*Work partially supported by the National Science Foundation under Grant No. GP30799X to Princeton University.

[1]See, e.g., S. Hawking, in *Black Holes*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973).

[2]For a physical interpretation of this result as well as for a classical analog see, e.g., D. Christodoulou and R. Ruffini, in *Black Holes*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973), and R. Ruffini and A. Treves, Astrophys. Lett. 13, 109 (1973).

[3]See, e.g., R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964).

[4]This uncertainty can be further reduced if the black hole is endowed with an electric charge. It then becomes advantageous to measure the mass of the black hole by a deflection suffered in uniform electric field ("oil-drop experiment").

[5]R. Squier Hanni and R. Ruffini, Phys. Rev. D 8, 3259 (1973).

[6]See, e.g., R. M. Barker and R. F. O'Connell, Phys. Rev. D 2, 1428 (1970).

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Conformal energy-momentum tensor in curved spacetime: Adiabatic regularization and renormalization*

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In preparation for an investigation of whether field-theoretic effects helped to make the early universe become isotropic, we seek to determine the physical (divergence-free) energy-momentum tensor through which the geometry of spacetime is influenced by a quantized scalar field with conformal ("new improved") coupling to the metric. The cosmological models studied are the Kasner-like (type I) metrics (homogeneous, spatially flat, nonrotating, but anisotropic), and also the isotropic Robertson-Walker metrics. The methods employed have previously been expounded in the context of a minimally coupled scalar field and a Robertson-Walker metric. Three divergent leading terms are extracted from an adiabatic expansion of the formal expressions for the expectation values of the energy density and pressures. In the Kasner case a slight reshuffling of the leading terms in the energy density displays all divergences to be proportional to either the metric tensor or a second-order curvature tensor which vanishes when the spacetime is isotropic; hence a finite energy-momentum tensor remains after renormalization of the cosmological constant and one other coupling constant in a generalized Einstein equation. In the Robertson-Walker cases, because of conformal flatness, there is no divergence beyond the usual quartically divergent constant vacuum energy; when the mass is not zero, however, a finite renormalization of the gravitational constant is suggested. The correctness of the methods is tested by considering a coordinate system in which flat spacetime assumes the form of a Kasner universe: The adiabatic definition of particle number and vacuum, which is basic to our expansion and renormalization methods, is seen to be consistent with the usual flat-space concepts.

I. INTRODUCTION

One of the contexts in which quantum effects may be of practical importance in cosmology is a hypothetical anisotropic stage in the early history of the universe. The observed fact that the spatial universe and its expansion in time are quite isotropic, as evidenced by the thermal background radiation, would be easier to understand if any initial anisotropy gave rise to creation of real or virtual particles,[1] and if the reaction of this process back on the gravitational field tended to drive the geometry toward isotropy. This idea has been pursued by Zel'dovich[2] in the context of an anisotropic but spatially flat spacetime (Bianchi type I) and by Hu[3] for a closed anisotropic spacetime (mixmaster universe, type IX). The main purpose of the present work is to lay the theoretical foundation for a detailed calculation to establish (or refute) the existence of the anisotropy-damping effect in the first of these cases. We will not be dealing here with interparticle interactions, which could also give rise to anisotropy damping, but only with the interaction between the particle fields and gravity.

The key step in constructing a theory of the interaction between quantized matter fields and a classical gravitational field is identifying the energy-momentum tensor of the quantized fields, which acts as the source of the gravity field. This quantity is presumably obtainable from the divergent expression for the energy-momentum tensor that results from formal Lagrangian field theory. Progress toward an understanding of this problem has been reported in two recent papers,[4,5] the ideas of which are central to the present work. The model considered there was a Robertson-Walker (homogeneous and isotropic) spacetime containing a scalar field ϕ minimally coupled to the geometry—i.e., obeying the most obvious covariant generalization of the Klein-Gordon equation,

