ is any path connecting the two external vertices A and B of T. We can always choose a path entirely contained within T; then we have, using (91),

$$\begin{aligned} A_m&=-\frac{1}{U}\sum_{j\in T}\eta_{jP}z_jB_{mj}\\ &=A_m^S\left(-\frac{U^S}{U}\sum_{j\in T}\eta_{jP}z_jB_{ij}^T\right)\\ &=\frac{U^SU^T}{U}A_m^SA_i^T \end{aligned} \tag{93}$$

and

$$A_j=\frac{U^SU^T}{U}A_j^T. \tag{94}$$

Substituting (94) into (93), we obtain

$$A_m=A_m^SA_i. \tag{95}$$

G. Further topological functions

There are topological functions besides U and $B_{\alpha\beta}$ that can be defined over a chain diagram. For instance, in carrying out the operation (7) one often encounters the following function:

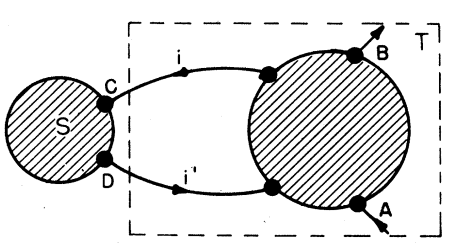

FIG. 6. Insertion of a self-energy diagram S in the line i of the diagram T.

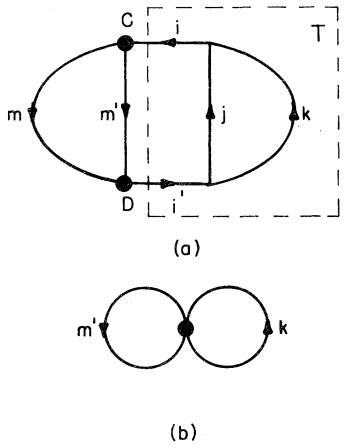

FIG. 7. (a) An example of a diagram of the type drawn in Fig. 6. (b) Reduced diagram obtained by shrinking the loop $\{i, m', i', k\}$ in the diagram in (a).

$$B_{\alpha\beta,\gamma\delta} = \frac{1}{U}\begin{vmatrix} B_{\alpha\beta} & B_{\gamma\beta} \\ B_{\alpha\delta} & B_{\gamma\delta} \end{vmatrix}. \tag{96}$$

Of course this function is known once $B_{\alpha\beta}$'s are given, but the evaluation is much easier if one uses a direct topological formula analogous to (51) (see Appendix for derivation):

$$B_{\alpha\beta,\gamma\delta} = \sum_{r,s}{}'(\eta_{\alpha r}\eta_{\beta r}\eta_{\gamma s}\eta_{\delta s} - \eta_{\alpha r}\eta_{\beta s}\eta_{\gamma s}\eta_{\delta r})U_{r*s}, \tag{97}$$

where the prime is attached to emphasize that the summation is over all topologically distinct unions of loops r and s, and U_{r*s} is the U function for the diagram obtained by shrinking the loops r and s to points.

As an example of application of (97) let us calculate $B_{\eta\delta,\gamma\beta}$ for the diagram of Fig. 2(d). The product $\eta_{\eta r}\eta_{\delta r}\eta_{\gamma s}\eta_{\beta s}$ is nonvanishing for $r = c = \{\alpha, \delta, \eta\}$, $s = c' = \{\alpha, \beta, \gamma\}$, or $r = \{\beta, \gamma, \eta, \delta\}$, $s = \{\alpha, \beta, \gamma\}$, or $r = \{\alpha, \delta, \eta\}$, $s = \{\beta, \gamma, \eta, \delta\}$. Since $r \cup s = \{\alpha, \beta, \gamma, \delta, \eta\}$ in all cases, they give rise to only one topologically distinct contribution $U_{c*c'}$ to (97). Similarly the only contribution to the product $\eta_{\eta r}\eta_{\delta s}\eta_{\gamma s}\eta_{\beta r}$ may arise from the cases corresponding to $r \cup s = \{\alpha, \beta, \gamma, \delta, \eta, \epsilon\}$. However, we find $U_{r*s} = 0$ in this case since all lines are shrunk to points. Thus we obtain

