Frequency shifts in matter falling into a black hole

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We consider luminous test particles which are falling radially into a Schwarzschild black hole from rest at infinity. For an observer traveling with the particles inside of the horizon, we calculate the frequency shift of light received from sources in various directions and at positions up to $r = 5m$. The differential equations for the particles and the photons are set up in Kruskal coordinates. They are valid both inside and outside the horizon. We find exclusively red shifts in radial directions. For light from off-radial directions we find blue shifts in the vicinity of the observer, but increasing red shifts for sources further out. This behavior of frequency shifts is similar to that seen by comoving observers outside of the horizon.

I. INTRODUCTION

In recent years there has been a great increase in interest in black holes and in physical phenomena occurring in their environs. Though the existence of black holes has not yet been confirmed by conclusive data, a variety of research covering a broad range of theoretical aspects has been conducted throughout the world. Among the topics which have been investigated are questions of the stability of black holes,[1-3] of electromagnetic and gravitational radiation arising from the interaction of black holes with external matter,[4-11] and of the deflection of material particles and photons in the vicinity of black holes.[12] A comprehensive discussion of particle and photon orbits near a Schwarzschild black hole has been given by Darwin.[13] These studies are concerned with physical phenomena as viewed by external observers, i.e., those who are outside the event horizon. On the other hand, there are discussions of the effects of a black hole on an observer who passes the horizon in free fall, such as the tidal forces he is subjected to and the amount of proper time spent in reaching the singularity.

In this paper we determine the frequency shifts which would be measured by observers inside a cloud of freely falling light sources. We restrict ourselves to radially falling test particles in a spherically symmetric gravitational field which is otherwise free of matter (i.e., it satisfies Einstein's empty-space field equations) and, in addition, we assume the particles to fall from rest at spatial infinity. This last assumption has been made for convenience in the numerical calculations. Though we could extend our analysis to the case where particles are falling from rest at a given finite distance or even have nonradial motion, we feel that this would just mean introducing more *ad hoc* chosen parameters. Our assumption corresponds to a physical situation where a black hole has been formed as a condensation of matter in a homogeneous cosmological model. In such a model, particles at the interface between vacuum and matter may be caused to fall radially into the black hole. Of course, this process constitutes a fall from a finite distance. However, if the ratio of the Schwarzschild radius R_s to the radius R of the spherical cavity around the black hole is small, one may treat the matter as if it were falling from rest at infinity. An estimate in Newton's theory shows that $R_s/R \sim 10^{-38} M^{2/3}$ for the condensation of a mass M out of matter of density 10^{-30} g/cm^3. On the other hand, a particle which falls from rest at infinity would have a velocity $v/c \sim 10^{-19} M^{1/3}$ at R. Even for a black hole as massive as $M \sim 10^6 M_\odot$ one would have $R_s/R \sim 10^{-12}$ and $v/c \sim 10^{-6}$.

For an event of observation we calculate the frequency shifts of light which comes from a specified spatial direction and originates from sources at various distances. We are considering observers who are falling inside the horizon and who measure frequency shifts of light received from sources both inside and outside the horizon. The world lines of our sources and observers are in the half plane $u+v>0$. We do not consider photons which might be received inside of the horizon from sources whose world lines originate in the section $u<0$, $|u|>|v|$, i.e. in the "other half" of the Kruskal diagram. Only those photons are considered whose null rays can be connected to events outside of the photon sphere $r=3m$ (see also the closing remarks in Sec. III). For that reason we cannot use Schwarzschild coordinates which provide a coordinate system for the spherically symmetric field only outside the horizon. Instead we will carry out our analysis entirely in Kruskal space-time. We feel that there is intrinsic merit in developing calculational techniques to deal with observations by formulating the equations of motion for material particles and photons in Kruskal space-time.