$$\nabla^\mu \nabla_\mu \phi + m^2 \phi = 0 .$$

In this paper the same methods are applied to metrics of the generalized Kasner type (anisotropic type I), and also, in Sec. V, to the Robertson-Walker metrics. Furthermore, we take the scalar field to be conformally coupled. That is, the field equation is

$$\nabla^\mu \nabla_\mu \phi + m^2 \phi + \tfrac{1}{6} R \phi = 0 ,$$

where R is the scalar curvature of the spacetime, and the corresponding (classical or formal quantum) energy-momentum tensor is the covariant generalization[6] of the "new improved energy-momentum tensor" of Callan, Coleman, and Jackiw.[7,8] This field theory is conformally invariant when $m = 0$. Since the Robertson-Walker spacetimes are conformally flat but the anisotropic homogeneous universes are not, anisotropy is a very significant factor in a conformally invariant or approximately invariant theory. In the massless limit of the scalar field theory with conformal coupling, there will be no particle creation in an isotropic spacetime; the field theory can be mapped conformally into the free scalar field theory in Minkowski space.[9] When the space is anisotropic, on the other hand, particles will be created. A scalar field is most likely to produce an anisotropy-damping effect, therefore, when the coupling is of the conformal type and the mass is zero or small (cf. Ref. 2).

The paper begins by applying the approach of Ref. 4, "adiabatic regularization," to the anisotropic Kasner metric (2.1). The divergences in the energy-momentum tensor are isolated in the three leading terms of an asymptotic expansion corresponding to a limit of slow time dependence of the metric. The quantity expanded is the expectation value of the energy-momentum tensor with respect to an approximate vacuum state which is defined asymptotically and physically motivated in Ref. 4. The Kasner model with conformally coupled scalar field has been studied previously by Zel'dovich and Starobinsky.[10] Our results in Sec. II overlap somewhat with those of Ref. 10 but go beyond them.

Section III is devoted to renormalization, following Ref. 5. We show that all the divergent terms in the energy density can be written, after some rearrangement, in the form

$$A g_0{}^0 + B\, {}^{(-)}H_0{}^0 , \tag{1.1}$$

where A and B are divergent integrals and ${}^{(-)}H_\mu{}^\nu$ is a certain tensor formed from derivatives and quadratic products of the Riemann tensor [see Eqs. (A28)–(A30) and (3.5) and (3.6)], which vanishes in the isotropic case. If Einstein's gravitational field equation is modified by adding a term proportional to ${}^{(-)}H_\mu{}^\nu$, then the time-time component of the equation is

$$\begin{aligned} G_0{}^0 + \Lambda g_0{}^0 + \sigma_-\, {}^{(-)}H_0{}^0 &= -8\pi G T_0{}^0 \\ &= -8\pi G (A g_0{}^0 + B\, {}^{(-)}H_0{}^0 + \rho_{\rm ren}) , \end{aligned} \tag{1.2}$$

where $\rho_{\rm ren}$ is finite, which is equivalent to

$$G_0{}^0 + (\Lambda + 8\pi G A) g_0{}^0 + (\sigma_- + 8\pi G B)\, {}^{(-)}H_0{}^0 = -8\pi G \rho_{\rm ren} . \tag{1.3}$$

Thus one obtains an equation without infinities if the modified values of the cosmological constant, $\Lambda + 8\pi GA$, and of the new coupling constant, $\sigma_- + 8\pi GB$, are assumed to be finite and experimentally relevant (the original "bare" values, Λ and σ_-, being unobservable). A finite term in $T_0{}^0$ proportional to $G_0{}^0$ may be interpreted as causing a finite renormalization of the effective value of the gravitational constant G [see Eqs. (5.37)–(5.39)].

