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Asymptotic freedom and finiteness of the wave-function renormalization constant*

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It is shown that a particle field is asymptotically free in the ultraviolet limit if the wave-function renormalization constant corresponding to the field is a nonvanishing function of coupling constants. In the proof we postulate the positivity of the Källén-Lehmann spectral function and the renormalizability of the theory.

I. INTRODUCTION

The renormalization-group equation[1] or the Callan-Symanzik equation[2] approach is now standard in the investigation of the ultraviolet behavior of renormalizable field theories. Asymptotically free theories are requested in connection with Bjorken scaling.[3] Non-Abelian gauge theories have recently been found to be ultraviolet-stable at the origin of the coupling constant,[4,5] whereas other familiar field theories would not satisfy the asymptotic freedom[6] in ordinary four-dimensional space-time. In an asymptotically free theory, an N-point Green's function $\Gamma^{(N)}$ will behave in the ultraviolet region as

$$\Gamma^{(N)}(lp_i,g,m) \underset{l\gg 1}{\longrightarrow} l^{D-N\gamma(\bar{g}(l))}\Gamma^{(N)}(p_i,\bar{g}(l),m) \times(\text{possible logarithmic factor of } l) \tag{1.1}$$

for the nonexceptional momenta p_i. Here D denotes the canonical dimension of $\Gamma^{(N)}$. The effective coupling constant $\bar{g}(l)$ satisfies the differential equation

$$l\frac{d\bar{g}(l)}{dl}=\beta(\bar{g}(l)) \,, \tag{1.2}$$

with the initial condition $\bar{g}(1)=g$, and approaches zero as $l\to\infty$. The functions β and γ are coefficient functions in the Callan-Symanzik equation.

In this paper we shall present a theorem on asymptotic freedom in the theory with a finite wave-function renormalization constant, say, Z_3. Our theorem can be regarded as a generalization of the Federbush-Johnson theorem[7]: If a two-point Green's function coincides with a free propagator (i.e., $Z_3=1$), then the corresponding field to the two-point function should satisfy the free-field equation.

Our present theorem reads as follows.

Theorem. If a wave-function renormalization constant Z_3 is a nonvanishing function of coupling constants and satisfies the Källén-Lehmann positivity requirement $0<Z_3\leq 1$, then the corresponding field to our Z_3 is asymptotically free.

By asymptotic freedom we mean that the coupling constants which connect our field with other fields effectively approach zero in the ultraviolet limit when the renormalized coupling constants are taken sufficiently small. A complete proof of the theorem shall be given in subsequent sections. Here we wish to offer an intuitive interpretation of our theorem stated above. Imagine a cloud of virtually created pairs around a bare particle. In the ultraviolet region particle masses may be considered effectively small and hence pair creation will occur very frequently. Unless the effective interaction strengths get weaker in the ultraviolet region so as to suppress the increasing pair creations, the bare particle in the thick cloud will grow small. In the extreme limit we can scarcely sight it, i.e., the probability of finding the bare particle in the physical particle Z_3 tends to zero. Conversely, if we suppose that $Z_3\neq 0$, the interaction strengths or effective coupling constants should vanish in the ultraviolet limit.

In Sec. II we give a proof of our theorem in a simple case where only one charge is involved. The theorem for the multicharge case shall be verified in Sec. III. An example, two-dimensional quantum electrodynamics, is demonstrated in Sec. IV. Section V is devoted to our conclusion and discussion.

II. PROOF OF THE THEOREM—ONE-CHARGE CASE

In this section we prove the theorem for the one-charge case in order to obtain essential features in the general proof avoiding the complexity in the multicharge case. Let us presuppose that the theory under consideration is renormalizable in the usual sense. The first postulate in our theorem is that one of the wave-function renormalization constants, say, Z_3, appearing in the theory is a finite function of a coupling constant g as a result of the complete summation of all order contributions in perturbation theory. It is irrelevant here whether our Z_3 may or may not be cutoff-dependent in the finite-order approximation. It is well known that the renormalizability of the theory leads to the renormalization-group equations[1] or the Callan-Symanzik equations[2] for the renormalized Green's functions. For the present purpose we do not need the whole of them. We will simply exploit the Callan-Symanzik–type equation for the wave-function renormalization constant Z_3:

