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Critical behavior in a class of O(N)-invariant field theories in two dimensions*

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Critical behavior in a class of two-dimensional field theories which exhibit dynamical symmetry breaking at zero temperature is analyzed in the $1/N$ approximation. We show that, in the case of an O(N)-invariant theory of massless, N-component, Fermi fields, a phase transition takes place in the limit as N goes to infinity. The critical temperature, above which the model becomes symmetric, is given in terms of the induced fermion mass at zero temperature, m_f^0, as $m_f^0\beta_c = 1.764$. The equivalence between the critical parameters of the theory and those predicted by the BCS theory of superconductivity is established. We show that the BCS gap equation follows from the stability conditions imposed on the effective potential. The phase transition is discussed in a thermodynamical analog of the model. The analysis of the symmetry behavior of the theory is carried out by functional methods.

I. INTRODUCTION

Recently, Gross and Neveu analyzed a class of two-dimensional field theories of N-component, massless fermions with O(N)-invariant quartic interactions.[1] They showed that the fermions acquired a mass via dynamical symmetry breaking.

The possibility for the restoration of certain symmetries as a consequence of finite-temperature effects has recently been considered by several authors,[2-4] who found critical behavior in some cases of spontaneous symmetry violation.

In this paper we investigate the behavior of the two-dimensional O(N) fermion theories and show the existence of a second-order phase transition. The study of the symmetry behavior of the model is carried out by use of the effective potential formalism. Since the methods of computation as well as the physical meaning of the effective potential and its role in the investigation of symmetry breaking have been treated extensively in the literature,[5-7] we will avoid detailed calculations and definitions of the methods employed. In the $1/N$ approximation,[8] which seems to be consistent in the theory treated here, calculations are greatly simplified by the use of a combinatoric trick.[1,9] To avoid possible inconsistencies, we use the imaginary-time formalism in our finite-temperature calculations.[4,10,11]

The paper is organized as follows: In Sec. I we obtain the finite-temperature generalization of the O(N) fermion model. We exhibit the symmetry-breaking solution as well as the critical temperature, above which the model regains its symmetry. Here we also obtain an equation for the temperature-dependent mass, and solve it explicitly, in the limits $\beta \gtrsim \beta_c$ and $\beta \gg \beta_c$. In Sec. II we identify the temperature-dependent fermion mass with the BCS gap function, Δ_β. With this identification established, we show that the BCS gap equation is obtained as the solution to the stability condition for the effective potential. Also in this section we draw a thermodynamical analog of the model and briefly discuss the phase transition in this context.

Throughout our investigation we will consider only those states for which the vacuum is translationally invariant. Therefore, we will take the classical fields, and hence the effective potential, to be space-time independent.

II. THE O(N) FERMION MODEL

This model describes a theory of massless, N-component Fermi fields, ψ, with O(N)-invariant quartic self-interactions. The Lagrangian density for the theory is given by

$$\mathcal{L}[\psi] = i\bar{\psi}\not{\partial}\psi + \frac{\lambda}{2N}(\bar{\psi}\psi)^2 . \quad (1.1)$$

The symmetry which prevents the fermions from acquiring a mass in perturbation theory is[12]

$$\psi \to \gamma^5\psi . \quad (1.2)$$

The analysis of the symmetry behavior of the theory is considerably simplified by using the following combinatoric trick.[1,9] One introduces a constraint field, $\sigma(x)$, in the Lagrangian (1.1),

$$\begin{aligned}\mathcal{L}[\sigma,\psi] &= \mathcal{L}[\psi] - \tfrac{1}{2}[\sigma + (\lambda/N)^{1/2}\bar{\psi}\psi]^2 \\ &= i\bar{\psi}\not{\partial}\psi - (\lambda/N)^{1/2}\sigma\bar{\psi}\psi - \tfrac{1}{2}\sigma^2 . \end{aligned} \quad (1.3)$$