$$B_{\eta\delta,\gamma\beta} = U_{c*c'} = z_\epsilon. \tag{98}$$

We recall here that the argument leading to (97) applies to diagrams with three or more independent loops. The evaluation of $B_{\eta\delta,\gamma\beta}$ from the determinant (96) involves about 30 terms of the form $z_\alpha z_\beta z_\gamma z_\delta$ which reduces to 16 terms by cancellation. These terms then factor into the form $z_\epsilon U$. The advantage of the topological formula (97) is thus evident.

If not all indices of $B_{\alpha\beta,\gamma\delta}$ are different, we have a simpler formula[23] analogous to (60):

$$B_{\alpha\alpha}B_{\beta\gamma} - B_{\alpha\beta}B_{\alpha\gamma} = U\frac{\partial B_{\beta\gamma}}{\partial z_\alpha}. \tag{99}$$

The simplest chain diagram to which this formula applies is the two-loop diagram of Fig. 2(b). In this case we have

$$B_{\alpha\alpha,\beta\gamma} = 1 \tag{100}$$

analogous to (54).

We shall also need a corresponding relation for B'_{ij}. Noting that $(1 - z_i\,\partial/\partial z_i)$ applied on any expression linear in z_i is equivalent to setting $z_i = 0$, we obtain

$$\frac{z_i}{U}(B'_{ik}B'_{il} - B'_{kl}B'_{ii}) = B'_{kl}\big|_{z_i=0} \tag{101}$$

from (99).

V. EXAMPLE

We shall illustrate the parametric Feynman-Dyson rules of Sec. II by applying them to the vertex diagram of Fig. 1(a). The corresponding chain diagram is shown in Fig. 1(b). The chains are $\{1, 6, 9\}$, $\{2, 4, 5\}$, $\{3\}$, $\{7\}$, and $\{8\}$. For simplicity let us introduce the following notation for chain parameters:

$$z_\alpha = z_{169} = z_1 + z_6 + z_9, \text{ etc.} \tag{102}$$

1. We read off B_{ij} from Fig. 3(b),

$$\begin{aligned} B_{33} &= (z_7 + z_{245})(z_8 + z_{169}) + z_8 z_{169}, \\ B_{37} &= -z_{245}(z_8 + z_{169}) - z_8 z_{169}, \end{aligned} \tag{103}$$

etc., until all necessary B_{ij} are found.

2. The simplest expression for the function U is obtained by choosing the loop $\{3, 7\}$:

$$U = z_3 B_{33} - z_7 B_{37}. \tag{104}$$

If we choose the loop $\{1, 6, 9, 8\}$, we obtain another simple result

$$U = z_{169}B_{11} - z_8 B_{18}. \tag{105}$$

3. In this example the parametric functions $Q_i'^\mu$ depend on the external momenta p, q as $Q_i'^\mu = A_i p^\mu + a_i q^\mu$. Let us temporarily ignore q^μ in Fig. 1(a) and consider the p^μ routing only. A possible routing is $q_9 = -p$ and $q_i = 0$ for all other lines. This choice yields the simplest result (setting $p^2 = m^2 = 1$)

$$\begin{aligned} A_i &= z_9 B_{1i}/U, \quad i = 1, 2, \ldots, 8 \\ A_9 &= A_1 - 1. \end{aligned} \tag{106}$$

Next, letting q go through lines 5, 6 and $-\frac{1}{2}q$ through the line 9, we obtain

$$a_i = -(z_5 B'_{5i} + z_6 B'_{6i} - \tfrac{1}{2} z_9 B'_{9i})/U . \tag{107}$$

In the static limit ($q^\mu \to 0$) V depends only on A_i. Routing p^μ through the line 9 we obtain a very simple expression ($p^2 = m^2 = 1$)

$$V = \sum_{i=1}^{9} z_i m_i^2 + z_9 A_9 . \tag{108}$$

4. We have three loop integrations and nine lines. Thus the parametric integral is given by

$$-2\left(\frac{i}{16\pi^2}\right)^3 \int \frac{dz}{U^2 V^3}, \tag{109}$$

where the integration domain is given by

$$dz = \delta\left(1 - \sum_{j=1}^{9} z_j\right) \prod_{i=1}^{9} dz_i . \tag{110}$$