$$(\beta\partial/\partial g - 2\gamma)Z_3(\Lambda/\mu, g) = 0 . \qquad (2.1)$$

Here β and γ are the same coefficient functions which depend on g as those in the Callan-Symanzik equations for Green's functions (see Appendix A). We remark that if Z_3 were divergent, contrary to our postulate, Eq. (2.1) should be replaced by the differential equation

$$(\mu\partial/\partial\mu + \beta\partial/\partial g - 2\gamma)Z_3(\Lambda/\mu, g) = 0$$

in the cutoff regularization scheme, or should be replaced by

$$\{[\beta + \tfrac{1}{2}(n-4)g]\partial/\partial g - 2\gamma\}Z_3(n, g) = 0$$

in the dimensional-regularization method, with n being the space-time dimension.

From the Källén-Lehmann spectral representation of Z_3,

$$Z_3^{-1} = 1 + \frac{1}{\pi}\int_{\text{th}}^{\infty} ds\rho(s) , \qquad (2.2)$$

and the positivity of the spectral function ρ we can immediately deduce two things. One is that $0 < Z_3 \leqslant 1$ ($Z_3 \neq 0$ for $g \neq 0$ by the assumption), and the other is that the anomalous dimension $\gamma(g)$ is positive in the vicinity of the origin and vanishes at the origin. Here it should be noted that the one-loop contribution to γ in perturbation theory, which is evidently positive by the positivity of the norm of the intermediate states, dominates the function γ for a sufficiently small coupling constant. More compactly speaking, the functions $\ln[Z_3^{-1}(g)]$ and $\gamma(g)$ are positive-definite for all values of g and positive-definite in the neighborhood of the origin of coupling constant, respectively (see Appendix B for definition of positive-definiteness used here). These are consequences of the second postulate, the requirement of the positivity for spectral function in our theorem.

The following mathematical theorem on the stability of the nonlinear differential equation[8] is a key to the proof of our theorem which has been stated in the Introduction. Consider the differential equations of an autonomous system $\dot{x}_i = X_i(x_1x_2\cdots x_N)$ $(i = 1, \ldots, N)$ with a singularity at the origin and suppose that we have found a positive-definite function $V(x_1x_2\cdots x_N)$ in a region about the origin satisfying $V(00\cdots 0) = 0$. We define

$$U(x_1x_2\cdots x_N) = \sum_{i=1}^{N} X_i(x_1x_2\cdots x_N)\frac{\partial}{\partial x_i} \times V(x_1x_2\cdots x_N) .$$

Theorem **(Lyapunov). If U is negative-semidefinite** (negative-definite), then the motion of $x_i(t)$ is stable (asymptotically stable) at the origin (see Appendix B for definitions).

Armed with the Lyapunov stability theorem we can prove our theorem in the case where only one coupling constant is contained. From the positivity of the spectral function the function $\ln[Z_3^{-1}(g)]$ has been shown to be positive-definite, and therefore we can use it as a Lyapunov's function V. The Callan-Symanzik–type equation (2.1) can be converted into the form

$$U(g) \equiv \beta\frac{\partial}{\partial g}\ln[Z_3^{-1}(g)] = -2\gamma .$$

Since γ is a positive-definite function of g as we have discussed, the system of the differential equation,

$$d\bar{g}(t)/dt = \beta(\bar{g}(t)) , \qquad (2.3)$$

is asymptotically stable at the origin by the Lyapunov stability theorem. In other words the effective coupling constant $\bar{g}(t)$ which appears in the formulas for the asymptotic Green's functions tends to zero as t approaches infinity or $l \to \infty$ in Eqs. (1.1) and (1.2). This is just the conclusion of our theorem for one charge. For a general case we shall present its proof in the next section.