This new field has no effect in the dynamics of the theory, since from the point of view of functional integration the integral over $\sigma(x)$ merely multiplies the generating functional of the theory by an irrelevant constant. The simplification comes from the fact that the Feynman rules for the theory are now different. The introduction of σ makes it possible to distinguish all algebraically distinct graphs.[9] Thus, since no new dynamics emerge from the new Lagrangian, given by Eq. (1.3), it generates the same theory as Eq. (1.1). In this formulation, each vertex carries a factor of $N^{-1/2}$ and each closed fermion loop a factor of N. Thus, to order $(1/N)^0$, the only radiative corrections are to the σ propagator (which, in momentum space, is simply $-i$). Also, as will be seen, $\sigma \simeq N^{1/2}$ near the minimum (see Eq. 1.10). The temperature-dependent effective potential[4,11] is given, to this order, by

$$V^\beta(\sigma^2) = \tfrac{1}{2}\sigma^2 + i\int (dk)\ln\det[\not{k} - (\lambda/N)^{1/2}\sigma] , \quad (1.4)$$

where

$$\int (dk) = \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{(2\pi)}$$

and

$$k_0 = \begin{cases}(2n+1)\pi i/\beta & \text{(fermions)} \\ (2n)\pi i/\beta & \text{(bosons)}\end{cases} ,$$

and β is defined as usual, $\beta = 1/k_BT$. Hence,

$$V^\beta(\sigma^2) = \tfrac{1}{2}\sigma^2 - \frac{N}{\beta}\sum_n \int_{-\infty}^{\infty}\frac{dk_1}{(2\pi)}\ln\left[-\frac{(2n+1)^2}{\beta^2} - E_\sigma^{\,2}\right] , \quad (1.5)$$

where

$$E_\sigma^{\,2} = k_1^{\,2} + \frac{\lambda}{N}\sigma^2 .$$

Summing over n (see Ref. 4) gives

$$\begin{aligned}V^\beta(\sigma^2) &= \tfrac{1}{2}\sigma^2 - \frac{N}{\pi}\int_{-\infty}^{\infty} dk_1\left[\frac{E_\sigma}{2} + \frac{1}{\beta}\ln(1 - e^{\beta E})\right] \\ &= V^0(\sigma^2) + \bar{V}^\beta(\sigma^2) , \end{aligned} \quad (1.6a)$$

where

$$\begin{aligned}V^0(\sigma^2) &= \tfrac{1}{2}\sigma^2 - \frac{N}{2\pi}\int_{-\infty}^{\infty} dk_1\left(k_1^{\,2} + \frac{\lambda}{N}\sigma^2\right)^{1/2} \\ &= V^0_{\text{tree}} + V^0_1 \end{aligned} \quad (1.6b)$$

and

$$\bar{V}^\beta(\sigma^2) = -\frac{2N}{\pi\beta^2}\int_0^\infty dx \ln\left\{1 + \exp\left[-\left(x^2 + \frac{\lambda}{N}\beta^2\sigma^2\right)^{1/2}\right]\right\} . \quad (1.6c)$$

To see that this agrees with the zero-temperature result obtained in Ref. 1,

$$V(\sigma^2) = \frac{iN}{(2\pi)^2}\int d^2k \ln(k_0^{\,2} - E_\sigma^{\,2}) , \quad (1.7a)$$

notice that, up to a constant,[4]

$$\int_{-\infty}^{\infty} dk_0 \ln(-k_0^{\,2} + E_\sigma^{\,2} - i\epsilon) = iE_\sigma . \quad (1.7b)$$

Also, from Eq. (1.6c), $\bar{V}^\beta = 0$ at zero temperature. Thus, using these results, the renormalized, temperature-dependent effective potential is given by

$$\begin{aligned}V^\beta(\sigma^2) = \tfrac{1}{2}\sigma^2 &+ \frac{\lambda}{4\pi}\sigma^2\left[\ln\left(\frac{\sigma^2}{\sigma_0^{\,2}}\right) - 3\right] \\ &- \frac{2N}{\pi\beta^2}\int_0^\infty dx \ln\left\{1 + \exp\left[-\left(x^2 + \frac{\lambda}{N}\beta^2\sigma^2\right)^{1/2}\right]\right\} . \end{aligned} \quad (1.8)$$

We remark that all the terms in Eq. (1.8) are of order N. Now, at zero temperature, the symmetry $\psi \to \gamma^5\psi$ and $\sigma \to -\sigma$ is broken dynamically by the appearance of the σ bound state.[1] This can be seen from the first two terms in Eq. (1.8): The minima of V^0 occur away from the origin, which is a local maximum, at the points given by (see Fig. 1)

$$\sigma_A^0 = \pm\sigma_0 \exp(1 - \pi/\lambda) . \quad (1.9)$$

This gives the fermions a mass:

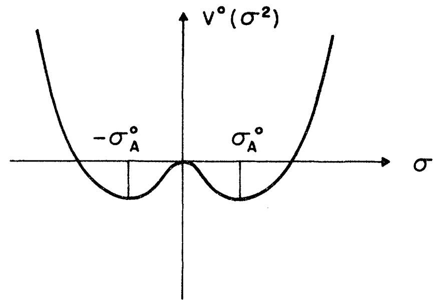

FIG. 1. $V^\beta(\sigma^2)$ at zero temperature.

$$m_F^0 = (\lambda/N)^{1/2}|\sigma_A{}^0| = (\lambda/N)^{1/2}\sigma_0 \exp(1-\pi/\lambda)\,. \quad (1.10)$$

What we will now investigate is whether or not the symmetry can be restored at some finite temperature.

At this point we prefer to keep the result given by Eq. (1.8), since we will not have to integrate the last term to carry out our planned analysis of the symmetry behavior of the theory.

Before proceeding to any calculations, we will state what it is we are looking for.

For a phase transition to take place, the broken-symmetry local maximum found before at the origin in σ space should change to an absolute minimum. Hence, what we must first do is look for all solutions to

$$\frac{\delta V^\beta(\sigma^2)}{\delta\sigma} = 0\,. \quad (1.11)$$

If a phase transition does occur it must be true that, for any temperature above the critical point,

$$\left.\frac{\delta V^\beta}{\delta\sigma}\right|_{\sigma=0} = 0 \quad (1.12a)$$

and

$$\left.\frac{\delta^2 V^\beta}{\delta\sigma^2}\right|_{\sigma=0} > 0\,. \quad (1.12b)$$

These conditions, however, only specify a local minimum, but as we shall see, for $\beta<\beta_c$, $\delta V^\beta/\delta\sigma|_{\sigma>0}>0$ away from the origin, so that $V^\beta(\sigma^2)$ is a monotonically increasing function of σ. Thus, in this temperature range the only real solution to Eq. (1.12a) is $\sigma=0$. We will see that Eq. (1.12b) is also satisfied at $\sigma=0$. Furthermore, we will find that $|\delta V^\beta/\delta\sigma|$ is itself monotonically increasing, which means that $V^\beta(\sigma^2)$ is concave upwards. From Eq. (1.8)

$$\frac{\delta V^\beta}{\delta\sigma} = \frac{\lambda}{\pi}\sigma\left[\frac{\pi}{\lambda} - 1 + \tfrac{1}{2}\ln\left(\frac{\sigma^2}{\sigma_0{}^2}\right) + 2\mathfrak{F}\left(\frac{\lambda}{N}\sigma^2\beta^2\right)\right], \quad (1.13)$$

where

$$\mathfrak{F}(a^2) = \int_0^\infty dx\, \frac{1}{1+\exp[(x^2+a^2)^{1/2}]}\,\frac{1}{(x^2+a^2)^{1/2}}\,. \quad (1.14)$$

$\mathfrak{F}(a^2)$ can be expanded in the following way[4]:

$$\mathfrak{F}(a^2) = -\tfrac{1}{4}\ln\frac{a^2}{\pi^2} + f(a^2)\,, \quad (1.15)$$

where

$$f(a^2) = \sum_{n=0}^\infty \frac{1}{(2n+1)}\left\{1 - \left[1 + \frac{a^2}{(2n+1)^2\pi^2}\right]^{-1/2}\right\}. \quad (1.16)$$

Therefore,

$$\frac{\delta V^\beta}{\delta\sigma} = \frac{\lambda}{\pi}\sigma\left[c + \tfrac{1}{2}\ln\frac{N\pi^2}{\lambda\sigma_0{}^2\beta^2} + 2f\left(\frac{\lambda}{N}\sigma^2\beta^2\right)\right], \quad (1.17)$$

where

$$c = \frac{\pi}{\lambda} - 1 - \gamma$$

and $\gamma = 0.577\ldots$ is the Euler constant.