A crucial test of a new theory or method of calculation is that it give physically sensible results in a special case where the answer is already known. In Sec. IV we study ordinary Minkowski space, whose metric takes, in a certain curvilinear coordinate system, the Kasner form [Eq. (4.2)]. We show that in this degenerate Kasner universe the adiabatic definition of particle annihilation and creation operators (which leads to the asymptotic concept of vacuum used in Sec. II) coincides, to within its inherent imprecision, with the ordinary notion of free scalar particles in flat space. Calculations are under way to verify that the term of leading adiabatic order (viz., sixth) retained in the renormalized energy density $\rho_{\rm ren}$ is zero in this case, at least when $m = 0$. (For the massive case, as in Ref. 5, our present approach yields extra terms in $\rho_{\rm ren}$ which are convergent, but whose covariance is suspect.) This result is necessary for agreement with conventional Minkowski-space theory, where the renormalized vacuum energy vanishes by definition.

In Sec. V we consider the conformal scalar field in the Robertson-Walker universes. In these isotropic models there is no divergence beyond the leading constant term. In fact, in the massless case this term constitutes the entire vacuum energy density, in agreement with the reduction of this system to the flat-space theory by a conformal transformation (Ref. 6). When $m\neq 0$ there is a finite term proportional to $G_\mu{}^\nu$ in the case of flat three-space, which suggests a finite renormalization of G. The convergent terms in the expansion of the pressure are related to the corresponding terms in the energy density as they should be through the condition $\nabla_\nu T_\mu{}^\nu=0$.

The notation is the same as in Refs. 4 and 5. Our sign conventions are (1) metric signature $(+---)$, (2) $R^\alpha_{\beta\gamma\delta}=+\Gamma^\alpha_{\beta\gamma,\delta}-\cdots$, (3) $R_{\mu\nu}=R^\alpha_{\mu\alpha\nu}$. The usual summation convention is in effect over Greek (spacetime) indices. Summations over Latin (three-space) indices are indicated explicitly, but an index may be omitted from the summation sign when there is no chance of confusion. The nth derivative of a quantity A with respect to the time variable η [Eq. (2.15)] is denoted by $A^{(n)}$ if $n>2$. The units are such that $\hbar=c=1$.

II. ADIABATIC REGULARIZATION IN A KASNER-TYPE UNIVERSE

A. Equation of motion and energy-momentum tensor

We shall consider chiefly the class of metrics of the form

$$ds^2=dt^2+\sum_{i,j} g_{ij}\,dx^i\,dx^j \qquad [t\equiv x^0,\ \vec{x}\equiv(x^1,x^2,x^3)], \tag{2.1a}$$

with

$$g_{ij}=-a_i(t)^2\delta_{ij}, \tag{2.1b}$$

where the a_i are arbitrary functions.[11] This is the most general homogeneous, nonrotating cosmological model with flat three-space. For convenience we call these "Kasner metrics." The Kasner solutions in a stricter sense[12] are the special cases

$$a_i(t)=t^{p_i},\ \sum p_i=1,\ \sum p_i{}^2=1; \tag{2.2}$$

they satisfy the vacuum Einstein equation. The components of the Einstein tensor $G_\mu{}^\nu$ and other geometric quantities for the metric (2.1) are recorded in Appendix A.

The matter in our model is represented by a quantized neutral scalar field with the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(-g)^{1/2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{6}R\phi^2-m^2\phi^2). \tag{2.3}$$

The resulting field equation,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi+\tfrac{1}{6}R\phi+m^2\phi=0, \tag{2.4}$$

takes for the metric (2.1) the form

$$\partial_0{}^2\phi+(\partial_0V/V)\partial_0\phi-\sum_i a_i{}^{-2}\partial_i{}^2\phi+(\tfrac{1}{6}R+m^2)\phi=0, \tag{2.5}$$

where $V=(-g)^{1/2}=a_1a_2a_3$. The canonical commutation relations,

$$\begin{aligned}&[\phi(\vec{x},t),\phi(\vec{x}',t)]=[\pi(\vec{x},t),\pi(\vec{x}',t)]=0,\\&[\phi(\vec{x},t),\pi(\vec{x}',t)]=i\delta(\vec{x}-\vec{x}'),\end{aligned} \tag{2.6}$$

hold for the field and its conjugate momentum,

$$\pi\equiv\partial\mathcal{L}/\partial(\partial_0\phi)=V\partial_0\phi. \tag{2.7}$$

