Field theory of spinning strings*

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We present the field theory of spinning strings, which is a natural generalization of the field theory of Dirac particles. Because we place spinors along a string defined at each space-time point, the theory is at once multilocal and reproduces an infinite-component field theory. We introduce interactions by allowing the string to execute well-defined topological transformations and show that we reproduce the usual dual models of Neveu, Schwarz, and Ramond. We give the full Lagrangian which yields S matrices which are dual, factorizable, Lorentz-invariant, crossing-symmetric, and probably renormalizable.

I. INTRODUCTION

Local field theory has long been the standard tool through which we explore the dynamics of elementary-particle physics. But when we discuss simple phenomenological properties of strong interactions, such as resonances, Regge trajectories, duality, etc., local field theory becomes a cumbersome theoretical formalism. Attempts to construct generalizations of local field theory, such as nonlocal field theory and infinite-component field theory, have all been plagued with theoretical problems which render them unsatisfactory.

In previous papers[1] we have presented an alternative to local field theory which easily incorporates desirable features such as Regge poles, duality, linear trajectories, and which also reproduces the dual resonance model.[2] We defined a field functional defined not at space-time points but along a multilocal string[3] which is allowed to execute well-defined vibrations producing the resonances of strong interactions. This multilocal field theory represents a drastic departure from local field theory, yet has most of its desirable features.

In our previous papers the string carried only orbital modes, and no intrinsic spin, and hence could not incorporate fermions. In this paper we extend our previous work and add spinors onto our string, so that we reproduce the dual models of Neveu and Schwarz[4] and Ramond.[5]

Our spinning string can execute several topological transformations:

(1) It can propagate freely as an open string, in which case it can be identified with mesons, fermions, and antifermions.

(2) It can exist as a closed string, in which case it is a Pomeron.

(3) It can perform at most five topological deformations: (a) the open string can break at an interior point into two smaller strings (see Fig. 1); (b) the closed string may pinch at an interior point and fission into two smaller closed strings (Fig. 2), (c) the open string's end points may join and form a loop (Fig. 3), (d) two strings may touch at an interior point and form two other strings (Fig. 4), (e) one open string may overlap with itself and pinch off a closed string (Fig. 5).

Notice that these five primitive interactions are all compatible with *local* deformations of the string.

II. FREE THEORY

We start by defining a string variable parametrized by σ: $X_\mu(\sigma)$, where $0<\sigma<\pi\alpha$. In addition, we now place a conformal spinor $S_1(\sigma)$, $S_2(\sigma)$ at each point on the string. We now write

$$\Phi(X_\mu(\sigma_1), X_\mu(\sigma_2), \ldots, X_\mu(\sigma_N); S_{i\mu}(\sigma_1), S_{i\mu}(\sigma_2), \ldots, S_{i\mu}(\sigma_N)) \quad (i=1,2) \tag{2.1}$$

or simply

$$\Phi[X,S], \tag{2.2}$$

where the functional Φ loses its σ dependence. We have thus superimposed a multilocal spinor onto our string.

From the first-quantized functional theory of the spinning string,[6] we know that the correct boundary conditions are

$$\begin{aligned}
&S_1(0)=-S_2(0), \quad S_1(\pi\alpha)=S_2(\pi\alpha) \quad \text{for mesons (no } q),\\
&S_1(0)=S_2(0), \quad S_1(\pi\alpha)=-S_2(\pi\alpha) \text{ for mesons } (q\bar{q}),\\
&S_1(0)=-S_2(0), \quad S_1(\pi\alpha)=-S_2(\pi\alpha) \text{ for fermions},\\
&S_1(0)=S_2(0), \quad S_1(\pi\alpha)=S_2(\pi\alpha) \quad \text{for antifermions}.
\end{aligned} \tag{2.3}$$

If we associate arrows on the ends of the string as follows: forward arrow, $S_1=-S_2$, and backward arrow, $S_1=S_2$, then we recover the usual duality

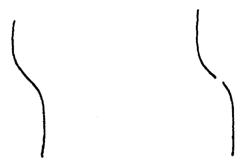

FIG. 1. One string splits into two strings.

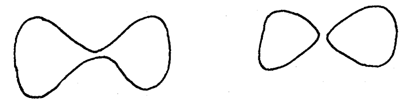

FIG. 2. One ring fissions into two rings.

diagrams. In Fig. 6, for example, we show how these arrows are used to determine three-string vertices.

We make the standard decomposition into normal modes (we suppress Lorentz indices):

$$
\begin{aligned}
X(\sigma) &= x_0 + \sum_{n=1}^{\infty} 2X_n \cos(n\sigma/\alpha)\,,\\
P(\sigma) &= -i\,\frac{\delta}{\delta X(\sigma)}\\
&= \frac{1}{\pi\alpha}\left[\,p_0 + \sum_{n=1}^{\infty} P_n \cos(n\sigma/\alpha)\right]\,,\\
S_1(\sigma) &= \frac{1}{\sqrt{2\alpha}} \sum_n b_n e^{in\sigma/\alpha}\,,\\
S_2(\sigma) &= \pm\frac{1}{\sqrt{2\alpha}} \sum_n b_n e^{-in\sigma/\alpha}\,,
\end{aligned}
\qquad (2.4)
$$

where the spinors are summed over all half-integers for mesons and over all integers for fermions [use (−)] and for antifermions [use (+)]. Imposing

$$(P_n, X_m)_- = -i\delta_{n,m} \text{ and } (b_n, b_m)_+ = \delta_{n,m} \qquad (2.5)$$

we arrive at the following:

$$
\begin{aligned}
&(P_\mu(\sigma), X_\nu(\sigma'))_- = -i\delta(\sigma-\sigma')\delta_{\mu\nu}\,,\\
&(S_i(\sigma), S_j(\sigma'))_+ = \pi\delta(\sigma-\sigma')\delta_{ij}\,.
\end{aligned}
\qquad (2.6)
$$

We would now like to write the Klein-Gordon equation of motion for the string. As a guide, we first take the first-quantized form of the model, calculate Dirac's ϕ condition, and then apply the ϕ condition onto state vectors. (See Appendix A for the difference between first- and second-quantized formalisms.)

Following Iwasaki and Kikkawa,[7] we take

$$\int_0^{\pi\alpha} d\sigma \left\{ P^2(\sigma) + \frac{1}{(2\pi)^2} X'^2(\sigma) - \frac{i}{(2\pi)^2}\left[S_1(\sigma)\overleftrightarrow{\partial}_\sigma S_1(\sigma) - S_2(\sigma)\overleftrightarrow{\partial}_\sigma S_2(\sigma)\right]\right\} |\phi\rangle = 0\,. \qquad (2.7)$$

To convert this to a differential equation, we take $P(\sigma) = -i\delta/\delta X(\sigma)$ and $b_n \equiv \delta/\delta\xi_n + \xi_{-n}$, where ξ_n are *totally anticommuting c numbers*:

$$(\xi_n, \xi_m)_+ = \left(\frac{\delta}{\delta\xi_n}, \frac{\delta}{\delta\xi_m}\right)_+ = 0\,, \quad \xi_n^\dagger = \xi_{-n}\,. \qquad (2.8)$$

At this point, we must state that the fully relativistic Lagrangian for the spinning string has not yet been found. The difficulty in constructing a manifestly relativistic multilocal Lagrangian lies in the fact that it is nontrivial to construct multilocal Lagrangians which possess two infinite classes of gauge invariances. Because we have not yet found the relativistic Lagrangian, we will go into the light-cone gauge, so that no invariances (or ghosts) remain in our formalism. We take the gauge of Goddard, Goldstone, Rebbi, and Thorn[8] and Iwasaki and Kikkawa:

$$
\begin{aligned}
&X_+ = i\tau\,,\\
&S_{i+} = 0\,.
\end{aligned}
\qquad (2.9)
$$

The "Schrödinger equation" for the model now reads

$$\left[\,2p_+\frac{\delta}{\delta\tau} - \tfrac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{\delta^2}{\delta\vec{X}_n^{\,2}} + n^2\vec{X}_n^{\,2}\right) + \sum_{m=1/2}^{\infty} m\left(\frac{\delta}{\delta\vec{\xi}_m} + \vec{\xi}_{-m}\right)\cdot\left(\frac{\delta}{\delta\vec{\xi}_{-m}} + \vec{\xi}_m\right)\right]|\phi\rangle = 0\,. \qquad (2.10)$$

FIG. 3. The ends of a string touch and form a ring.

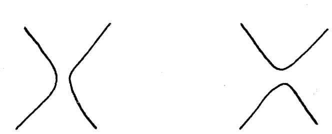

FIG. 4. Two strings touch and rearrange topology.