Now,

$$\left.\frac{\delta^2 V^\beta}{\delta\sigma^2}\right|_{\sigma=0} = \frac{\lambda}{\pi}\left(c + \tfrac{1}{2}\ln\frac{N\pi^2}{\lambda\sigma_0{}^2\beta^2}\right), \quad (1.18)$$

since

$$f(a^2)|_{a=0} = f'(a^2)|_{a=0} = 0\,.$$

Thus, we see that Eq. (1.12b) can be satisfied for any $\beta<\beta_0$, where β_0 is given by

$$\left(\frac{\lambda}{N}\right)^{1/2}\sigma_0\beta_0 = \pi e^c\,. \quad (1.19)$$

Before asserting that a phase transition has taken place, we must first look for all other solutions to Eq. (1.12a) for $\beta<\beta_0$. These are given by the roots of

$$f\left(\frac{\lambda}{N}\sigma^2\beta^2\right) = -\frac{1}{2}\left(c + \tfrac{1}{2}\ln\frac{N\pi^2}{\lambda\sigma_0{}^2\beta^2}\right) = -\frac{1}{2}\left.\frac{\delta^2 V^\beta}{\delta\sigma^2}\right|_{\sigma=0}. \quad (1.20)$$

But we just proved that for $\beta<\beta_0$, $\delta^2 V^\beta/\delta\sigma^2|_{\sigma=0}>0$. And, since $f(a^2)$ converges for positive, finite a^2, then, for all $a^2>0$, $f(a^2)>0$, which implies that Eq. (1.20) has no real solutions.

Hence we can conclude safely that the temperature given by Eq. (1.19) is indeed the critical temperature we were looking for.

Also, from Eq. (1.17), it is clear that $|\delta V^\beta/\delta\sigma|$ is itself monotonically increasing as a function of σ for $\beta<\beta_0=\beta_c$, since $f'(a^2)>0$ for all $a^2\neq 0$. Thus

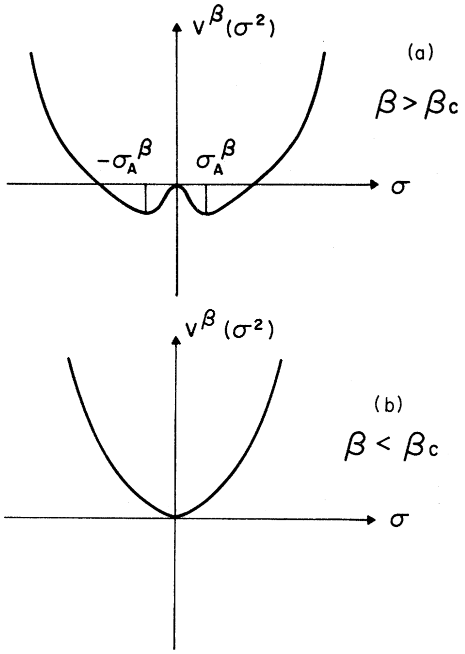

FIG. 2. $V^{\beta}(\sigma^2)$ for (a) $\beta > \beta_c$ and (b) $\beta < \beta_c$.

$V^{\beta}(\sigma^2)$ is concave upwards (Q.E.D.). (See Fig. 2.)

In order to express the critical temperature in terms of the only physical (dimensional) parameter of the theory [namely the induced fermion mass at zero temperature, given by Eq. (1.10)], we fix the renormalization point, σ_0. A convenient choice, which fixes the coupling constant as $\lambda = \pi$, is

$$\sigma_0{}^2 = \frac{N}{\lambda}(m_f^0)^2 . \tag{1.21}$$

Using this value in Eq. (1.19), we obtain

$$m_f^0 \beta_c = \pi e^{-\gamma} = 1.764 . \tag{1.22}$$

The temperature-dependent fermion mass is defined as $m_f^{\beta} = (\lambda/N)^{1/2}\sigma_A^{\beta}$, where σ_A^{β} is the solution of

$$\left.\frac{\delta V^{\beta}}{\delta\sigma}\right|_{\sigma=\sigma_A^{\beta}} = 0 \tag{1.23a}$$

and

$$\left.\frac{\delta^2 V^{\beta}}{\delta\sigma^2}\right|_{\sigma=\sigma_A^{\beta}} = 0 . \tag{1.23b}$$

Using Eq. (1.17) for $\sigma_A^{\beta} \neq 0$, we obtain

$$2f((m^{\beta})^2\beta^2) = \gamma + \ln\frac{m^0\beta}{\pi} . \tag{1.24}$$

(To simplify notation, we drop the subscript f.) The roots of this equation can only be obtained numerically. Instead of doing this, we will solve it in the approximation $\beta/\beta_c \gtrsim 1$ (i.e., for temperatures below but very nearly equal to the critical temperature). But we found that the solution to Eqs. (1.23a) and (1.23b) for $\beta = \beta_c$ is just $\sigma_A = 0$, so the approximation is equivalent to setting $m^{\beta} \ll 1$.