The metric is to be related to an expectation value of the energy-momentum tensor of the conformally coupled scalar field through the Einstein equation

$$G_\mu{}^\nu\equiv R_\mu{}^\nu-\tfrac{1}{2}\delta_\mu{}^\nu R=-8\pi G\langle\Lambda_\mu{}^\nu\rangle. \tag{2.8}$$

The conformal energy-momentum tensor $\Lambda_\mu{}^\nu$ (see Ref. 6) is given by

$$\begin{aligned}\Lambda_{\mu\nu}=&(\partial_\mu\phi)(\partial_\nu\phi)-\tfrac{1}{2}g_{\mu\nu}g^{\lambda\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi)+\tfrac{1}{2}g_{\mu\nu}m^2\phi^2\\&-\tfrac{1}{6}\nabla_\mu\partial_\nu(\phi^2)+\tfrac{1}{6}g_{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\partial_\sigma(\phi^2)-\tfrac{1}{6}\phi^2G_{\mu\nu}.\end{aligned} \tag{2.9}$$

In this and similar formulas to follow, terms which involve noncommuting quantities are understood to be symmetrized. Calculating the covariant derivatives, one obtains

$$\begin{aligned}\Lambda_{\mu\nu}=&\tfrac{2}{3}(\partial_\mu\phi)(\partial_\nu\phi)-\tfrac{1}{3}\phi\partial_\mu\partial_\nu\phi+\tfrac{1}{3}\phi\Gamma^\lambda_{\mu\nu}\partial_\lambda\phi\\&-\tfrac{1}{6}g_{\mu\nu}g^{\lambda\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi)+\tfrac{1}{3}g_{\mu\nu}\phi g^{\lambda\sigma}\partial_\lambda\partial_\sigma\phi\\&-\tfrac{1}{3}g_{\mu\nu}\phi g^{\lambda\sigma}\Gamma^\alpha_{\lambda\sigma}\partial_\alpha\phi+\tfrac{1}{2}g_{\mu\nu}m^2\phi^2-\tfrac{1}{6}\phi^2G_{\mu\nu}.\end{aligned} \tag{2.10}$$

Hence, using Eqs. (A2) for the Christoffel symbols, the field equation (2.5), and the identity

$$\frac{\dot{V}}{V}=\sum\frac{\dot{a}_i}{a_i}, \tag{2.11}$$

one finds that

$$\begin{aligned}\Lambda_0{}^0=&\tfrac{1}{2}(\partial_0\phi)^2+\tfrac{1}{3}(\dot{V}/V)\phi\partial_0\phi+\tfrac{1}{6}\sum a_i{}^{-2}(\partial_i\phi)^2\\&-\tfrac{1}{3}\sum a_i{}^{-2}\phi\partial_i{}^2\phi+\tfrac{1}{2}m^2\phi^2-\tfrac{1}{6}G_0{}^0\phi^2\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}-\Lambda_i{}^i=&\tfrac{1}{6}(\partial_0\phi)^2+\tfrac{1}{3}(\dot{a}_i/a_i)\phi\partial_0\phi+\tfrac{2}{3}a_i{}^{-2}(\partial_i\phi)^2\\&-\tfrac{1}{6}\sum a_j{}^{-2}(\partial_j\phi)^2-\tfrac{1}{3}a_i{}^{-2}\phi\partial_i{}^2\phi\\&+\tfrac{1}{6}(-m^2+\tfrac{1}{3}R+G_i{}^i)\phi^2.\end{aligned} \tag{2.13}$$

(We use a dot to indicate differentiation with respect to t.)

In studying the conformally coupled scalar field it is convenient to introduce a modified field variable,

$$\chi = V^{1/3}\phi, \tag{2.14}$$

and a new time variable,

$$\eta = \int^t V^{-1/3} dt'. \tag{2.15}$$

Note that if the metric (2.1) is isotropic (i.e., $a_i = V^{1/3}$ for all i), the transformation (2.15) converts it to the conformally flat form

$$ds^2 = V(\eta)^{2/3}\left[d\eta^2 - \sum_i (dx^i)^2\right]. \tag{2.16}$$

We denote differentiation with respect to η by a prime or by ∂_η.