5. Now we collect all the factors (rules a through g) to obtain

$$M = -ie_0 \sqrt{Z_3}\, Z_2 (-\tfrac{1}{32}) \left(\frac{\alpha}{\pi}\right)^3 \epsilon_\nu \mathsf{F}^\nu \int \frac{dz}{U^2 V^3}, \tag{111}$$

where

$$\mathsf{F}^\nu = \bar{u}(p + \tfrac{1}{2}q)\gamma^\alpha (\not{p}_6 + m)\gamma^\beta (\not{p}_5 + m)\gamma^\nu (\not{p}_4 + m)\gamma^\lambda (\not{p}_3 + m)$$
$$\times \gamma_\lambda (\not{p}_2 + m)\gamma_\beta (\not{p}_1 + m)\gamma_\alpha u(p - \tfrac{1}{2}q) . \tag{112}$$

6. The F^ν operation on $1/V^3$ yields an F_0^ν term identical in form with F^ν, except that all $\not{p}_i$ are replaced by $\not{q}'_i$. The computation of F_1^ν term in (7) is indicated schematically in Fig. 8(a). It is readily recognized that it is best to leave this computation to a computer[24] because as many as 5×4 distinct terms may be generated, although the final result will have only B_{11}, B_{12}, B_{13}, and B_{23} terms appearing in it because of the relations like $B_{1i} = B_{6i}$, $B_{2i} = B_{4i} = B_{5i}$. To clarify the meaning of Fig. 8(a), we give the first term of F_1^ν in Dirac notation:

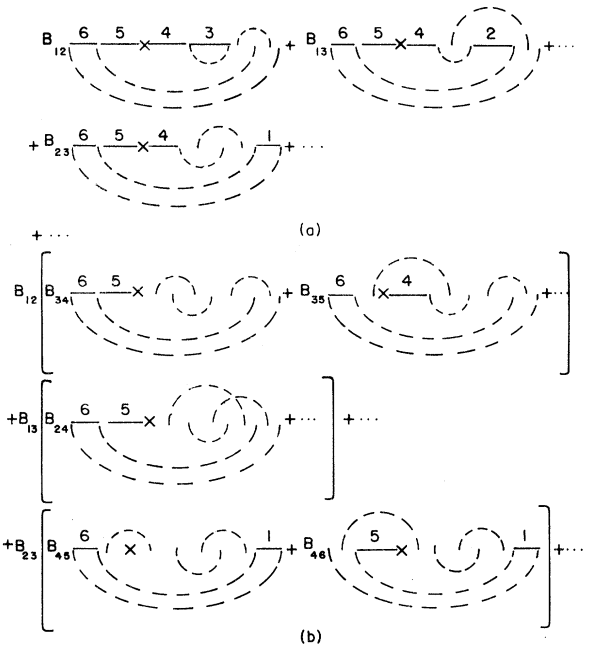

FIG. 8. (a) Graphical representation of the terms contributing to $-2F_1$. (b) Graphical representation of the terms contributing to $4F_2$. Note that each distinct double contraction appears only once.

$$F_1^\nu = \bar{u}(p + \tfrac{1}{2}q)[-\tfrac{1}{2} B_{12}\gamma_\alpha (\not{q}'_6 + m)\gamma_\beta (\not{q}'_5 + m)\gamma^\nu (\not{q}'_4 + m)$$
$$\times \gamma^\lambda (\not{q}'_3 + m)\gamma_\lambda \gamma^\mu \gamma^\beta \gamma_\mu \gamma^\alpha + \cdots]$$
$$\times u(p - \tfrac{1}{2}q) . \tag{113}$$