We now write down the Lagrangian which reproduces the Schrödinger equation:

$$\mathcal{L}_0=\int_0^\infty dp_+\int_0^{2\pi p_+}d\sigma\int\mathfrak{D}\vec{X}\mathfrak{D}\vec{S}_1\mathfrak{D}\vec{S}_2\Big[\Phi^\dagger_{p_+}[X,S]\,2p_+ i\,\frac{\delta}{\delta X_+}\,\Phi_{p_+}[X,S]$$
$$-(\pi\alpha)^2\Phi^\dagger_{p_+}[X,S]\left(-\frac{\delta^2}{\delta\vec{X}(\sigma)^2}+\frac{1}{(2\pi)^2}\vec{X}'^2(\sigma)\right.$$
$$\left.-\frac{i}{(2\pi)^2}[\vec{S}_1(\sigma)\cdot\bar{\partial}_\sigma\vec{S}_1(\sigma)-\vec{S}_2(\sigma)\cdot\bar{\partial}_\sigma\vec{S}_2(\sigma)]\right)\Phi_{p_+}[X,S]\Big],\tag{2.11}$$

where S_1 and S_2 are taken to be operators.

We can construct the canonical momentum to our field:

$$\Pi[X,S]=\frac{\delta\mathcal{L}}{\delta(\delta\Phi[X,S]/\delta\tau)}=i\Phi^\dagger[X,S]\,.\tag{2.12}$$

(Notice that taking $X_+=i\tau$ allows us to make a one-to-one correspondence between the field and its conjugate. Multitime quantization, however, prevents this identification and leads to serious theoretical difficulties.)

Our canonical quantization relations read

$$[\Phi_{p_+}[X,S,\tau_1],\Phi^\dagger_{q_+}[X',S',\tau_2]]_{\tau_1=\tau_2}=\prod_{i=1,2}\prod_\sigma\delta(\vec{X}(\sigma)-\vec{X}'(\sigma))\delta(\vec{S}_i(\sigma)-\vec{S}_i'(\sigma))\delta(p_+-q_+)\,.\tag{2.13}$$

The solution to our differential equation in X_n is, of course, normalized Hermite polynomials. But the solution of the equation for the spin modes requires some care, because we are dealing with totally anticommuting c numbers. At this point, before we write down the solution to our field equations, we must make use of the theory of the *calculus defined on a Grassmann algebra*.[9] The usual equations of calculus must be modified (e.g., $e^A=1+A$ if $A^2=0$). We shall use the following convention:

$$\int\mathfrak{D}\xi_n=0\,,\quad\int\xi_n\mathfrak{D}\xi_n=1\,,\quad(\xi_n,\xi_m)_+=(\xi_n,\mathfrak{D}\xi_m)_+=(\mathfrak{D}\xi_n,\mathfrak{D}\xi_m)_+=0\,.\tag{2.14}$$

A few mathematical preliminaries will prove useful. Given a Grassmann algebra with N generators $\{\xi_n\}$, the most general function defined in this space is a linear combination of monomials

$$f(\xi)=f_0+\sum_{i=1}^N f_i\xi_i+\sum_{i,j=1}^N f_{ij}\xi_i\xi_j+\cdots+f_N\xi_1\xi_2\cdots\xi_N\,.\tag{2.15}$$

We have

$$\int f(\xi)\mathfrak{D}\xi_N\mathfrak{D}\xi_{N-1}\cdots\mathfrak{D}\xi_1=f_N\,.\tag{2.16}$$

We will make constant use of the following equations:

$$\int\exp\left(\sum_{n=1}^N\lambda_n\xi_n\xi_{-n}\right)\prod_{n=1}^N(\mathfrak{D}\xi_{-n}\mathfrak{D}\xi_n)=\int\frac{1}{N!}\left(\sum_{n=1}^N\lambda_n\xi_n\xi_{-n}\right)^N\prod_{n=1}^N(\mathfrak{D}\xi_{-n}\mathfrak{D}\xi_n)=\prod_{i=1}^N\lambda_i$$

and (2.17)

$$\int\exp\left(\sum_{n,m=1}^N\xi_n a_{nm}\xi_m+\sum_{n=1}^N\xi_n b_n\right)\mathfrak{D}\xi_N\cdots\mathfrak{D}\xi_1=[\det(2a)]^{1/2}\exp\left(-\tfrac{1}{2}\sum_{n,m=1}^N b_n(a^{-1})_{nm}b_m\right).$$

To prove the last formula, we "diagonalize" the a_{nm} matrix to the following form:[9]

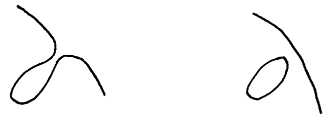

FIG. 5. One string overlaps with itself and produces a string and a ring.

$$
\begin{bmatrix}
0 & \lambda_1 & 0 & 0 & \\
-\lambda_1 & 0 & 0 & 0 & \\
0 & 0 & 0 & \lambda_2 & \cdots \\
0 & 0 & -\lambda_2 & 0 & \\
 & & \vdots & &
\end{bmatrix},
$$

$$a_{nm} = -a_{mn}\,, \quad (b_n, b_m)_+ = 0\,. \tag{2.18}$$

Let us now introduce *two* independent sets of generators $\{\xi_n\}$ and $\{\xi'_n\}$, $-N<n<N$. In this strange algebra, the δ function between primed and unprimed variables $\{\xi_n\}$ and $\{\xi'_n\}$ is given by

$$\prod_{i=1}^{N} (\xi_i - \xi'_i)(\xi_{-i} - \xi'_{-i}) = \delta(\xi - \xi')\,, \tag{2.19}$$

so that

$$
\begin{aligned}
&\int \delta(\xi - \xi') \prod_{i=1}^{N} (\mathfrak{D}\xi_{-i} \mathfrak{D}\xi_i) = 1\,, \\
&\int f(\xi)\delta(\xi - \xi') \prod_{i=1}^{N} (\mathfrak{D}\xi_{-i}\, \mathfrak{D}\xi_i) = f(\xi')\,,
\end{aligned} \tag{2.20}
$$

where $f(\xi)$ is a Grassmann monomial.

Now that the mathematical preliminaries are out of the way, we can write down the "Hermite" polynomials which are solutions to our differential equation[10]:

$$
\begin{aligned}
J_{1,n}(\xi) &= \frac{1}{\sqrt{2}} e^{\xi_n \xi_{-n}} = \frac{1}{\sqrt{2}} (1 + \xi_n \xi_{-n})\,, \\
J_{2,n}(\xi) &= \sqrt{2}\, \xi_{-n} J_{1,n}(\xi) = \xi_{-n}\,, \\
J_{-1,n}(\xi) &= \frac{1}{\sqrt{2}} (1 - \xi_n \xi_{-n})\,, \\
J_{-2,n}(\xi) &= \xi_n\,.
\end{aligned} \tag{2.21}
$$

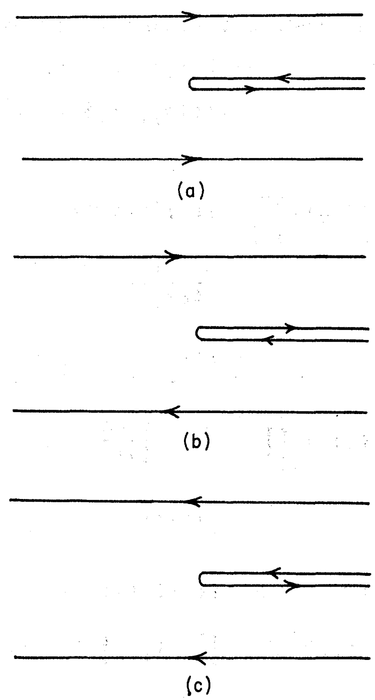

FIG. 6. (a) Dual diagram for a fermion going into a fermion and a meson. (b) Dual diagram for a meson going into a fermion and an antifermion. (c) Dual diagram for an antifermion going into a meson and an antifermion.