III. GENERAL PROOF

We now verify our theorem in the general case where we have an arbitrary number (say, N) of coupling constants. We classify the N coupling constants $g_1, g_2, \ldots, g_N$ into two groups: "direct coupling constants" $g_1, g_2, \ldots, g_m$ and "indirect coupling constants" $g_{m+1}, g_{m+2}, \ldots, g_N$, with respect to the wave-function renormalization constant Z_3 of the considered field. They are defined as follows. Direct coupling constants of Z_3 are such coupling constants that any member of them connects the considered field (which corresponds to our Z_3) with other fields or itself in the interaction Hamiltonian. The other coupling constants are called "indirect" ones.

It is important to observe that the function $\ln[Z_3^{-1}(g_1g_2\cdots g_N)]$ and the anomalous dimension $\gamma(g_1g_2\cdots g_N)$ of our field are both positive-definite as functions of the direct coupling constants $g_1, g_2, \ldots, g_m$ if we take all direct and indirect coupling constants $g_1, g_2, \ldots, g_m, g_{m+1}, \ldots, g_N$ sufficiently small. One may be easily convinced of this by drawing lowest-order Feynman diagrams which contribute to Z_3 and γ and by recalling the positivity of the spectral function in that order in perturbation theory.

The Callan-Symanzik-type equation for a finite wave-function renormalization constant Z_3 can be generally written as follows (see Appendix A):

$$\left(\sum_{i=1}^{N} \beta_i \partial/\partial g_i - 2\gamma\right) Z_3(g_1 g_2 \cdots g_N) = 0 . \qquad (3.1)$$

It will be more convenient to rewrite Eq. (3.1) in the form

$$\sum_{\substack{i=1 \\ \text{direct}}}^{m} \beta_i \frac{\partial}{\partial g_i} \ln Z_3^{-1}(g_1 g_2 \cdots g_N) = -\sum_{i=m+1}^{N} \beta_i \frac{\partial}{\partial g_i} \ln Z_3^{-1}(g_1 g_2 \cdots g_N) - 2\gamma(g_1 g_2 \cdots g_N) . \qquad (3.2)$$

As we noted before, the functions

$$\ln[Z_3^{-1}(g_1 g_2 \cdots g_m ; g_{m+1} \cdots g_N)]$$

and

$$\gamma(g_1 g_2 \cdots g_m ; g_{m+1} \cdots g_N)$$

are positive-definite functions of $g_1, g_2, \ldots, g_m$ if we take $g_1, g_2, \ldots, g_m ; g_{m+1}, \ldots, g_N$ sufficiently small. The right-hand side of Eq. (3.2) is dominated by the second term and therefore is negative-definite as a function of the direct coupling constants $g_1, g_2, \ldots, g_m$ for sufficiently small $g_1, g_2, \ldots, g_m ; g_{m+1}, \ldots, g_N$. By the usage of the Lyapunov stability theorem we can see that the system of the differential equations,

$$l\, d\bar{g}_i(l)/dl = \beta_i(\bar{g}_1(l)\bar{g}_2(l) \cdots \bar{g}_m(l); g_{m+1} \cdots g_N) , \quad (i=1, \ldots, m) \qquad (3.3)$$

is asymptotically stable at the origin of the direct coupling constants $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_m$ for the small fixed parameters $g_{m+1}, \ldots, g_N$. In order to see the implications of the asymptotic stability of the differential equations (3.3), we write the Callan-Symanzik equation for an asymptotic Green's function such that at least one of the external lines corresponds to the field under consideration:

$$\left(\mu\frac{\partial}{\partial\mu} + \sum_{i=1}^{m} \beta_i \frac{\partial}{\partial g_i} - \gamma_\Gamma\right) \Gamma^{\text{asy}}(p_j ; g_1 g_2 \cdots g_m ; g_{m+1} \cdots g_N) = -\sum_{i=m+1}^{N} \beta_i \frac{\partial}{\partial g_i} \Gamma^{\text{asy}}(p_j ; g_1 g_2 \cdots g_m ; g_{m+1} \cdots g_N) . \qquad (3.4)$$