In describing our work, we proceed as follows: In Sec. II we present the 4-velocity field for particles in free radial fall from rest at infinity in a form valid both inside and outside the horizon. Section III contains the formulas relating to the null geodesics. We give the general form of the first-order differential equations for geodesic null rays in Kruskal space-time, again valid inside and outside the horizon, and discuss the initial-value problem for the null rays. In Sec. IV, a discussion of the cases which were selected for numerical calculation is given.

II. THE 4-VELOCITY OF PARTICLES FALLING RADIALLY IN KRUSKAL SPACE-TIME FROM REST AT INFINITY

As already mentioned in the Introduction, we are dealing with Kruskal space-time which is furnished with the metric[14]

$$ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu = K(du^2 - dv^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \quad (1)$$

Here, the function $r(u^2 - v^2)$ which is the radial coordinate in Schwarzschild space-time is defined by the transcendental equation

$$u^2 - v^2 = (r/2m - 1)\exp(r/2m)\,, \quad (2)$$

and the "Kruskal function" K is given by

$$K = \frac{32m^3}{r}\exp(-r/2m)\,; \quad (3)$$

m is, as usual, the mass parameter $m = GM/c^2$. The coordinates are numbered according to $(u, \theta, \phi, v) = (x^1, x^2, x^3, x^4)$.

Particles which are in radial free fall from rest at infinity have the 4-velocity W^λ with

$$W^1 = \frac{-\sqrt{2mr}\,u + rv}{4m(r-2m)}\,, \quad (4a)$$

$$W^2 = W^3 = 0\,, \quad (4b)$$

$$W^4 = \frac{ru - \sqrt{2mr}\,v}{4m(r-2m)}\,. \quad (4c)$$

These expressions hold true on the Kruskal space-time where $u + v > 0$. One may obtain them in standard fashion from the variational principle for timelike geodesics. The 4-velocity vector field satisfies everywhere the normalization condition

$$g_{\lambda\mu}W^\lambda W^\mu = -1\,.$$

At first glance, the 4-velocity field seems to be singular at the horizon ($r = 2m$, $u = v$). Therefore, we note that at the horizon its nonvanishing components have finite values given by

$$W^1 = \frac{v}{8m}\left(1 - \frac{e}{v^2}\right)\,, \quad (5a)$$

$$W^4 = \frac{v}{8m}\left(1 + \frac{e}{v^2}\right)\,. \quad (5b)$$

The world lines of three particles released at different times are pictured in Fig. 1.

It is interesting to note that one can draw a qualitative picture of the world lines without actually integrating the differential equations associated with the vector field (4). The components of the 4-velocity field approach the value zero at the singularity

$$(r = 0 \Longleftrightarrow u^2 - v^2 = -1)\,.$$

However, for the slope of the world lines at the singularity, we find

$$\lim_{r\to 0}\left(\frac{W^4}{W^1}\right) = \frac{v}{u} = \frac{1}{u}(1+u^2)^{1/2}\,. \quad (6)$$

This means that the world lines of our particles reach the singularity with negative, infinite, or positive slope if, respectively, $u < 0$, $u = 0$, or $u > 0$.

III. THE NULL GEODESICS

At a chosen event of observation we wish to calculate the frequency shifts in the light coming from all the sources in a given direction. Thus we will integrate the differential equations of the null geodesics by following the ray into the past.

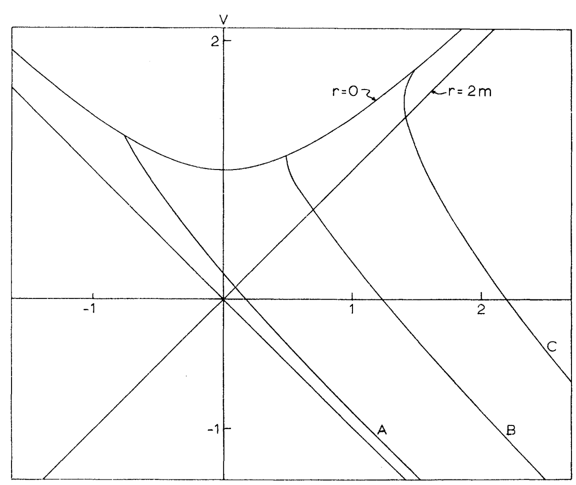

FIG. 1. The world lines of three test particles released at different times from rest at infinity (in Kruskal space-time).