We have the following orthonormality relations:

$$
\begin{aligned}
&\int J^\dagger_{i,n}(\xi) J_{j,n}(\xi)\, \mathfrak{D}\xi_{-n} \mathfrak{D}\xi_n = (\pm)\delta_{ij} \quad (\xi^\dagger_n = \xi_{-n})\,, \\
&\sum_{i=-2}^{2} (\pm) J_{i,n}(\xi) J^\dagger_{j,n}(\xi') = \delta(\xi_n - \xi'_n)\delta(\xi_{-n} - \xi'_{-n}) \\
&\qquad (+ \text{ if } i,j>0\,,\ - \text{ if } i,j<0)\,.
\end{aligned} \tag{2.22}
$$

Now that we know the solutions to spinor differential equations, we can write down the solution to our field equations for mesons:

$$
\Phi_{p_+}[X,S] = \int \frac{d\vec{p}}{2\pi} \sum_{\{n_a^{(i)}\},\{m_b^{(j)}\}} \prod_{\substack{a,b=1,\infty \\ i,j=1,D-2}} H_{\{n_a^{(i)}\}}(X_a^{(i)}) e^{-aX_a^{(i)2}} J_{m_b^{(j)},b}(\xi) A_{\{n_a^{(i)}\},\{m_b^{(j)}\},\vec{p},p_+}
\times \exp[i(\vec{p}\cdot\vec{x} - E_{\{n_a^{(i)}\},\{m_b^{(j)}\},\vec{p},p_+} X_+)]\,, \tag{2.23}
$$

where

$$
\begin{aligned}
&E_{\{n_a^{(i)}\},\{m_b^{(j)}\},\vec{p},p_+} = \frac{1}{\alpha} \sum_{a,b} [a(n_a^{(i)} + \tfrac{1}{2}) + b(r_b^{(j)} - \tfrac{1}{2})] - \vec{p}^{\,2} - \alpha_0 \quad (r_b^{(j)} = m_b^{(j)} - 1 = 0,1)\,, \\
&(A_{\{n_a^{(i)}\},\{m_b^{(j)}\},\vec{p},p_+}, A^\dagger_{\{n_a^{(i)}\}',\{m_b^{(j)}\}',\vec{q},q_+}) = \delta_{\{n_a^{(i)}\},\{n_a^{(i)}\}'}\, \delta_{\{m_b^{(j)}\},\{m_b^{(j)}\}'}\, \delta(\vec{p} - \vec{q})\delta(p_+ - q_+)\,.
\end{aligned} \tag{2.24}
$$

We can now construct the Green's functions for our model:

$$G(X,S,\tau_1;X',S',\tau_2)=G_0(X,\tau_1;X',\tau_2)G_1(S,\tau_1;S',\tau_2)$$
$$=\langle\langle 0|\Phi_{p_+}[X,S,\tau_1]\,\Phi^\dagger_{p_+}[X',S',\tau_2]|0\rangle\rangle\,,$$

where

$$G_0(X,\tau_1;X',\tau_2)=\prod_{l=1}^{\infty}\{\sinh[l(\tau_1-\tau_2)/\alpha]\}^{(D-2)/2}$$
$$\times\left(\frac{\alpha}{4\pi(\tau_1-\tau_2)}\right)^{(D-2)/2}\exp\left(\frac{-l}{\sinh[l(\tau_1-\tau_2)/\alpha]}\{(\vec{X}_l^{\,2}+\vec{X}_l'^{\,2})\cosh[l(\tau_1-\tau_2)/\alpha]-2\vec{X}_l\cdot\vec{X}_l'\}\right)$$
$$\times\exp\left[-\frac{\alpha}{4(\tau_1-\tau_2)}(\vec{x}_0-\vec{x}_0')^2\right]\,, \tag{2.25}$$

$$G_1(S,\tau_1;S',\tau_2)=\prod_{l=1/2}^{\infty}\left[\sinh\frac{l(\tau_1-\tau_2)}{2\alpha}+\cosh\frac{l(\tau_1-\tau_2)}{2\alpha}(\xi_l\,\xi_{-l}+\xi_l'\,\xi_{-l}')\right.$$
$$\left.-e^{l(\tau_1-\tau_2)/2\alpha}\xi_l\,\xi_{-l}'+e^{-l(\tau_1-\tau_2)/2\alpha}\xi_{-l}\,\xi_l'+\sinh\frac{l(\tau_1-\tau_2)}{2\alpha}\xi_l\,\xi_{-l}\,\xi_l'\,\xi_{-l}'\right]. \tag{2.26}$$

These Green's functions satisfy the sewing rule:

$$\int G(X,S,\tau_1;X',S',\tau_2)\prod_n \mathfrak{D}X_n'\prod_l(\mathfrak{D}\xi_{-l}'\,\mathfrak{D}\xi_l')\,G(X',S',\tau_2;X'',S'',\tau_3)=G(X,S,\tau_1;X'',S'',\tau_3)\,. \tag{2.27}$$

For small times, we recover the usual δ function:

$$\lim_{\tau_1\to\tau_2}G(X,S,\tau_1;X',S',\tau_2)=\prod_\sigma\delta(X(\sigma)-X'(\sigma))\prod_l\delta(\xi_l-\xi_l')\delta(\xi_{-l}-\xi_{-l}')\,. \tag{2.28}$$

At this point, we must clarify what we mean by the negative-energy states of (2.21). Clearly, the Lagrangian (2.11) only allows for positive-energy states. The reason for this doubling, of course, is that the representation of the Clifford algebra of the b's has been *doubled* when going over to the Grassmann algebra of (2.10) (see Ref. 9). A more physical representation would be to choose matrix spinor representations of J_1 and J_2 of (2.21):

$$J_{1,n}=\begin{pmatrix}0\\1\end{pmatrix}_n;\;J_{2,n}=\begin{pmatrix}1\\0\end{pmatrix}_n;\;b_n=\begin{pmatrix}0&0\\1&0\end{pmatrix}_n;\;b_n^\dagger=\begin{pmatrix}0&1\\0&0\end{pmatrix}_n.$$

Then the wave function (2.23) is only summed over positive-energy spinor states. An equivalent method within the present formalism is to simply insert the projection operator:

$$\prod_n\left[\sum_{i=1}^{2}J_{i,n}(\xi)J^\dagger_{i,n}(\xi')\right]\equiv P(\xi,\xi')\,,$$
$$\int f(\xi)P(\xi,\xi')\prod_n\mathfrak{D}\xi_{-n}\mathfrak{D}\xi_n=f_+(\xi')\,,$$

which projects out positive-energy states f_+. Inserting this projection operator at all propagators insures the positivity of all energies. We will, however, continue to use the doubled representation of the Grassmann algebra because it allows us to check our results with the functional formalism immediately, but at all times we must remember that only positive-energy states are selected out by the projection operator.

As in the usual theory of point particles, the Green's function found in the second-quantized formalism can be directly related to the functional average over paths defined in the first-quantized formalism. In particular, we find (see Appendix B)

$$\langle\langle 0|\Phi_{p_+}[X,S,\tau_1]\Phi^\dagger_{p_+}[X',S',\tau_2]|0\rangle\rangle\cong\int\mathfrak{D}\vec{X}\mathfrak{D}\vec{S}_1\mathfrak{D}\vec{S}_2\exp\left(\int_0^{\pi\alpha}d\sigma\int_{\tau_1}^{\tau_2}d\tau\,L\right)$$
$$\times\prod_{\sigma,\sigma'}\delta(\vec{X}(\sigma)-\vec{X}(\sigma,\tau_1))\,\delta(\vec{X}'(\sigma')-\vec{X}(\sigma',\tau_2))$$
$$\times\prod_{i=1,2}\prod_{\sigma,\sigma'}\delta(\vec{S}_i(\sigma)-\vec{S}_i(\sigma,\tau_1))\,\delta(\vec{S}_i'(\sigma')-\vec{S}_i(\sigma',\tau_2))\,, \tag{2.29}$$

where

$$L=\frac{1}{4\pi}\left[-\dot{\vec{X}}^2-\vec{X}'^2-\vec{S}_1\cdot(\partial_\tau+i\partial_\sigma)\vec{S}_1-\vec{S}_2\cdot(\partial_\tau-i\partial_\sigma)\vec{S}_2\right].$$

In a similar manner, it is possible to construct the wave functions for Pomerons (we find a doubling of harmonic oscillators) with slope $\frac{1}{2}$ that of the meson trajectories. We use the same Lagrangian, except the Riemann surface of interest is now a tube rather than a strip.

When we quantize the fermions, however, we encounter some problems. If we naively quantize the model in the way shown above, we find that the propagator is a Klein-Gordon operator, rather than the Dirac operator. This is because, in (2.29) and (2.7), we have taken the Hamiltonian and Lagrangian of the Klein-Gordon operator, which is quadratic in both the X and the S fields. But, instead of using (2.7), we could equally have started with the F operator [which is the "square root" of (2.7)], which is linear in both X and S and corresponds to the Dirac equation. But if we use the true Dirac operator to represent the propagator, then we are forced to modify our vertices. An exactly analogous situation holds for the familiar harmonic-oscillator formalism. The formalism with Klein-Gordon propagators is called R_1 and the formulation with the Dirac operator is called R_2, and both formalisms are related by the fact that the square of the propagator in $R_2(1/F_0)$ is equal to the propagator in the $R_1(1/L_0)$, with corresponding changes in the vertices.

In our paper we will adopt the formalism used in Ref. 13, which uses the Klein-Gordon formulation rather than the Dirac formulation. This means that we sum over integers in (2.26) rather than half integers, and that we adopt the vertex function of Mandelstam. (Alternatively, it is not hard to quantize the model in the Dirac formalism.)