For convenience we write the two kinds of β terms separately in both sides of Eq. (3.4). Solutions of Eq. (3.4) and the dimensional analysis lead to the formula

$$\begin{aligned} l^{-D}\Gamma^{\text{asy}}(lp_j ; g_1 g_2 \cdots g_m ; g_{m+1} \cdots g_N) &= \exp\left[-\int_1^l \frac{dl'}{l'} \gamma_\Gamma(\bar{g}_1(l')\bar{g}_2(l') \cdots \bar{g}_m(l'); g_{m+1} \cdots g_N)\right] \Gamma^{\text{asy}}(p_j ; \bar{g}_1(l)\bar{g}_2(l) \cdots \bar{g}_m(l); g_{m+1} \cdots g_N) \\ &+ \int_1^l \frac{dl'}{l'} l'^{-D} \exp\left(-\int_1^{l/l'} \frac{dl''}{l''} \gamma_\Gamma\right) \left[\sum_{i=m+1}^{N} \beta_i \left(\bar{g}_1\left(\frac{l}{l'}\right)\bar{g}_2\left(\frac{l}{l'}\right) \cdots \bar{g}_m\left(\frac{l}{l'}\right); g_{m+1} \cdots g_N\right)\right. \\ &\left.\times \frac{\partial}{\partial g_i} \Gamma^{\text{asy}}\left(l'p_j ; \bar{g}_1\left(\frac{l}{l'}\right)\bar{g}_2\left(\frac{l}{l'}\right) \cdots \bar{g}_m\left(\frac{l}{l'}\right); g_{m+1} \cdots g_N\right)\right]. \end{aligned} \qquad (3.5)$$

The first term will contain no problems. We are concerned with the large-l behavior of the second term of Eq. (3.5). Without loss of generality the fixed coupling constants $g_{m+1}, \ldots, g_N$ are supposed to be non-vanishing. Indeed, if some of them $g_{i0}, g_{i1}, \ldots$ are zero, we can simply drop the terms $\sum_{i=i0,i1,\ldots} \beta_i(\partial/\partial g_i)\Gamma^{\text{asy}}$ from the summation $\sum_{i=1}^{N} \beta_i(\partial/\partial g_i)\Gamma^{\text{asy}}$ in Eq. (3.5).

It is easy to verify the following formula which is a slight generalization of Eq. (3.5):

$$\begin{aligned} l_1^{-D}\Gamma^{\text{asy}}(l_1 p_j ; \bar{g}_1(l_2)\bar{g}_2(l_2) \cdots \bar{g}_m(l_2); g_{m+1} \cdots g_N) &= \exp\left[\int_1^{l_2} \frac{dl'}{l'} \gamma_\Gamma(\bar{g}_1(l')\bar{g}_2(l') \cdots \bar{g}_m(l'); g_{m+1} \cdots g_N)\right] \\ &\times \left\{\exp\left[-\int_1^{l_1 l_2} \frac{dl'}{l'} \gamma_\Gamma(\bar{g}_1(l')\bar{g}_2(l') \cdots \bar{g}_m(l'); g_{m+1} \cdots g_N)\right] \Gamma^{\text{asy}}(p_j ; \bar{g}_1(l_1 l_2)\bar{g}_2(l_1 l_2) \cdots \bar{g}_m(l_1 l_2); g_{m+1} \cdots g_N)\right. \\ &+ \int_1^{l_1} \frac{dl'}{l'} l'^{-D} \exp\left[-\int_1^{l_1 l_2/l'} \frac{dl''}{l''} \gamma_\Gamma(\bar{g}_1(l'')\bar{g}_2(l'') \cdots \bar{g}_m(l''); g_{m+1} \cdots g_N)\right] \\ &\times \left[\sum_{i=m+1}^{N} \beta_i \left(\bar{g}_1\left(\frac{l_1 l_2}{l'}\right)\bar{g}_2\left(\frac{l_1 l_2}{l'}\right) \cdots \bar{g}_m\left(\frac{l_1 l_2}{l'}\right); g_{m+1} \cdots g_N\right)\right. \\ &\left.\left.\times \frac{\partial}{\partial g_i} \Gamma^{\text{asy}}\left(l'p_j ; \bar{g}_1\left(\frac{l_1 l_2}{l'}\right)\bar{g}_2\left(\frac{l_1 l_2}{l'}\right) \cdots \bar{g}_m\left(\frac{l_1 l_2}{l'}\right); g_{m+1} \cdots g_N\right)\right]\right\}. \end{aligned} \qquad (3.6)$$