In the plane $\theta = \frac{1}{2}\pi$, let the null ray be given by $(u(y), \frac{1}{2}\pi, \phi(y), v(y))$, where y is an affine parameter which increases along the curve. Then the null vector

$$(k^\lambda) = (u', 0, \phi', v') = \left(\frac{du}{dy}, 0, \frac{d\phi}{dy}, \frac{dv}{dy}\right) \tag{7}$$

points backward on the past null cone. The requirement $k_\lambda k^\lambda = 0$ gives the condition

$$K(u'^2 - v'^2) + r^2\phi'^2 = 0 \tag{8}$$

for the components of the null vector. Two further first integrals of the conditions for a geodesic null ray are obtained from the fact that ϕ and t are cyclic coordinates. These integrals are

$$uv' - vu' = \frac{\beta}{8m^2} r \exp(r/2m) \equiv S \tag{9a}$$

and

$$r^2\phi' \equiv \alpha, \tag{9b}$$

where α and β are constants. With the help of (9b) we obtain from (8) and (9a) the explicit differential equations

$$u' = \frac{S}{u^2 - v^2}\left\{v - u\left[1 - \frac{T}{S^2}(u^2 - v^2)\right]^{1/2}\right\}, \tag{10a}$$

$$v' = \frac{S}{u^2 - v^2}\left\{u - v\left[1 - \frac{T}{S^2}(u^2 - v^2)\right]^{1/2}\right\}, \tag{10b}$$

where

$$T \equiv \frac{\alpha^2}{Kr^2}. \tag{10c}$$

Equations (10) are valid on both sides of the horizon. The null vector at the horizon is given by

$$u' = \frac{S}{2v}\left(\frac{v^2 T}{S^2} - 1\right), \tag{11a}$$

$$v' = \frac{S}{2v}\left(\frac{v^2 T}{S^2} + 1\right), \tag{11b}$$

with $r = 2m$ in S and T.

To complete the discussion of the null geodesics we now state the initial-value problem for the equations (9a), (10a), and (10b). Let the event of observation be P_0, with coordinates $(u_0, \frac{1}{2}\pi, 0, v_0)$. Since the affine parameter for the geodesic null ray is fixed only up to an arbitrary constant factor, we can and will set

$$(v')_0 = -1. \tag{12}$$

Equation (8) implies now that the initial values are restricted by

$$[(u')_0]^2 = 1 - \frac{r_0^2}{K_0}[(\phi')_0]^2 \quad (\geq 0), \tag{13}$$

and that we may choose $(\phi')_0$, the initial value of ϕ', subject only to the condition

$$(\phi')_0 < \sqrt{K_0}/r_0, \tag{14}$$

where r_0 and K_0 are the values of r and K at P_0. Then $(u')_0$ is determined by the null condition (8). According to this equation, to each value of $(\phi')_0$ there belong the two values

$$(u')_0 = \pm\left\{1 - \frac{r^2}{K}[(\phi')_0]^2\right\}^{1/2}.$$

We have considered only null rays for which $(u')_0$ is positive. These null rays are among those which can be connected to the exterior of the photon sphere. However, they do not comprise all of them. The reason for our restriction in considering all rays which could be received from the outside of the photon sphere lies merely in changes of the computational procedure which would be necessary in the range $(u')_0 < 0$.

We wish to see which angles of reception are covered by our calculation and how these angles are related to the angle under which photons are received from the photon sphere. All these angles are measured against the outward radial direction in the rest space of the observer.