In order to quantize the model in the Dirac formalism, we follow the work of Iwasaki and Kikkawa and write down the F gauges acting upon state vectors:

$$\int_0^{\pi\alpha} d\sigma\left[\left(P(\sigma)+\frac{1}{(2\pi)}X'(\sigma)\right)_\mu S_1(\sigma)_\mu+\left(P(\sigma)-\frac{1}{(2\pi)}X'(\sigma)\right)_\mu S_2(\sigma)_\mu\right]|\phi\rangle=0. \tag{2.30}$$

We will take a new gauge condition $S_{+i}=\gamma_+$, in order to keep the Dirac matrices as zero modes of the S's. In this gauge we have

$$\left[p_+\gamma_-+i\frac{\delta}{\delta X_+}\gamma_+-\int_{-\pi\alpha}^{\pi\alpha}d\sigma\left(\vec{P}(\sigma)+\frac{1}{(2\pi)}\partial_\sigma\vec{X}(\sigma)\right)\cdot\vec{S}_1(\sigma)\right]|\phi\rangle=0,\quad \vec{S}_1(-\sigma)\equiv\vec{S}_2(\sigma). \tag{2.31}$$

This is the multilocal Dirac equation, formulated in the R_2 system. In order to revert to the R_1 formalism, we merely square the operator appearing in (2.31) and obtain the operator (2.7), except for two differences: (1) We now must sum over integer modes rather than half-integer modes. (2) We must attach the on-shell spinor wave function $u(p_+,p_-,\vec{p})$ onto the solution (2.23) in order to obtain half-integral spins. The R_2 formalism, therefore, does not pose any new difficulties.

The solution of (2.31) (see Ramond, Ref. 5) can be found in terms of the solution of (2.7) for fermions. If $|\psi\rangle$ is a solution of (2.31) so that $F|\psi\rangle=0$, and if $|\phi\rangle$ is the fermionic solution to (2.7), then the solution of (2.31) is given by $|\psi\rangle=F|\phi\rangle$. We immediately check that $F|\psi\rangle=F^2|\phi\rangle=0$.

At this point, we must make a few remarks concerning the differences between the scalar and pseudoscalar mesons and the $q\bar{q}$ and "no q" mesons. The boundary conditions (2.3) tell us that the $q\bar{q}$ and the no-q mesons differ by the orientation of the duality arrows, but since (2.7) is invariant under $S_2\to-S_2$, the propagator for mesons propagates both $q\bar{q}$ as well as no-q mesons. What is the difference?

First of all, the fermions in this model are actually one-quark states, so fermion-meson couplings force us to have mesons which are quark-antiquark states or which do not have quarks at all. In Fig. 6(a), for example, we can consider the top line of the incoming fermion to be a one-quark state, so the meson must be a $q\bar{q}$ meson (the outgoing fermion's top line is also a quark line). But now look at Fig. 6(a) again, which now describes meson-antifermion scattering into antifermions. The quark line is now at the bottom of the antifermion strip, and hence the meson has no quark lines, and is therefore a no-q state.

Second, the parities of the $q\bar{q}$ and no-q mesons are defined by how they couple to fermions (which will be described in the next section), i.e., whether there is a γ_5 or not. We define the vacuum of the $q\bar{q}$ meson to be pseudoscalar. Then we find that $q\bar{q}$ and no-q mesons are of *different (same)* parities if they have *odd (even)* g parities. The vacuum of

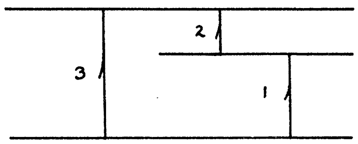

FIG. 7. String three breaks into strings one and two.

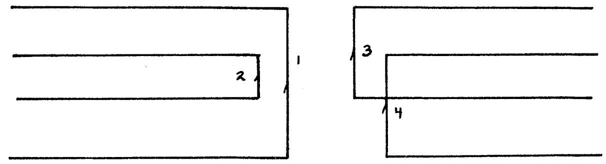

FIG. 8. Strings one and two join, rearrange topology, and come off as strings three and four.

the no-q meson is then a scalar.

It is not hard to find the second-quantized version of the Pomeron in this version, but there does exist a bit of ambiguity in how to construct the boundary conditions for the Pomeron tube (i.e., there seems to be more than one Pomeron, depending on its boundary conditions and depending on how it couples to other particles). Part of this ambiguity arises because the spinor $S(\sigma)$ can assume a factor ($\pm$) depending on how many times it circulates around the tube. Once the normal-mode decomposition of the ring is known, then its corresponding Green's function for finite times is easily calculable. The problem seems to be in defining all possible normal mode expansions of the tube and in defining how it couples to strings. [See L. Brink and D. B. Fairlie, 1973 (unpublished) and D. B. Fairlie and D. Martin, 1973 (unpublished).]

III. INTERACTIONS

The theory of spinning strings is not complete without a discussion of how to introduce interactions into the model. As mentioned earlier, the interacting spinning string can execute at most five basic topological transformations,[11] all consistent with *local* deformations of the string topology.

In our previous papers we demonstrated that a string breaks in the "smoothest" possible manner, i.e., via a δ functional. We also found that the finite-time matrix elements resulting from this interaction produced the matrix elements of Mandelstam,[12] who first solved the stringlike solutions for the interacting string. In this paper we also find that the interaction terms correspond to δ functions (with a small exception) and that these interactions reproduce the finite-time matrix elements found in Mandelstam's investigation of the spinning string.[13] We will now list the various δ functionals responsible for creating topological deformations of the spinning string:

(I) three-string interaction (see Fig. 7):

$$\delta(\mathrm{I}) \equiv \prod_{\sigma_3} \prod_{\vec{\Omega}=\vec{X},\vec{S}_1,\vec{S}_2} \delta(\vec{\Omega}_3(\sigma_3) - \vec{\Omega}_1(\sigma_1)\theta(\pi\alpha_1 - \sigma_3) - \vec{\Omega}_2(\sigma_2)\theta(\sigma_3 - \pi\alpha_1)),$$

$$0<\sigma_i<\pi\alpha_i,\ \ \alpha_i>0,\ \ \alpha_1+\alpha_2=\alpha_3,\ \ \sigma_3=\sigma_2+\pi\alpha_1(0<\sigma_2<\pi\alpha_2);\ \sigma_3=\sigma_1(0<\sigma_1<\pi\alpha_1). \quad (3.1)$$

(II) four-string interaction (Fig. 8):

$$\delta(\mathrm{II}) \equiv \int_{\pi(\alpha_1-\alpha_3)}^{\pi\alpha_4} d\sigma_0 \prod_{\sigma_1,\sigma_2} \prod_{\vec{\Omega}=\vec{X},\vec{S}_1,\vec{S}_2} \delta(\vec{\Omega}_1(\sigma_1) - \vec{\Omega}_3(\sigma_3)\theta(\sigma_1-\sigma_0) - \vec{\Omega}_4(\sigma_4)\theta(\sigma_0-\sigma_1))$$

$$\times\, \delta(\vec{\Omega}_2(\sigma_2) - \vec{\Omega}_3(\sigma_3)\theta(\sigma_0-\sigma_1) - \vec{\Omega}_4(\sigma_4)\theta(\sigma_1-\sigma_0)),$$

$$0<\sigma_i<\pi\alpha_i;\ \alpha_i>0;\ \alpha_1>\alpha_3>\alpha_2;\ \alpha_1>\alpha_4>\alpha_2;\ \alpha_1+\alpha_2=\alpha_3+\alpha_4,$$

$$\sigma_4=\sigma_1(0<\sigma_4<\pi\alpha_4);\ \sigma_1=\sigma_2+\pi(\alpha_1-\alpha_3);\ \sigma_1=\sigma_3+\pi(\alpha_1-\alpha_3). \quad (3.2)$$

(III) three-ring interaction (Fig. 9):

$$\delta(\mathrm{III}) \equiv \prod_{\sigma_3} \prod_{\vec{\Omega}=\vec{Y},\vec{S}_1,\vec{S}_2} \delta(\vec{\Omega}_3(\sigma_3) - \vec{\Omega}_1(\sigma_1)\theta(\pi\alpha_1-\sigma_3) - \vec{\Omega}_2(\sigma_2)\theta(\sigma_3-\pi\alpha_1)),$$

$$0<\sigma_i<\pi\alpha_i;\ \sigma_3=\sigma_1(0<\sigma_1<\pi\alpha_1);\ \sigma_3=\sigma_2+\pi\alpha_1(0<\sigma_2<\pi\alpha_2)$$

$$\alpha_i>0,\ \ \alpha_1+\alpha_2=\alpha_3,\ \ \vec{\Omega}_i(0)=\vec{\Omega}_i(\pi\alpha_i). \quad (3.3)$$

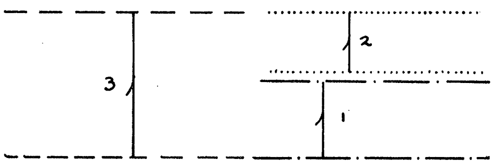

FIG. 9. One closed string fissions into two smaller closed strings. Riemann surface is sealed between two sets of matching dotted lines.