In this approximation,

$$f((m^{\beta})^2\beta^2) = \left(\frac{m^{\beta}\beta}{\pi}\right)^2 \frac{\alpha}{2} , \tag{1.25}$$

where

$$\alpha = \sum_n \frac{1}{(2n+1)^3} = \tfrac{7}{8}\zeta(3)$$

and ζ is the Riemann zeta function. Therefore,

$$\beta m^{\beta} \approx \frac{\pi}{\sqrt{\alpha}}\left(\ln\frac{m^0\beta}{\pi}e^{\gamma}\right)^{1/2} , \tag{1.26}$$

which can also be written as

$$\beta m^{\beta}|_{\beta \gtrsim \beta_c} \approx \frac{\pi}{\sqrt{\alpha}}\left(1 - \frac{\beta_c}{\beta}\right)^{1/2}$$

$$\approx 3.063\left(1 - \frac{\beta_c}{\beta}\right)^{1/2} . \tag{1.27}$$

We now look at the opposite limit, $\beta \gg \beta_c$. The solution for m^{β} is given by Eq. (1.13):

$$\ln\frac{m^0}{m^{\beta}} = 2\mathfrak{F}((m^{\beta})^2\beta^2) , \tag{1.28}$$

which can also be represented by

$$\ln\frac{m^0}{m^{\beta}} = 2\sum_{n=1}^{\infty}(-)^{n+1}K_0(n\beta m^{\beta}) , \tag{1.29}$$

where K_0 is the modified Bessel function of order zero. In this limit, $\beta m^{\beta} \gg 1$, if we use the asymptotic form for the Bessel functions (also, $m^0 \simeq m^{\beta}$),

$$m^{\beta} = m^0 - \left(\frac{2\pi m^0}{\beta}\right)^{1/2}\left(1 - \frac{1}{8m^0\beta}\right)e^{-\beta m^0} . \tag{1.30}$$

The temperature dependence of the function $m^{\beta}\beta_c$ is shown in Fig. 3.

We notice that if we identify the induced fermion mass with the gap function in the theory of superconductivity, our results given by Eqs. (1.22),

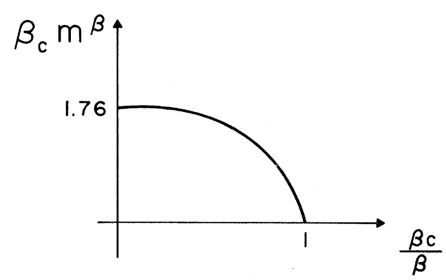

FIG. 3. The function $\beta_c m^{\beta}$.

(1.27), and (1.28) agree exactly with the corresponding predictions of the BCS theory. In fact, as we will show in the following section, the BCS gap equation follows directly from the stability conditions, Eqs. (1.23a) and (1.23b).

We remark here that our results are only valid in the limit as N goes to infinity.

III. STATISTICAL-MECHANICAL ANALOGY

A. Language analogy

In the past section we began making formal connections between the model and the BCS theory of superconductivity. However, to go any further, we must first establish an analogy between our variables and the thermodynamic functions. This analogy is not new; in fact, Lee and Wick have given all the relevant physical information.[7] The equation we will use to fix our variables is

$$P=-\left(\frac{\partial F}{\partial \mathcal{V}}\right)_{\beta}, \tag{2.1}$$

where P is the pressure, F is the Helmholtz free energy, and $\mathcal{V}$ is the volume. This equation corresponds to the equation of state, since $F=F(\mathcal{V},\beta)$.

The corresponding equation in the field-theoretic language is

$$J(\sigma)=-\frac{\delta V^{\beta}}{\delta\sigma}, \tag{2.2}$$

where $J(\sigma)$ is the Lagrange multiplier (source) used in defining the effective potential.[6] (An overall space-time volume has been removed.) We will thus call Eq. (2.2) the equation of state. (Two "isotherms" are shown in Fig. 4.)