For a radially falling observer with 4-velocity W^λ, a photon which is described by the null vector k^λ is received at an angle which satisfies

$$\cos\alpha = \frac{k^1W^4 - k^4W^1}{|k^1W^1 - k^4W^4|}.$$

To indicate the origin of this formula we remark that the outward-pointing radial unit vector r^λ in the rest frame of the observer has the components $(r^1, r^2, r^3, r^4) = (W^4, 0, 0, W^1)$. In the case of photons for which $(\phi')_0$ takes its maximal value, the formula specializes to $\cos\alpha_0 = W^1/W^4$.

For photons from the photon sphere $(r = 3m)$ we get

$$\cos\alpha_{3m} = \frac{(2d)^{1/2} - [1 - 27d^2(1-2d)]^{1/2}}{|1 - (2d)^{1/2}[1 - 27d^2(1-2d)]^{1/2}|}$$

where $d = m/r$.

Incidentally, α_{3m} is the half-angle of the cone inside which the observer sees that part of the "world" which is outside of the photon sphere. Some authors refer to the existence of this angle as the "porthole effect."[15]

The values of $(\phi')_0$ and the corresponding angles used in our integration are listed in Table II. This table shows that our rays cover most of the cone limited by photons from the sphere $r = 3m$.

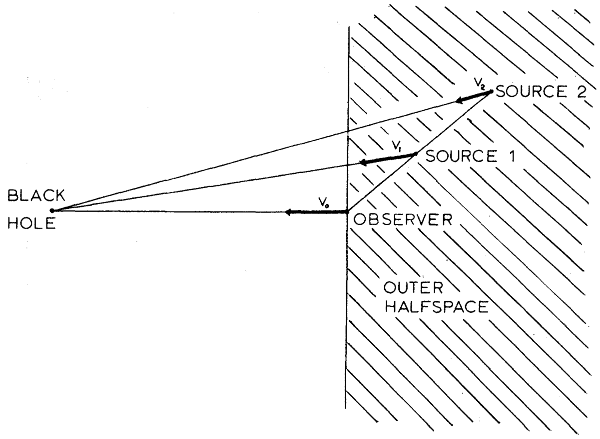

FIG. 2. The paths of photons from two sources along the same line of sight in a nearly flat region of Schwarzschild space. Here v_0, v_1, and v_2 are respectively the radial velocities of the observer, source 1, and source 2.

IV. THE FREQUENCY SHIFTS

In order to find the frequency shifts

$$Z = \frac{\lambda_{\text{received}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}}, \qquad (15)$$

we integrate the system of equations (9b), (10a), (10b), beginning at the event of observation. In terms of the null vector k and the 4-velocity W, the frequency shift is given by

$$Z = \frac{(k_\mu W^\mu)_{\text{source}}}{(k_\mu W^\mu)_{\text{observer}}} - 1. \qquad (16)$$

For sources in the immediate neighborhood of the observer one can predict the qualitative features in the directional dependence of the frequency shifts. First of all it is clear that any observer will see red shifts if he looks in the radial direction, regardless of whether it is inward or outward. Since the relative motion of neighboring particles with nearly equal values of the radial coordinate is towards each other, we expect to find blue shifts in the light from sources seen in lateral directions. Will these lateral blue shifts prevail if one looks at sources far away? A simple qualitative discussion of this question can be given for an observer who is at a large distance from the black hole, i.e., in a region where space-time is nearly flat. Let this observer look out along an off-radial direction into his outer half space (shaded region in Fig. 2). To him the light from nearby sources will appear blue-shifted, but the further the source is away, the smaller is the blue shift, and beyond a certain distance which depends on the direction in which he is looking the frequency shifts will be increasingly red. Only in the lateral direction will the frequency shift remain blue, although it decreases with increasing distance of the source. The signs of the frequency

TABLE I. Events of observation (P_0) and the reception angles corresponding to $\max(\phi')_0$ and the photon sphere.

	r/m	u	v	$\max(\phi')_0$	α_0 (deg)	α_{3m} (deg)
P_{01}	1.972	0.685	0.712	1.246	133.6	137.7
P_{02}	1.012	0.502	1.035	4.316	110.2	126.9
P_{03}	0.111	0.488	1.112	148.61	76.9	103.3

TABLE II. Initial values of ϕ' and the corresponding reception angles.