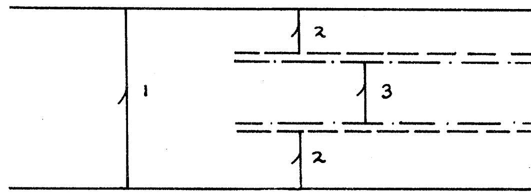

FIG. 10. String one breaks into string two and closed string three.

(IV) string-string-ring interaction (Fig. 10):

$$\delta(\mathrm{IV}) \equiv \int_0^{\pi(\alpha_1-\alpha_3)} d\sigma_0 \prod_{\sigma_1} \prod_{\substack{\vec{\Omega}=\vec{X},\\ \vec{S}_1,\vec{S}_2}} \delta\big(\vec{\Omega}_1(\sigma_1) - \vec{\Omega}_2(\sigma_2)\theta(\sigma_0-\sigma_1) - \vec{\Omega}_2(\sigma_2)\theta(\sigma_1-(\sigma_0+\pi\alpha_3)) - \vec{\Omega}_3(\sigma_3)\theta(\sigma_1-\sigma_0)\theta(\pi\alpha_3+\sigma_0-\sigma_1)\big),$$

$$0<\sigma_i<\pi\alpha_i;\ \alpha_1=\alpha_2+\alpha_3;\ \sigma_1=\sigma_3+\sigma_0;\ \vec{\Omega}_3(0)=\vec{\Omega}_3(\pi\alpha_3),$$

$$\sigma_1=\sigma_2\ (0<\sigma_2<\sigma_0);\ \sigma_1=\sigma_2+\pi\alpha_3+\sigma_0(\sigma_1>\sigma_0+\pi\alpha_3). \tag{3.4}$$

(V) string-ring interaction (Fig. 11):

$$\delta(\mathrm{V}) \equiv \prod_{\substack{\vec{\Omega}=\vec{X}\\ \vec{S}_1,\vec{S}_2}} \prod_{\sigma} \delta(\vec{\Omega}_1(\sigma) - \vec{\Omega}_2(\sigma)), \quad 0<\sigma<\pi\alpha;\ \vec{\Omega}_2(0)=\vec{\Omega}_2(\pi\alpha). \tag{3.5}$$

We can now write down the full Lagrangian for the interacting spinning string:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}} + \mathcal{L}_{\mathrm{II}} + \mathcal{L}_{\mathrm{III}} + \mathcal{L}_{\mathrm{IV}} + \mathcal{L}_{\mathrm{V}}, \tag{3.6}$$

where (notice that we only use meson, fermion, and antifermion fields only when they are compatible with the dual diagrams)

$$\mathcal{L}_{\mathrm{I}} = \tfrac{1}{2}g \sum_{\psi} \int \prod_{i=1}^{3} \mathfrak{D}\vec{X}_i \mathfrak{D}\vec{S}_{1,i} \mathfrak{D}\vec{S}_{2,i} \frac{dp_{+i}}{(2p_{+i})^{1/2}} \delta(p_{+1}+p_{+2}-p_{+3}) \times \psi^{\dagger}_{p_{+3}}[X_3,S_3]\,\psi_{p_{+1}}[X_1,S_1]\,\psi_{p_{+2}}[X_2,S_2]\,\delta(\mathrm{I}) \frac{\vec{\delta}}{\delta\vec{X}_3}(\pi\alpha_1)\cdot\vec{S}_{1,3}(\pi\alpha_1) + \mathrm{H.c.}$$

(We note that the symbol Y represents a closed string. Also, the summation over ψ is purely symbolic, and represents the sum over all fermion, meson, and Pomeron wave functions in the Lagrangian compatible with duality diagrams for the vertex, i.e., three-fermion interactions are not allowed, Pomerons only couple to mesons, the number of fermions is conserved in each vertex, etc.) The functional derivative is to be taken on the δ functional (or the wave functional):

$$\mathcal{L}_{\mathrm{II}} = \tfrac{1}{4}g^2 \sum_{\psi} \int \prod_{i=1}^{4} \mathfrak{D}\vec{X}_i \mathfrak{D}\vec{S}_{1,i} \mathfrak{D}\vec{S}_{2,i} \frac{dp_{+i}}{(2p_{+i})^{1/2}} \delta(p_{+1}+p_{+2}-p_{+3}-p_{+4}) \times \psi^{\dagger}_{p_{+1}}[X_1,S_1]\,\psi^{\dagger}_{p_{+2}}[X_2,S_2]\,\psi_{p_{+3}}[X_3,S_3]\,\psi_{p_{+4}}[X_4,S_4]\,\delta(\mathrm{II}) \frac{\vec{\delta}}{\delta\vec{X}_1}(\sigma_0)\cdot\vec{S}_{1,1}(\sigma_0) \frac{\vec{\delta}}{\delta\vec{X}_4}(\sigma_0)\cdot\vec{S}_{2,4}(\sigma_0) + \mathrm{H.c.},$$

$$\mathcal{L}_{\mathrm{III}} = \tfrac{1}{4}g^2 \sum_{\psi} \int \prod_{i=1}^{3} \mathfrak{D}\vec{Y}_i \mathfrak{D}\vec{S}_{1,i} \mathfrak{D}\vec{S}_{2,i} \frac{dp_{+i}}{(2p_{+i})^{1/2}} \delta(p_{+1}+p_{+2}-p_{+3})\psi^{\dagger}_{p_{+1}}[Y_1,S_1]\,\psi^{\dagger}_{p_{+2}}[Y_2,S_2]\,\psi_{p_{+3}}[Y_3,S_3] \times \delta(\mathrm{III}) \frac{\vec{\delta}}{\delta\vec{X}_1}(\pi\alpha_1)\cdot\vec{S}_{1,1}(\pi\alpha_1) \frac{\vec{\delta}}{\delta\vec{X}_3}(0)\cdot\vec{S}_{2,3}(0) + \mathrm{H.c.},$$

$$\mathcal{L}_{\mathrm{IV}} = \tfrac{1}{4}g^2 \sum_{\psi} \int \prod_{i=1}^{2} \mathfrak{D}\vec{X}_i \mathfrak{D}\vec{Y}_3 \mathfrak{D}\vec{S}_{1,i} \mathfrak{D}\vec{S}_{2,i} \mathfrak{D}\vec{S}_{1,3} \mathfrak{D}\vec{S}_{2,3} \frac{dp_{+i}}{(2p_{+i})^{1/2}} \frac{dp_{+3}}{(2p_{+3})^{1/2}} \delta(p_{+1}-p_{+2}-p_{+3}) \times \psi^{\dagger}_{p_{+1}}[X_1,S_1]\,\psi_{p_{+2}}[X_2,S_2]\,\psi_{p_{+3}}[Y_3,S_3]\,\delta(\mathrm{IV}) \frac{\vec{\delta}}{\delta\vec{X}_1}(\sigma_0)\cdot\vec{S}_{1,1}(\sigma_0) \frac{\vec{\delta}}{\delta\vec{X}_2}(\sigma_0)\cdot\vec{S}_{2,2}(\sigma_0) + \mathrm{H.c.},$$

$$\mathcal{L}_{\mathrm{V}} = \tfrac{1}{2}g \sum_{\psi} \int \mathfrak{D}\vec{X}\, \mathfrak{D}\vec{Y} \prod_{i=1}^{2} \mathfrak{D}\vec{S}_{1,i} \mathfrak{D}\vec{S}_{2,i} \frac{dp_{+}}{(2p_{+})^{1/2}} \delta(\mathrm{V})\psi^{\dagger}_{p_{+}}[X,S_1]\,\psi_{p_{+}}[Y,S_2] \frac{\vec{\delta}}{\delta\vec{X}}(0)\cdot\vec{S}_{1,1}(0) + \mathrm{H.c.}$$

The functional derivative with respect to the X variable can be shown to be equivalent to using $\dot{X}$.