Computation of F_2^ν is indicated in Fig. 8(b). In this example, however, the coefficient of the F_3^ν term in (7) is undefined. This reflects the presence of an overall ultraviolet divergence in M. As it stands, M also contains various ultraviolet divergences arising from divergent subdiagrams. In a subsequent publication we shall present a general procedure for removing such divergences.[14] For now we give a form of (111) regularized by the prescription (65):

$$M = -ie_0 \sqrt{Z_3}\, Z_2 \left(-\frac{e^2}{(4\pi)^\omega}\right)^3 \epsilon_\nu \int dz \left[\frac{\Gamma(9-3\omega)}{U^\omega V^{9-3\omega}} F_0^\nu + \frac{\Gamma(8-3\omega)}{U^{\omega+1} V^{8-3\omega}} F_1^\nu + \frac{\Gamma(7-3\omega)}{U^{\omega+2} V^{7-3\omega}} F_2^\nu + \frac{\Gamma(6-3\omega)}{U^{\omega+3} V^{6-3\omega}} F_3^\nu\right] . \tag{114}$$

ACKNOWLEDGMENT

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APPENDIX: DERIVATION OF FORMULA (97)

To derive (97) let us rewrite (96) as

$$B_{\alpha\beta,\gamma\delta} = \frac{1}{U} \sum_{ss'tt'}^{n} \eta_{\alpha s}\eta_{\beta s'}\eta_{\gamma t}\eta_{\delta t'} \begin{vmatrix} \Delta_{ss'} & \Delta_{ts'} \\ \Delta_{st'} & \Delta_{tt'} \end{vmatrix} , \tag{A1}$$

making use of (27), or equivalently

$$B_{\alpha\beta} = \sum_{s,s'=1}^{n} \eta_{\alpha s}\eta_{\beta s'}\Delta_{ss'} , \tag{A2}$$

where Δ_{st} is the cofactor of U_{st}:

$$\Delta_{st} = U(\mathfrak{U}^{-1})_{st} , \tag{A3}$$

and $\mathfrak{U}$ is the $n \times n$ matrix (U_{st}). Using Jacobi's theorem for determinants[25]:

$$\begin{vmatrix} \Delta_{ss'} & \Delta_{ts'} \\ \Delta_{st'} & \Delta_{tt'} \end{vmatrix} = (-1)^{s+s'+t+t'} U \det(\mathfrak{U}(st/s't')), \tag{A4}$$

where $\mathfrak{U}(st/s't')$ is the matrix obtained from $\mathfrak{U}$ by deleting the rows s, t and the columns s', t', we can put (A1) in the form

$$B_{\alpha\beta,\gamma\delta} = \sum_{ss'tt'}^{n} \eta_{\alpha s}\eta_{\beta s'}\eta_{\gamma t}\eta_{\delta t'}(-1)^{s+s'+t+t'} \times \det(\mathfrak{U}(st/s't')). \tag{A5}$$

Note that the U's in (A1) and (A4) have canceled out.

To identify (A5) with the circuit-representation formula (97), first note that, if we set $z_c = 0$, $z_{c'} = 0$ for any pair of loops c, c', we have

$$\det(\mathfrak{U}(st/s't')) = 0, \tag{A6}$$

except in one of the following four cases:

$$\begin{aligned} &s = s' = c, \quad t = t' = c'; \\ &s = t' = c, \quad s' = t = c'; \\ &s = s' = c', \quad t = t' = c; \\ &s = t' = c', \quad s' = t = c. \end{aligned} \tag{A7}$$

In other words, only those terms proportional to $\det(U(cc'/cc'))$ are nonvanishing. But we have

$$\det(U(cc'/cc'))|_{z_c = z_{c'} = 0} = U_{c*c'}, \tag{A8}$$

where $U_{c*c'}$ is the U function for the diagram obtained by shrinking the loops c and c' to points. Thus, for any pair of loops c, c', we find

$$\begin{aligned} B_{\alpha\beta,\gamma\delta}|_{z_c = z_{c'} = 0} = (&\eta_{\alpha c}\eta_{\beta c}\eta_{\gamma c'}\eta_{\delta c'} - \eta_{\alpha c}\eta_{\beta c'}\eta_{\gamma c'}\eta_{\delta c} \\ &+ \eta_{\alpha c'}\eta_{\beta c'}\eta_{\gamma c}\eta_{\delta c} - \eta_{\alpha c'}\eta_{\beta c}\eta_{\gamma c}\eta_{\delta c'}) \\ &\times U_{c*c'}. \end{aligned} \tag{A9}$$