P_{01} $(\phi')_0$	α (deg)	P_{02} $(\phi')_0$	α (deg)	P_{03} $(\phi')_0$	α (deg)
0.0	0.0	0.0	0.0	0.0	0.0
0.2	21.3	1.0	19.2	80.0	26.1
0.4	42.1	2.0	38.9	120.0	43.9
0.6	61.8	3.0	60.3	140.0	58.5
0.8	80.5	4.0	88.2	148.0	71.7
1.0	99.0	4.3	106.2		
1.2	120.9				

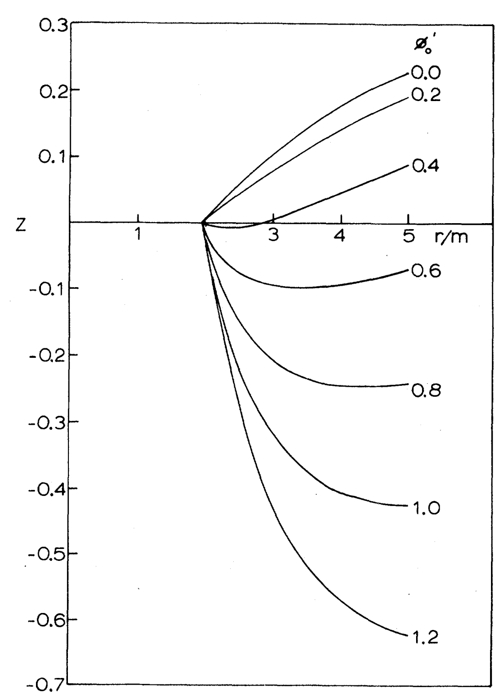

FIG. 3. Frequency shifts measured by an observer at $r = 1.96m$ as a function of the radial coordinates of the sources and the initial value of ϕ'.

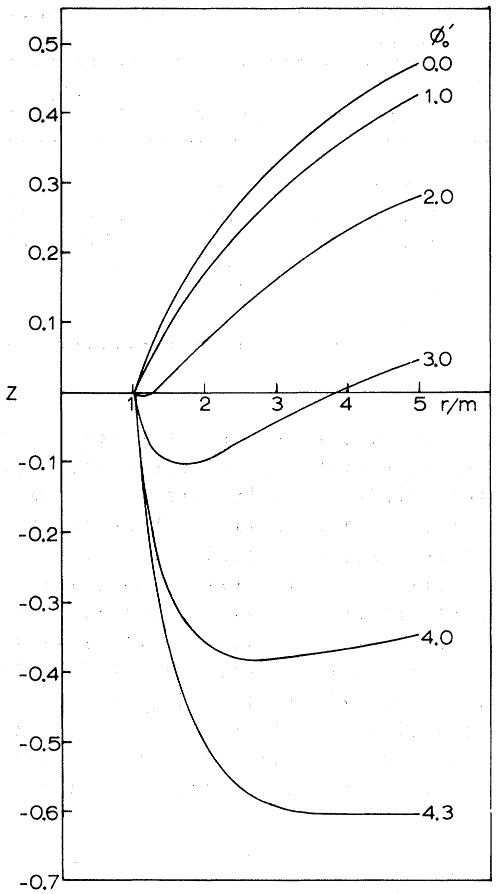

FIG. 4. Frequency shifts measured by an observer at $r=1.12m$ as a function of the radial coordinates of the sources and the initial values of ϕ'.