Notice that the G factor in the vertices is related to the G factor found by Mandelstam.[14]

It is now a simple matter to calculate finite-time matrix elements with arbitrary external particles. We simply sandwich the δ functional between the various Green's functions (for each leg) and functionally integrate over all intermediate string states. (We must not forget the term in the vertex proportional to the G operator.) The explicit calculation of the finite-time matrix elements is a trivial matter, simply using Eq. (2.17) a repeated number of times. We will not present these tedious but straightforward calculations, except in the case of the three-string interaction (which is presented in Appendix C). We will define some of our finite-time matrix elements as follows:

$$V_{\rm I}(\tau_1,\tau_2,\tau_3)=\int G(X_3',S_3',\tau_3;X_3,S_3,0)\delta({\rm I})\frac{\vec{\delta}}{\delta\vec{X}_3}(\pi\alpha_1)\cdot\vec{S}_{1,3}(\pi\alpha_1)\,G(X_1,S_1,0;X_1',S_1',\tau_1)G(X_2,S_2,0;X_2',S_2',\tau_2)$$
$$\times\mathfrak{D}\vec{X}_3\mathfrak{D}\vec{X}_1\mathfrak{D}\vec{X}_2\prod_{i=1}^{3}(\mathfrak{D}\vec{S}_{1,i}\mathfrak{D}\vec{S}_{2,i}),$$

$$V_{\rm II}(\tau_1,\tau_2,\tau_3,\tau_4)=\int\prod_{i=1}^{2}\prod_{j=3}^{4}G(X_i',S_i',\tau_i;X_i,S_i,0)\delta({\rm II})\frac{\vec{\delta}}{\delta\vec{X}_1}(\sigma_0)\cdot\vec{S}_{1,1}(\sigma_0)\frac{\vec{\delta}}{\delta\vec{X}_4}(\sigma_0)\cdot\vec{S}_{2,4}(\sigma_0)$$
$$\times G(X_j,S_j,0;X_j',S_j',\tau_j)\prod_{k=1}^{4}\mathfrak{D}\vec{X}_k\mathfrak{D}\vec{S}_{1,k}\mathfrak{D}\vec{S}_{2,k},$$

$$V_{\rm III}(\tau_1,\tau_2,\tau_3)=\int\prod_{i=1}^{2}G_p(Y_i',S_i',\tau_i;Y_i,S_i,0)\delta({\rm III})\frac{\vec{\delta}}{\delta\vec{X}_1}(\pi\alpha_1)\cdot\vec{S}_{1,1}(\pi\alpha_1)\frac{\vec{\delta}}{\delta\vec{X}_3}(0)\cdot\vec{S}_{2,3}(0)$$
$$\times G(Y_3,S_3,0;Y_3',S_3',\tau_3)\prod_{i=1}^{3}\mathfrak{D}\vec{Y}_i\mathfrak{D}\vec{S}_{1,i}\mathfrak{D}\vec{S}_{2,i}\ \text{etc.}$$

As we shall see in Appendix C, the integration over all intermediate string modes yields matrix elements which are related to the Neumann function defined over the integration region (i.e., the Riemann surface). By sewing more and more complicated Riemann surfaces together, it is not hard to see that we simply reproduce the various Neumann functions defined over the surface, in agreement with the results found by Mandelstam. Thus, we can sew these finite-time vertices to reproduce the four-point interactions of the dual pion model, in agreement with the results of Neveu, Schwarz, and Ramond.

There are some as-yet-unresolved ambiguities concerning the exact form of the interaction vertices. The most serious ambiguity concerns whether or not these vertices are able to reproduce spinor Neumann functions which are "double sheeted" when going around a loop. Consider, for example, the single-loop diagram, consisting of a long strip with an internal horizontal slit. The Neumann function defined over this Riemann sheet can be obtained by using our sewing prescription. At each joint, we insert and integrate over a complete set of spinor eigenstates and use the δ functional as the vertex. The resulting spinor Neumann function defined over this surface is a continuous, smooth function of ρ and ρ'. Now keep ρ fixed and let ρ' vary continuously around the internal slit. If ρ' makes a complete trip around the hole and returns to its original position, then does the Neumann function acquire a factor of (±)? Since this Riemann sheet is conformally equivalent to a Pomeron tube disappearing into the "vacuum," and since we know that the Pomeron Neumann function *does* acquire a (−) under certain circumstances, we are faced with the problem of obtaining a "double-sheeted" Neumann function constructed out of single-sheeted vertex and propagator functions. This double-sheeted spinor Neumann function, which acquires a (±) when ρ' circulates around a hole of the Riemann surface, may possibly arise out of sewing single-sheeted Neumann functions. We do not know. An alternative prescription is to change our sewing rules such that S of one surface maps onto $-S$ of the other connecting surface. This ambiguity has not been resolved.

Another ambiguity arises out of using the R_1 and R_2 formalisms. It is *not* true that they are exactly equivalent. When going from the R_2 to the R_1 formalism, there appears a factor $(2m)^{-1}$ in front of all trees. But because we wish to take m vanishing, we are faced with a problem of defining

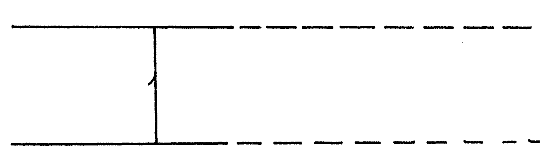

FIG. 11. One string turns into a ring (Pomeron).

this limit. In practice the limit can be smoothly defined if one keeps m finite in both the spinors and the operators and takes the limit at the end of the calculation. So the equivalence of the R_1 and R_2 formalisms is not a simple one. It is possible that a simple renormalization of the neutrino spinor wave function will also absorb this ambiguity.

IV. CONCLUDING REMARKS

In this paper we have presented the Lagrangian responsible for describing the interacting spinning string. The theory is a natural generalization of the Dirac and Klein-Gordon equation extended to multilocal strings. We can easily apply our functional rules to calculate all finite-time matrix elements, which are found to agree with those found by Mandelstam.

There are some unresolved theoretical points, however. First, we have not been able to find the manifestly Lorentz-invariant Lagrangian, mainly because of the difficulty of handling two infinite sets of gauge invariances, and because of the problem of multitime quantization. Because of this, we have been forced to work in the light-cone gauge from the start, and prove Lorentz invariance by constructing the Lorentz generators. In this light-cone formalism, however, we cannot go off the mass shell or construct currents, because the Lorentz generators are *defined* on the mass shell. The Lagrangian, strictly speaking, is therefore *not* Lorentz-invariant (at least the off-shell Lorentz generators have not yet been found) but does produce relativistic S-matrix elements.

Second, we have not yet resolved the question of measure. In all the functional integrations, we are left with determinants of Neumann matrices. We can show generally that these determinants have symmetry properties in various limits, similar to the Jacobian, but we have not shown that the determinant is, in fact, the Jacobian of the transformation from light-cone coordinates to Koba-Nielsen coordinates.

Third, we have not yet fully explored the implications of taking various limits on our Lagrangian. For example, in the zero-width limit, the Lagrangian should reduce to a Yang-Mills particle interacting with a Dirac particle. And if we make a power expansion of the matrix elements in the coupling constant, we should be able to reproduce quantized gravity in the light-cone gauge.[15]

And fourth, there are still questions of ambiguity concerning the double-sheeted Riemann surface associated with even-G-parity Pomerons and with loops of mesons.

ACKNOWLEDGMENTS

We are happy to acknowledge numerous fruitful and beneficial conversations with Professor K. Kikkawa concerning very important details of the field theory of strings and the generally covariant dual pion model. We also wish to acknowledge valuable conversations with Professor B. Sakita on the question of Lorentz invariance of the theory of strings and the quantum mechanics of point particles.

After this work was completed, we found that Marshall and Ramond have also presented a field theory of spinning strings, though their model differs from ours in many fundamental respects.[16] (See also P. Ramond's treatment of the free-field theory of strings [in *Proceedings of the Johns Hopkins Workshop on Current Problems in High Energy Particle Theory, 1974,* edited by G. Domokos *et al.* (Johns Hopkins Univ., Baltimore, 1974)].)

We would also like to thank J. Shapiro and D. Olive on the questions of ambiguities within the formalism, i.e., with respect to double-sheeted Riemann surfaces and with the R_1 and R_2 formalisms.

We have also learned that Cremmer and Gervais[17] have completed the proof that the measure of integration is nothing but the Jacobian from the light-cone variables to the Koba-Nielsen variables. This result can be carried over to the case of the dual pion model as well.

And lastly, we wish to thank the hospitality of the Aspen Center for Physics where part of this work was done.