We insert the expression written above into the second term of Eq. (3.5) and iterate this procedure. It reads

$$l^{-D}\Gamma^{\text{asy}}(lp_j;g_1g_2\cdots g_N)$$
$$=\exp\left[-\int_1^l \frac{dl'}{l'}\gamma_\Gamma(\bar g_1(l')\bar g_2(l')\cdots\bar g_m(l');g_{m+1}\cdots g_N)\right]\Gamma^{\text{asy}}(p_j;\bar g_1(l)\bar g_2(l)\cdots\bar g_m(l);g_{m+1}\cdots g_N)$$
$$+\int_1^l \frac{dl'}{l'}\exp\left[-\int_1^{l/l'}\frac{dl''}{l''}\gamma_\Gamma(\bar g_1(l'')\bar g_2(l'')\cdots\bar g_m(l'');g_{m+1}\cdots g_N)\right]$$
$$\times\sum_{i=m+1}^N \beta_i\left(\bar g_1\left(\frac{l}{l'}\right)\bar g_2\left(\frac{l}{l'}\right)\cdots\bar g_m\left(\frac{l}{l'}\right);g_{m+1}\cdots g_N\right)\frac{\partial}{\partial g_i}$$
$$\times\left\{\exp\left[\int_l^{l/l'}\frac{dl''}{l''}\gamma_\Gamma(\bar g_1(l'')\bar g_2(l'')\cdots\bar g_m(l'');g_{m+1}\cdots g_N)\right]\Gamma^{\text{asy}}(p_j;\bar g_1(l)\bar g_2(l)\cdots\bar g_m(l);g_{m+1}\cdots g_N)\right\}$$
$$+\cdots\;. \tag{3.7}$$

It should be noticed in Eq. (3.7) that each iterated term contains $\Gamma^{\text{asy}}(p_j;\bar g_1(l)\cdots)$, which depends on l but not on the integration variable l'. If we recall that the function

$$\Gamma^{\text{asy}}(p_j;\bar g_1(l)\bar g_2(l)\cdots\bar g_m(l);g_{m+1}\cdots g_N)$$

contains at least one of the direct coupling constants $\bar g_1(l), \bar g_2(l), \ldots, \bar g_m(l)$ as a factor, we can immediately observe that each term in Eq. (3.7) will vanish in the limit $l\to\infty$. All the arguments stated above amount to the fact that the right-hand side of Eq. (3.5) vanishes in the limit $l\to\infty$. Hence we conclude that the considered field whose wave-function renormalization constant Z_3 is finite will be asymptotically decoupled in the Green's functions in such a way that the direct coupling constants $\bar g_1, \bar g_2, \ldots, \bar g_m$ effectively approach zero in the ultraviolet limit. This is just what is alluded to.

We have a corollary of our theorem.

Corollary. If all the wave-function renormalization constants in the theory are finite functions of coupling constants, then the theory is asymptotically free. This can be also shown directly when we use the positive-definite function $\sum_k \ln[Z_3^{(k)-1}(g_1g_2\cdots g_N)]$ as a Lyapunov function with the positive-definiteness of the sum of the anomalous $\sum_k\gamma^{(k)}(g_1g_2\cdots g_N)$ kept in mind where the summations extend over all kinds of fields. In the next section we shall demonstrate an example of our theorem in a soluble model.