The analysis of the phase transition in terms of the compressibility, K, defined as

$$K(\mathcal{V})=-\frac{1}{\mathcal{V}}\left(\frac{\partial P}{\partial \mathcal{V}}\right)_{\beta}^{-1}, \tag{2.3}$$

will thus be carried out in terms of the corresponding function, $K(\sigma)$, given by

$$K(\sigma)=-\frac{1}{\sigma}\left(\frac{\delta J(\sigma)}{\delta\sigma}\right)^{-1}=\frac{1}{\sigma}\left(\frac{\delta^2 V^{\beta}}{\delta\sigma^2}\right)^{-1}. \tag{2.4}$$

The stability conditions for the thermodynamic system demand that $K(\mathcal{V})\geq 0$, i.e., $(\partial P/\partial\mathcal{V})_\beta\leq 0$. With the results of the past section, we see that $K(\sigma)\geq 0$ for $\sigma<-\sigma_A$ and $\sigma>\sigma_A$. In this region, outside the interval $(-\sigma_A,\sigma_A)$, the system exists in single, well-defined phases. For $\sigma\in(-\sigma_A,\sigma_A)$, the phases are mixed. However, for $\beta\leq\beta_c$ there is a single phase for all σ. Thus β_c here has the same meaning as in thermodynamics.

Therefore, with the proper identification of the corresponding variables, the analysis of the phase transition for the field-theoretic model is identical to the usual method in thermodynamics and need not be repeated here. Moreover, as we will see in the next subsection, the identification of our mass parameters with the corresponding quantities in the BCS theory enables us to calculate the discontinuity in the specific heat and thus establishes the nature of the transition.

B. Analogy with the BCS theory

As we mentioned above, for the weak-coupling case the BCS gap equation follows from the stability conditions, Eqs. (1.23a) and (1.23b). We will now show this explicitly.

The general BCS gap equation is a nonlinear integral equation for the gap function $\Delta(\beta)$. The nontrivial solutions, $\Delta\neq 0$, are those which correspond to the superconducting states. The general solutions of this equation are impossible to obtain. However, certain approximations have been developed for a few cases.[13] One such case is to consider an infinite superconductor at zero applied magnetic field. In this case, the gap equation reduces to

$$\Delta(\beta)=\frac{g}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\,\frac{\Delta(\beta)}{\omega_n{}^2+E_k{}^2}, \tag{2.5}$$

where

$$E_k{}^2=\xi_k{}^2+\Delta^2(\beta),\quad \omega_n=(2n+1)\pi/\beta,$$

and ξ_k is the energy relative to the Fermi energy (g is the coupling strength).

Equation (2.5) can be summed to give

$$1=g\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2E_k}\tanh(\tfrac{1}{2}\beta E_k). \tag{2.6}$$

One then introduces the standard approximation

$$\int\frac{d^3k}{(2\pi)^3}\sim N(0)\int_0^{\omega_D}d\xi, \tag{2.7}$$

where $N(0)$ is the density of states at the Fermi surface and ω_D is the Debye frequency.

Equation (2.6) becomes

$$1=gN(0)\int_0^{\omega_D}\frac{d\xi}{(\xi^2+\Delta^2)^{1/2}}\tanh[(\xi^2+\Delta^2)^{1/2}\tfrac{1}{2}\beta], \tag{2.8}$$

which reduces in the weak coupling limit $[\omega_D\gg\Delta(T=0)\equiv\Delta_0]$ to

$$\ln\frac{\Delta_0}{\Delta}=2\int_0^{\beta\omega_D}\frac{dx}{(x^2+\beta^2\Delta^2)^{1/2}}\,\frac{1}{1+e^{(x^2+\beta^2\Delta^2)^{1/2}}}. \tag{2.9}$$

This is to be compared with Eq. (1.13), with the condition $\delta V/\delta\sigma=0$, $\sigma\neq 0$ (for convenience, we also set $\lambda=\pi$):

$$\ln\frac{m_0}{m^{\beta}}=2\int_0^{\infty}\frac{dx}{[x^2+\beta^2(m^{\beta})^2]^{1/2}}\,\frac{1}{1+e^{[x^2+\beta^2(m^{\beta})^2]^{1/2}}}. \tag{2.10}$$

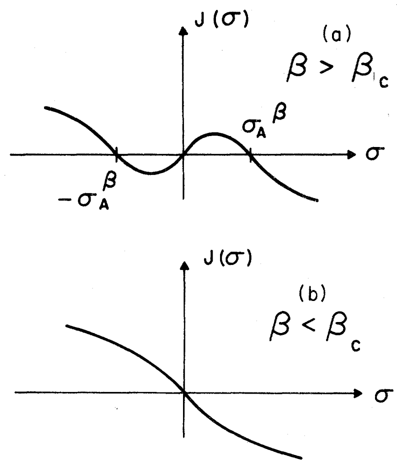

FIG. 4. $J(\sigma)$ for (a) $\beta > \beta_c$ and (b) $\beta < \beta_c$.