APPENDIX A

In this section we wish to make some elementary comments on the difference between first- and second-quantized formalisms. We start with the simplest of all systems, the first-quantized point particle, which has the action

$$I=-m\int_{X_i}^{X_f} dS = -m\int_0^T d\tau\left[\left(\frac{dX_\mu(\tau)}{d\tau}\right)^2\right]^{1/2}. \quad \text{(A1)}$$

Notice that if we make the identification $X_0(\tau)=\tau$ then we obtain the usual

$$I=\int_0^T -m(1-v^2)^{1/2}\,d\tau\,. \quad \text{(A2)}$$

Also notice that the Lagrangian (A1) is invariant under an arbitrary change in the parameter τ. This gauge invariance should be reflected in the fact that not all the momenta are independent. Taking derivatives with respect to $\dot{X}$, we find the following constraint among the momenta:

$$P_\mu \equiv \frac{\partial L}{\partial \dot{X}_\mu} = -m[\dot{X}_\mu/(\dot{X}^2)^{1/2}] ,$$
$$P^2 - m^2 = \left(\frac{\partial L}{\partial \dot{X}_\mu}\right)^2 - m^2 = 0 . \tag{A3}$$

We can quantize the model by introducing quantization relations

$$[X_\mu, P_\nu] = i g_{\mu\nu} . \tag{A4}$$

Equation (A3) shows us that not all of the momenta are independent. We can now write down the Klein-Gordon equation for the model by letting Dirac's ϕ condition (A3) act on a state vector:

$$(P^2 - m^2)|\phi\rangle = 0 .$$

We now *second*-quantize the model by imposing canonical quantization relations:

$$[\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t')]_{t=t'} = i\delta(\vec{x}-\vec{y}) . \tag{A5}$$

Because we have a gauge degree of freedom, we could have taken the light-cone gauge

$$X_+ = \frac{X_0 + X_3}{\sqrt{2}} = k\tau , \tag{A6}$$

in which case

$$L = -m(2k\dot{X}_- - \dot{\vec{X}}^2)^{1/2} ,$$
$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{X}}} = m\dot{\vec{X}}/(2k\dot{X}_- - \dot{\vec{X}}^2)^{1/2} ,$$
$$P_+ = -\frac{\partial L}{\partial \dot{X}_-} = mk/(2k\dot{X}_- - \dot{\vec{X}}^2)^{1/2} , \tag{A7}$$
$$H = \vec{P}\cdot\dot{\vec{X}} - P_+\dot{X}_- - L = \frac{k}{2P_+}(\vec{P}^2 + m^2) \equiv H_{P_+} .$$

Both first- and second-quantized versions of the quantum mechanics of point particles have identical Green's functions and light-cone quantizations. The main difference appears when we add *interactions*. In the first-quantized form we must introduce interactions which are functions of the X's, and therefore represent the interaction of the *point particle with external fields*.

Since we are aiming at a theory in which strings "fission," we first wish to produce a theory in which a point particle in the first-quantized formalism can fission. In this way we can reproduce Feynman's rules for ϕ^3 and ϕ^4 theories. In order to reproduce Feynman's rules for a first-quantized theory, we must add in the *topology of each diagram by hand*. Equation (A5) therefore can reproduce ϕ^3 and ϕ^4 field theory once we specify the topology of each diagram by hand. (Notice that unitarity cannot be proved in the first-quantized formalism, where interactions are introduced by arbitrarily specifying the topology of each diagram, since unitarity requires a Hermitian Hamiltonian.)

An exactly analogous situation occurs in the dual theory. In the functional form of the model the Born term corresponds to taking functional averages over a disk (or strip), rather than a line (as in the first-quantized approach for point particles). In order to introduce interactions into the model at the loop level, we are forced to sum over different topologies (i.e., spheres with N handles) which are *inserted by hand*. Again, unitarity cannot be proved in the first-quantized approach because we do not have a Hermitian Hamiltonian. In the first-quantized dual model, the counting of topological surfaces is somewhat random, guided only by the unitarity prescriptions at the first loop level.

In the second-quantized version of the dual model, we have an explicitly self-adjoint Hamiltonian which reproduces all the interaction vertices of the first-quantized model, except now the counting of diagrams is exact, and unitarity, at least formally, can be proved.

APPENDIX B

In order to prove (2.29) and establish the relationship between the first- and second-quantized formalisms, we first evaluate the functional integral by completing the square:

$$S_i(\rho) \to S_i(\rho) + \tfrac{1}{2}\int d\rho' K_{ij}(\rho, \rho')\eta_j(\rho') , \tag{B1}$$

such that

$$(\partial_\tau + i\partial_\sigma)K_{1j}(\rho, \rho') = 2\pi\delta_{1j}\,\delta^2(\rho-\rho') ,$$
$$(\partial_\tau - i\partial_\sigma)K_{2j}(\rho, \rho') = 2\pi\delta_{2j}\,\delta^2(\rho-\rho') . \tag{B2}$$

Now, by inserting (B1) into Eq. (2.29), the spinor integration yields

$$\int \mathfrak{D}\vec{S}\exp\left[\int\left(-\frac{1}{2\pi}[S_1(\overleftrightarrow{\partial}_\tau + i\overleftrightarrow{\partial}_\sigma)S_1 + S_2(\overleftrightarrow{\partial}_\tau - i\overleftrightarrow{\partial}_\sigma)S_2] + \eta_i(\rho)S_i(\rho)\right)d\rho\right]$$
$$\sim \prod_{i=1}^{2}\exp\left[\left(-\frac{1}{2\pi}\right)\left(\tfrac{1}{2}\int(\partial_\tau + i\partial_\sigma)[S_1(\rho)K_{1j}(\rho,\rho')\eta_j(\rho')d\rho\,d\rho'] + \tfrac{1}{2}\int(\partial_\tau - i\partial_\sigma)[S_2(\rho)K_{2j}(\rho,\rho')\eta_j(\rho')d\rho\,d\rho']\right)\right.$$
$$\left. + \tfrac{1}{4}\int\eta_j(\rho)K_{ji}(\rho,\rho')\eta_i(\rho')d\rho\,d\rho'\right] . \tag{B3}$$

If we perform the σ integration, the σ surface term vanishes if we take the usual boundary conditions (with a corresponding one for the K's). Performing the τ integration leaves the following remainder:

$$\sum_{i,j=1,2} \int_0^{\pi\alpha} d\sigma\, S_i(\tau,\sigma) K_{ij}(\tau,\sigma;\rho')\eta_j(\rho')d\rho' \Big|_{\tau=\tau_1}^{\tau=\tau_2} . \tag{B4}$$

This term can be eliminated if we set

$$\begin{aligned} &S_i(\tau_1,\sigma) = S_i(\tau_2,\sigma), \\ &K_{ij}(\tau_1,\sigma;\rho') = K_{ij}(\tau_2,\sigma;\rho') . \end{aligned} \tag{B5}$$

Given (B5) and (B6), we can now solve for $K_{ij}(\rho,\rho')$.

Let us first calculate the K's for meson-meson scattering, and then conformally map the upper-half plane onto the infinitely long strip. In the upper half-plane, the K's are as follows:

$$K_{11}(z,z') = \frac{1}{z-z'} \quad . \tag{B6}$$

(The other K's can be found by taking complex conjugates of the z's.) Under a conformal transformation, K transforms like

$$K_{11}(\rho,\rho') = \left(\frac{\partial\rho}{\partial z}\right)^{-1/2}\left(\frac{\partial\rho'}{\partial z'}\right)^{-1/2} K_{11}(z,z') . \tag{B7}$$

Taking $\rho = \ln z = (\tau + i\sigma)/\alpha$, we get

$$\begin{aligned} K_{11}(\rho,\rho') &= \frac{(zz')^{1/2}}{z-z'} \\ &= \sum_{n=1/2}^{\infty} \exp\left[\frac{n}{\alpha}(\tau'+i\sigma') - \frac{n}{\alpha}(\tau+i\sigma)\right] \quad (\tau>\tau') \\ &= -\sum_{n=1/2}^{\infty} \exp\left[\frac{n}{\alpha}(\tau+i\sigma) - \frac{n}{\alpha}(\tau'+i\sigma')\right] \quad (\tau'>\tau) . \end{aligned}$$