Now let us define

$$R_{\alpha\beta,\gamma\delta} = \sum_{r,s}{}' (\eta_{\alpha r}\eta_{\beta r}\eta_{\gamma s}\eta_{\delta s} - \eta_{\alpha r}\eta_{\beta s}\eta_{\gamma s}\eta_{\delta r}) U_{r*s}, \tag{A10}$$

where the prime is attached to emphasize that the summation is over all topologically distinct unions of loops r and s. Using (A9) we find that $R_{\alpha\beta,\gamma\delta} - B_{\alpha\beta,\gamma\delta}$, which is a homogeneous form of degree $n-2$ in z, vanishes for $z_c = z_{c'} = 0$. [If the summation in (A10) is unrestricted, $R_{\alpha\beta,\gamma\delta}$ would not reduce correctly to $B_{\alpha\beta,\gamma\delta}$ for $z_c = z_{c'} = 0$ whenever c and c' overlap.] Since there are n-independent loops in the diagram, however, no homogeneous form of degree $n-2$ can vanish for $z_c = z_{c'} = 0$ for arbitrary pair c, c' unless it vanishes identically. Thus we obtain $R_{\alpha\beta,\gamma\delta} - B_{\alpha\beta,\gamma\delta} = 0$, which proves (97).

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New approach to the separation of ultraviolet and infrared divergences of Feynman-parametric integrals*

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A scheme for systematically separating ultraviolet divergences of Feynman amplitudes in parametric space is developed. It is summarized by an explicit formula which enables us to incorporate readily the ultraviolet-finite remainders thus constructed into the usual renormalization scheme. It is shown further that infrared divergences can be treated in a very similar way. Our method is particularly suitable for numerical integration.

I. INTRODUCTION

In order to evaluate Feynman integrals of higher orders numerically, it is necessary to locate and subtract the ultraviolet (UV) and infrared (IR) divergences beforehand. Since the removal of UV divergences is the essential aspect of the renormalization procedure, various prescriptions have been proposed in the literature for the extraction of UV-finite parts, although they vary in mathematical rigor and practicality depending on the purpose for which they have been formulated. On the other hand, the treatment of IR divergences has been relatively underdeveloped, particularly in the Feynman-parametric form. Thus we have found it necessary to develop some workable scheme.[1a] The purpose of this article is to present a general and systematic scheme for separating both UV and IR divergences of Feynman integrals, following the line first suggested in Ref. 1b. This method has been applied to the evaluation of sixth-order contributions to the electron magnetic moment.[2,3]

Our method is based on the parametric representation of Feynman integrals summarized in the preceding article,[4] hereafter referred to as I. It is particularly suited for numerical calculation because of the following properties:

(i) After the removal of divergences the integral is almost as simple as the original divergent integral.

(ii) The singularity is subtracted at each point of the domain of integration (rather than having cancellation of contributions from different parts of the domain).

(iii) Subtraction terms introduce no new singularities. (Note that the standard renormalization introduces infrared divergences.)

(iv) Subtraction terms are factorizable into lower-order expressions. Thus they are easier to evaluate analytically or numerically than the original integral.

(v) Our construction of UV and IR subtraction terms is also useful for crosschecking of trace calculation.

In Sec. II we review the UV power-counting rule for arbitrary Feynman integrals and propose a method for removing all leading UV singularities of parametric integrands. In Sec. III we apply it to QED and derive an expression for Dyson-Salam-renormalized amplitudes in terms of finite integrals. A power-counting rule for the degree of superficial IR divergence is developed in Sec. IV for arbitrary QED amplitudes by examining the properties of their denominators. In Sec. V it is extended to the whole integrand, taking account of the structure of numerator functions. A method for removing all IR divergences of QED