The correspondence that we claim now amounts to requiring that $\omega_D \beta \to \infty$, which is appropriate to the low-temperature approximation. (Here low temperature means $T \simeq T_c$, and $\omega_D \gg 1/\beta_c$. We have called this high temperature in the context of field theory.)

Thus we see that the fundamental equations are identical in this limit if one makes the identification $m^\beta = \Delta(\beta)$ and, hence, all information which can be obtained from Eq. (2.13) in superconductivity, particularly the critical variables, can also be obtained from our simple treatment—for example, the specific heat discontinuity. Therefore, we can safely say that the phase transition which the model exhibits is a second-order transition, as is the case with superconductors.

IV. CONCLUSIONS AND COMMENTS

We have found an interesting result concerning phase transitions in the context of a two-dimensional field-theoretic model. The reasoning behind the correspondence between the covariant two-dimensional theory and the nonrelativistic BCS theory of superconductivity lies basically in the fact that, owing to the approximation given by Eq. (2.7), the phase space is identical in both cases.

This work also shows another example of the relevance of studying two-dimensional models. [The wealth of information which can be obtained from this simple and attractive model is nonexistent in the two-dimensional O(N) scalar model,[9] which is symmetric at all temperatures.[14]]

Note added. After submitting the manuscript for publication we received a report from Dashen, Ma, and Rajaraman,[15] whose results agree with ours in the limit as N goes to infinity. They also show that for finite N the results are substantially modified, a conclusion reached independently by Martin.[16] We also learned of the work of Harrington and Yildiz[17] and of Wada,[18] whose results are identical to ours. We would also like to remark here that this work is not a counterexample of the discussion given in Ref. 4, where a specific model that shows dynamical symmetry breaking is analyzed.[19] The symmetry breaking in the model discussed in Ref. 4 is due to the non-Fredholm character of the Bethe-Salpeter kernel as a consequence of ultraviolet singularities, which is contrary to our case.

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Statistics of classical blackbody radiation with ground state

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A classical model of blackbody radiation is investigated in which the field is produced by N_T incoherent thermal sources and N_0 incoherent sources of a ground state with Lorentz-invariant spectral energy density proportional to the frequency ω. Bundles of $2\mathfrak{N}$ nearly monochromatic, nearly parallel cavity modes are considered, and general formulas for all moments of the energy density ρ_b and the energy H_b in a bundle are derived by the use of the central limit theorem. It is found that the energy H of one cavity mode is a fluctuating quantity because of interference of the field contributions from different sources. This causes an energy distribution $W(H) = \langle H\rangle^{-1}\exp(-H/\langle H\rangle)$ which is formally different from the canonical energy distribution. The moments $\langle H^q\rangle$ obtained from $W(H)$ agree with those obtained by quantum statistics for $q = 1$ and $q = 2$, but disagree for $q > 2$; the higher moments are also inconsistent with some general relations of the canonical ensemble theory. Suggestions are made for achieving full equivalence between the classical model and quantized blackbody radiation.

INTRODUCTION

This paper is concerned with recent attempts to find a classical foundation of the quantum theory. The following three assumptions are at the bottom of the new development.

1. The interaction between radiation and matter can be correctly described by the semiclassical radiation theory in which particle dynamics is quantized, while radiation is treated as a classical electromagnetic field. The power of this theory has been demonstrated by the analysis of numerous phenomena usually considered strong evidence for the quantization of the radiation field. They include spontaneous emission,[1] the photoelectric effect,[2,3] the Compton effect,[4] resonance fluorescence,[2] and even Lamb-shift phenomena.[3,5-7]

2. The quantum mechanics of particles is a classical stochastic theory in which a Brownian particle motion is superimposed on the smooth Newtonian motion of conventional classical physics. This assumption is justified by the work of several authors[8-14] who have shown that the complex Schrödinger equation may be interpreted as a pair of real, classical stochastic equations of the Fokker-Planck type. The theory has been investigated for one and several particles, with and without spin, and for relativistic and nonrelativistic situations. In the past, the origin of the Brownian motion was usually considered unknown, and ascribed to collisions with unknown particles[15] which are randomly distributed throughout the