IV. EXAMPLE

In this section we wish to give an example in order to demonstrate how our theorem works in a soluble model. Our example is quantum electrodynamics in two-dimensional space-time, which was first considered by Schwinger.[9] Crewther *et al.*[10] recently discussed the short-distance behavior of the two-dimensional QED in the context of the renormalization-group approach according to Wilson.[11] They define a dimensionless coupling constant e_λ such that

$$e_\lambda{}^2=e_0{}^2/(\lambda^2+e_0{}^2/\pi)\;, \tag{4.1}$$

where e_0 is the bare electromagnetic coupling constant with the dimension of mass and λ is a subtraction mass. The full photon propagator $D'_{F\mu\nu}(q)$ is solved to be

$$D'_{F\mu\nu}(q)=-\frac{g_{\mu\nu}}{q^2}-\frac{e_0{}^2}{\pi}+\text{gauge term}\;. \tag{4.2}$$

From Eq. (4.2) we obtain the photon wave-function renormalization constant Z_3^λ with the subtraction performed at $q^2=-\lambda^2$ as follows:

$$Z_3^\lambda=1-e_\lambda{}^2/\pi\;. \tag{4.3}$$

Anomalous dimension of the photon field $\gamma(e_\lambda)$ is easily gotten by the formula

$$\gamma=\tfrac{1}{2}[\lambda(\partial/\partial\lambda)\ln Z_3^\lambda]_{e_0\text{ fixed}}\;.$$

That is,

$$\gamma(e_\lambda)=e_\lambda{}^2/\pi\;. \tag{4.4}$$

We can observe here that the finiteness of Z_3^λ is realized in Eq. (4.3), with $0<Z_3^\lambda\le 1$ for $0\le e_\lambda{}^2<\pi$, and that the photon anomalous dimension $\gamma(e_\lambda)$ is positive definite in e_λ. Hence our theorem teaches us that two-dimensional QED should be asymptotically free. On the other hand we can explicitly compute the coefficient function β $[=(\lambda\partial e_\lambda/\partial\lambda)_{e_0\text{ fixed}}]$ in order to see the asymptotic freedom. We find that

$$\beta(e_\lambda)=-e_\lambda(1-e_\lambda{}^2/\pi)\;. \tag{4.5}$$

We can also check the Callan-Symanzik-type equation for the wave-function renormalization constant Z_3^λ,

$$(\beta\partial/\partial e_\lambda-2\gamma)Z_3^\lambda=0\;, \tag{4.6}$$

using Eqs. (4.3), (4.4), and (4.5). The expression

(4.5) is of the asymptotically free type since $\beta'(e_\lambda)|_{e_\lambda=0}<0$. This confirms our theorem in the case of two-dimensional QED.

V. CONCLUSION AND DISCUSSION

We have shown that if the wave-function renormalization constant of a certain field is a finite function of renormalized coupling constants, then the field under consideration should be asymptotically free. In the proof of our theorem the positivity of the spectral function and the renormalizability of the theory have been essential. The former leads to the inequalities $0<Z_3\leqslant 1$ and $\gamma\geqslant 0$, which hold for sufficiently small coupling constants. The latter works in the form of the Callan-Symanzik–type equation for the wave-function renormalization constant:

$$\left(\sum_i \beta_i \frac{\partial}{\partial g_i} - 2\gamma\right) Z_3 = 0 \; .$$

They are combined in the use of the Lyapunov stability theorem to show our main theorem. A simple example of our theorem has been demonstrated in the model of two-dimensional quantum electrodynamics. There we have $Z_3=1-e^2/\pi$, $\beta=-e(1-e^2/\pi)$, and $\gamma=e^2/\pi$ in the sense of Crewther *et al.*

We would like to emphasize that we have not *a priori* assumed the existence of the Gell-Mann–Low eigenvalue. If assumed, our theorem would be rather similar to the arguments which were used by the authors of Ref. 3. It should be pointed out that our theorem cannot be applied to the non-Abelian gauge theories which are known to be asymptotically free because in such theories negative metric will emerge in Green's function off the mass shell; hence the positivity does not hold. On the basis of our theorem we put forward an argument on the properties of the wave-function renormalization constants in the ordinary field theories, e.g., ϕ^4 theory. Since it is known that the ordinary nongauge theories are not asymptotically free,[6] one might conclude that none of the wave-function renormalization constants should be finite in those theories using the contraposition of our theorem. One way to escape this consequence would be for some of the renormalization factors to be essentially singular at the origin of the coupling constants. We have implicitly assumed the analyticity of Z_3 near the origin of coupling constants in the proof.