Notice that $K_{11}(\rho,\rho') = -K_{11}(\rho',\rho)$. Now that we have found the K's defined on an infinitely long strip, it is a simple matter to match boundary conditions to find the K's defined on a finite rectangular surface:

$$K_{11}(\tau,\sigma;\tau',\sigma') = \sum_{n=1/2}^{\infty}\left\{(-1+a_n)\exp\left[\frac{n}{\alpha}(\tau+i\sigma) - \frac{n}{\alpha}(\tau'+i\sigma')\right] - a_n \exp\left[-\frac{n}{\alpha}(\tau+i\sigma) + \frac{n}{\alpha}(\tau'+i\sigma')\right]\right\},$$

$$a_n \equiv \frac{1}{1-\exp[-(n/\alpha)(\tau_1-\tau_2)]} \quad .$$

If we use the Green's function to calculate finite-time matrix elements, we get [compare with Eq. (2.26)]

$$\begin{aligned} \int d\rho\, d\rho' [\eta_i(\rho)\delta(\tau-\tau_1) + \eta_i'(\rho)\delta(\tau-\tau_2)] K_{ij}(\rho,\rho')[\eta_j'(\rho')\delta(\tau-\tau_2) + \eta_j(\rho')\delta(\tau-\tau_1)] \\ = \prod_{n=1/2}^{\infty} \exp\left[\frac{\pi}{2}\left(-\tanh\frac{n(\tau_1-\tau_2)}{2\alpha}(\xi_n\xi_{-n} + \xi_n'\xi_{-n}') - \frac{e^{n(\tau_1-\tau_2)/2\alpha}}{\sinh[n(\tau_1-\tau_2)/2\alpha]}\xi_{-n}\xi_n' + \frac{e^{-n(\tau_1-\tau_2)/2\alpha}}{\sinh[n(\tau_1-\tau_2)/2\alpha]}\xi_n\xi_{-n}'\right)\right] , \end{aligned}$$

where

$$\eta_1 = \sum_n \frac{1}{\sqrt{2\alpha}}\xi_n e^{in\sigma/\alpha},\ \eta_2 = \sum_n \frac{1}{\sqrt{2\alpha}}\xi_{-n}e^{in\sigma/\alpha}, \quad n = (\ldots, -\tfrac{1}{2}, \tfrac{1}{2}, \ldots) .$$

APPENDIX C

In order to calculate the finite-time matrix element between three spinning strings, we must make repeated use of the identity (2.17). In addition, we must be careful to include the contribution of the factor $\vec{S}\cdot\delta/\delta\vec{X}$ taken at the point of contact of the two strings.[15] The integrations are easy to perform:

$$\int G(X_3', S_3', \tau_3; X_3, S_3, 0)\delta(I)\frac{\overleftrightarrow{\delta}}{\delta\vec{X}_3}(\pi\alpha_1)\cdot\vec{S}_{1,3}(\pi\alpha_1)\prod_{i=1}^{2} G(X_i, S_i, 0; X_i', S_i', \tau_i)\prod_{j=1}^{3}\mathfrak{D}\vec{X}_j\mathfrak{D}\vec{S}_{1,j}\mathfrak{D}\vec{S}_{2,j} .$$

We will only keep spinor contributions. We define

$$S_{1,i} = \frac{1}{(2\alpha_i)^{1/2}}\sum_n \xi_n^i e^{in\sigma_i/\alpha_i} ,$$

$$S_{2,i} = \frac{1}{(2\alpha_i)^{1/2}}\sum_n \xi_{-n}^i e^{in\sigma_i/\alpha_i} ,$$

$$M_{n,m}^{13}=\left(\frac{\alpha_1}{\alpha_3}\right)^{1/2}\int_{-\pi\alpha_1}^{\pi\alpha_1} d\sigma_3 e^{-in\sigma_1/\alpha_2+im\sigma_3/\alpha_3},$$

$$M_{n,m}^{23}=\left(\frac{\alpha_2}{\alpha_3}\right)^{1/2}\left(\int_{\pi\alpha_1}^{\pi\alpha_3}+\int_{-\pi\alpha_3}^{-\pi\alpha_1}\right)d\sigma_3\, e^{-in\sigma_2/\alpha_2+im\sigma_3/\alpha_3},$$

$$\xi_n^1=\sum_m M_{nm}^{13}\xi_m^3,$$

$$\xi_n^2=\sum_m M_{nm}^{23}\xi_m^3.$$

Now,

$$\left(\prod_{l=1/2}^{\infty}\prod_{i=1}^{3}\sinh\frac{lT_i}{2\alpha_i}\right)^{D-2}\int\prod_{i=1}^{3}\prod_{n=1/2}^{\infty}(\mathfrak{D}\xi_{-n}^i\mathfrak{D}\xi_n^i)\exp\left\{\sum_{i=1}^{3}\sum_{l=1/2}^{\infty}\left[a_l^i(\xi_l^i\xi_{-l}^i+\xi_l^{i\prime}\xi_{-l}^{i\prime})+b_l^i\xi_l^i\xi_{-l}^{i\prime}+c_l^i\xi_{-l}^i\xi_l^{i\prime}\right]\right\}$$
$$\times\left|\prod_n\delta\left(\xi_n^1-\sum_m M_{nm}^{13}\xi_m^3\right)\delta\left(\xi_n^2-\sum_m M_{nm}^{23}\xi_m^3\right)\right.$$
$$=\left(\prod_{l=1/2}^{\infty}\prod_{i=1}^{3}\sinh\frac{lT_i}{2\alpha_i}\right)^{D-2}(\det 2A)^{(D-2)/2}\exp\left(-\tfrac{1}{2}B\frac{1}{A}B\right),$$

where

$$a_{l,l'}^i\equiv\delta_{l,l'}A_{l'}(-1)^{l'}/2,$$

$$A_{l,l'}=a_{l,l'}^3+\sum_{\substack{m\\ i=1,2}}M_{ln}^{3i}a_{nm}^iM_{ml'}^{i3},$$

$$B_l=\xi_{-l}^3b_l^3+\sum_{\substack{i=1,2\\ l'}}M_{l,l'}^{3i}b_{l'}^i\xi_{l'}^i,$$

$$a_l^i\equiv\cosh(lT_i/2\alpha_i),$$

$$b_l^i\equiv\frac{-e^{-lT_i/2\alpha_i}}{\sinh(lT_i/2\alpha_i)},$$

$$c_l^i\equiv\frac{e^{lT_i/2\alpha_i}}{\sinh(lT_i/2\alpha_i)},$$

$$T_1=-\tau_1,\quad T_2=-\tau_2,\quad T_3=\tau_3.$$

(Unless otherwise specified, the summation is always over both positive and negative half-integers.) (We have neglected the contribution of the G factor in the vertex. Suitable functional differentiations of the vertex will yield the contribution of the G factor.)

Mandelstam found that the interaction vertices can all be related to the Neumann functions (both orbital and spinor) defined over the appropriate two-dimensional Riemann surface. We reproduce this result.

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[11]In our previous works the presence of more interaction terms in our Lagrangian could not be strictly ruled out. But an interaction representing strings touching at more than one point (1) can probably be separated into a sequence of primitive interactions via a Lorentz transformation, and (2) such an interaction would have to have zero measure, which is unlikely.

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Critical behavior in a class of O(N)-invariant field theories in two dimensions*

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Critical behavior in a class of two-dimensional field theories which exhibit dynamical symmetry breaking at zero temperature is analyzed in the $1/N$ approximation. We show that, in the case of an O(N)-invariant theory of massless, N-component, Fermi fields, a phase transition takes place in the limit as N goes to infinity. The critical temperature, above which the model becomes symmetric, is given in terms of the induced fermion mass at zero temperature, m_f^0, as $m_f^0\beta_c = 1.764$. The equivalence between the critical parameters of the theory and those predicted by the BCS theory of superconductivity is established. We show that the BCS gap equation follows from the stability conditions imposed on the effective potential. The phase transition is discussed in a thermodynamical analog of the model. The analysis of the symmetry behavior of the theory is carried out by functional methods.

I. INTRODUCTION

Recently, Gross and Neveu analyzed a class of two-dimensional field theories of N-component, massless fermions with O(N)-invariant quartic interactions.[1] They showed that the fermions acquired a mass via dynamical symmetry breaking.

The possibility for the restoration of certain symmetries as a consequence of finite-temperature effects has recently been considered by several authors,[2-4] who found critical behavior in some cases of spontaneous symmetry violation.

In this paper we investigate the behavior of the two-dimensional O(N) fermion theories and show the existence of a second-order phase transition. The study of the symmetry behavior of the model is carried out by use of the effective potential formalism. Since the methods of computation as well as the physical meaning of the effective potential and its role in the investigation of symmetry breaking have been treated extensively in the literature,[5-7] we will avoid detailed calculations and definitions of the methods employed. In the $1/N$ approximation,[8] which seems to be consistent in the theory treated here, calculations are greatly simplified by the use of a combinatoric trick.[1,9] To avoid possible inconsistencies, we use the imaginary-time formalism in our finite-temperature calculations.[4,10,11]

The paper is organized as follows: In Sec. I we obtain the finite-temperature generalization of the O(N) fermion model. We exhibit the symmetry-breaking solution as well as the critical temperature, above which the model regains its symmetry. Here we also obtain an equation for the temperature-dependent mass, and solve it explicitly, in the limits $\beta \gtrsim \beta_c$ and $\beta \gg \beta_c$. In Sec. II we identify the temperature-dependent fermion mass with the BCS gap function, Δ_β. With this identification established, we show that the BCS gap equation is obtained as the solution to the stability condition for the effective potential. Also in this section we draw a thermodynamical analog of the model and briefly discuss the phase transition in this context.

Throughout our investigation we will consider only those states for which the vacuum is translationally invariant. Therefore, we will take the classical fields, and hence the effective potential, to be space-time independent.