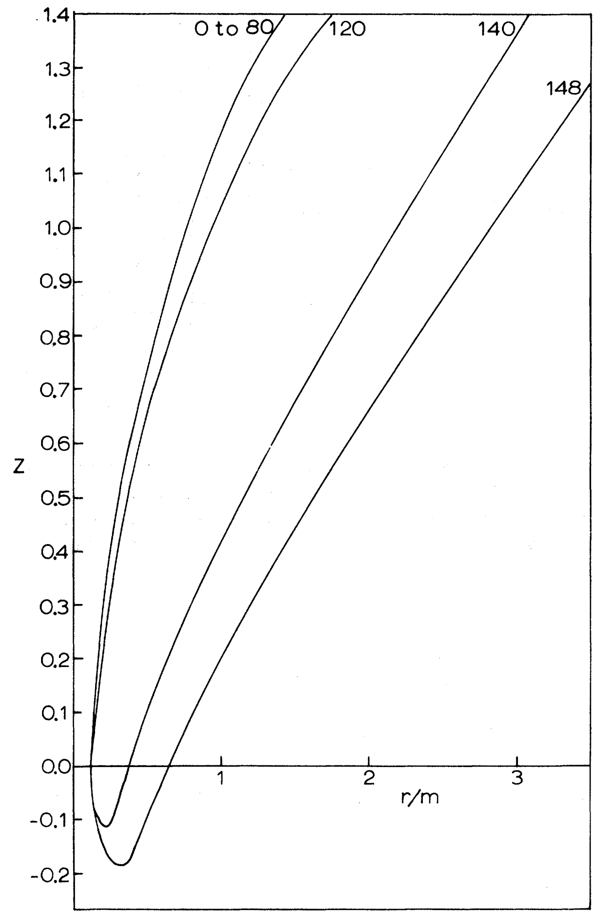

FIG. 5. Frequency shifts measured by an observer at $r=0.111m$ as a function of the radial coordinates of the sources and the initial values of ϕ'.

shifts can be inferred from Fig. 2 by projecting the velocity vectors onto the line of sight.

Can we expect that this picture remains valid if the observer is inside the horizon of a black hole in whose neighborhood the orbits of photons can undergo large deflections? Our calculations show that this is true.

For the numerical calculation of the frequency shifts we select three events of observation along the world line of a freely falling observer (curve *B* in Fig. 1). These events and the initial conditions at each are given in Table I. [We choose several values for $(\phi')_0$ between zero and the maximal value at each point; these values and the corresponding angles are given in Table II.]

The integration of the differential equations for the null geodesics was done using Hamming's modified predictor-corrector method.[16] The results are shown in Figs. 3–5. In all the cases the integration was terminated at $r=5m$. The behavior of frequency shifts expected by observers far from the black hole remains qualitatively unaltered as the observer passes the horizon and approaches the singularity. This finding is most clearly shown in Fig. 5, but it is also exhibited in Fig. 3 [$(\phi')_0=0.4$] and Fig. 4 [$(\phi')_0=3.9$]. Another interesting facet of this result is that inside the horizon this behavior of frequency shifts is seen in a much larger portion of the sphere of sight around the observer. This is due to the aberration effect brought about by the high velocities reached inside the black hole.

V. CONCLUSION

The 4-velocity field for test matter falling radially into a Schwarzschild black hole from rest at infinity has been presented in Kruskal space-time. We have also obtained in Kruskal space-time the general form of the tangent vector for null geodesics. Integrating backward along the null ray from a chosen event of observation we have calculated the frequency shifts in the light from sources in different directions and at various distances. As expected, we found pure red shifts in the light from sources in the radial direction. Blue shifts are found in off-radial directions for sources close to the observer. However, for light from sources at sufficient distances the shift of frequencies was found to be to the red, with the red shifts increasing with the distance of the source. Clearly, the calculated frequency shift is composed of both a gravitational shift (due to the location of source and observer) and a Doppler shift (due to the relative motion of source and observer). Yet, since no static observer can exist inside the horizon of a black hole, there is no unambiguous way to separate these two effects. Hence, no explanation of our findings in terms of them can be given.

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