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APPENDIX A

Here we wish to outline a formal derivation of the Callan-Symanzik equation for Green's functions and the Callan-Symanzik–type equation for a wave-function renormalization constant which played an important role in the text. An unrenormalized proper Green's function $\Gamma_U^{(N)}(g_0,\mu_0,p_i)$ is related to the renormalized one $\Gamma_R^{(N)}(g,\mu,p_i)$ as follows (we take the $g\phi^4$ model for simplicity):

$$\Gamma_U^{(N)}(g_0,\mu_0,p_i)=Z_3{}^{-N/2}\Gamma_R^{(N)}(g,\mu,p_i) \; . \tag{A1}$$

Taking the derivative with respect to the bare mass μ_0 in both sides of Eq. (A1) with the bare coupling constant kept fixed, we obtain

$$\left[Zm\left(\mu_0\frac{\partial \ln m}{\partial\mu_0}\right)_{g_0}\frac{\partial}{\partial m}+Z\left(\mu_0\frac{\partial g}{\partial\mu_0}\right)_{g_0}\frac{\partial}{\partial g} - Z\left(\mu_0\frac{\partial \ln Z_3{}^{1/2}}{\partial\mu_0}\right)_{g_0}\right]\Gamma_R^{(N)}=Z\mu_0\frac{\partial}{\partial\mu_0}\,\Gamma_U^{(N)} \; . \tag{A2}$$

Z is another renormalization constant which should be chosen so that $(\partial\ln m/\partial\ln\mu_0)_{g_0}=1$ holds. We define coefficient functions

$$\beta=Z(\mu_0\partial g/\partial\mu_0)_{g_0}\,, \tag{A3}$$

$$\gamma=Z(\mu_0\partial\ln Z_3{}^{1/2}/\partial\mu_0)_{g_0}\,. \tag{A4}$$

Writing the right-hand side of Eq. (A2) as $-i\Delta\Gamma_R^{(N)}$ we have the Callan-Symanzik equation

$$\left(m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial g}-N\gamma\right)\Gamma_R^{(N)}=-i\Delta\Gamma_R^{(N)} \tag{A5}$$

in the standard form.

If we assume that Z_3 is a finite function of g, we can rewrite Eq. (A4), using Eq. (A3), as follows:

$$\gamma=Z\left(\mu_0\frac{\partial g}{\partial\mu_0}\right)_{g_0}\frac{\partial\ln Z_3{}^{1/2}(g)}{\partial g} =\beta\frac{\partial}{\partial g}\left[\ln Z_3{}^{1/2}(g)\right] \; . \tag{A6}$$

It is straightforward to generalize Eq. (A6) to the many-charge case:

$$\left(\sum_{i=1}^{N}\beta_i\frac{\partial}{\partial g_i}-2\gamma\right)Z_3(g_1g_2\cdots g_N)=0 \; . \tag{A7}$$

This equation is just the wanted one.

APPENDIX B

We give the definitions of frequently used notions in the text: positive- (negative-) definiteness of a function and the stability of a differential equation.[8]

Positive- (negative-) definiteness. A function is positive- (negative-) definite in a region if it is positive (negative) throughout the region and vanishes only for zero values of the variables. A function is positive- (negative-) semidefinite in a region if it also vanishes at other points. For example, the function $V(X_1, X_2) = X_1^2 + X_2^2$ is positive-definite for all values of X_1 and X_2. The function $W(X_1, X_2) = X_1^2$ is positive-semidefinite since $W(X_1, X_2)$ vanishes for any point $X_1 = 0$ with an arbitrary value of X_2.

Stability, asymptotic stability. The motion of $X_i(t)$ $(i = 1, \ldots, N)$ is called *stable* if, given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that whenever the initial conditions satisfy $|X_i(X_0)| \leq \delta$ $(i = 1, \ldots, N)$, then $|X_i(t)| \leq \epsilon$ for all $t > t_0$. If, moreover, $X_i(t) \to 0$ as $t \to \infty$ we call the motion of $X_i(t)$ *asymptotically stable*.

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