Substituting

$$\partial_0\phi = V^{-2/3}[\partial_\eta\chi - \tfrac{1}{3}(V'/V)\chi] \tag{2.17}$$

into Eqs. (2.12) and (2.13), we obtain

$$\Lambda_0{}^0 = V^{-2/3}\left[\tfrac{1}{2}V^{-2/3}(\partial_\eta\chi)^2 + \tfrac{1}{6}\sum a_i{}^{-2}(\partial_i\chi)^2 - \tfrac{1}{3}\sum a_i{}^{-2}\chi\partial_i{}^2\chi + \tfrac{1}{2}(m^2 - V^{-2/3}Q)\chi^2\right], \tag{2.18}$$

$$-\Lambda_i{}^i = V^{-2/3}\left\{\tfrac{1}{6}V^{-2/3}(\partial_\eta\chi)^2 + \tfrac{1}{3}V^{-2/3}\left(\frac{a_i'}{a_i} - \frac{1}{3}\frac{V'}{V}\right)\chi\partial_\eta\chi \right.$$
$$\left. + \tfrac{2}{3}a_i{}^{-2}(\partial_i\chi)^2 - \tfrac{1}{6}\sum a_j{}^{-2}(\partial_j\chi)^2 - \tfrac{1}{3}a_i{}^{-2}\chi\partial_i{}^2\chi + \tfrac{1}{6}\left[-m^2 + V^{-2/3}\partial_\eta\left(\frac{a_i'}{a_i} - \frac{1}{3}\frac{V'}{V}\right) - V^{-2/3}Q\right]\chi^2\right\}. \tag{2.19}$$

Here Q is defined by Eq. (A13) and has been introduced through Eq. (A18):

$$Q = \tfrac{1}{18}\sum_{i<j}\left(\frac{a_i'}{a_i} - \frac{a_j'}{a_j}\right)^2,$$

$$G_0{}^0 = V^{-2/3}[3Q - \tfrac{1}{3}(V'/V)^2].$$

From these two equations it is clear that Q represents the contribution to $G_0{}^0$ from the anisotropy of the universe, while the $(V'/V)^2$ term is the part of $G_0{}^0$ due to isotropic expansion. In arriving at Eq. (2.19) we have also used Eqs. (A17) and (A19).

In view of Eqs. (2.17) and (A18), the field equation (2.5) is equivalent to

$$\partial_\eta{}^2\chi - V^{2/3}\sum a_i{}^{-2}\partial_i{}^2\chi + (V^{2/3}m^2 + Q)\chi = 0. \tag{2.20}$$

We write the solution of this equation in the form

$$\chi = (2\pi)^{-3/2}\int d^3k[A_{\vec{k}}\chi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + A_{\vec{k}}^\dagger\chi_{\vec{k}}^*(t)e^{-i\vec{k}\cdot\vec{x}}], \tag{2.21}$$

where

$$\chi_{\vec{k}}'' + (\Omega_{\vec{k}}{}^2 + Q)\chi_{\vec{k}} = 0, \tag{2.22}$$

$$\Omega_{\vec{k}}{}^2 \equiv V^{2/3}\omega_{\vec{k}}{}^2 = V^{2/3}\left(\sum\frac{k_i{}^2}{a_i{}^2} + m^2\right). \tag{2.23}$$

The $A_{\vec{k}}$ will satisfy

$$[A_{\vec{k}}, A_{\vec{k}'}] = [A_{\vec{k}}^\dagger, A_{\vec{k}'}^\dagger] = 0,$$
$$[A_{\vec{k}}, A_{\vec{k}'}^\dagger] = \delta(\vec{k} - \vec{k}'), \tag{2.24}$$

provided that

$$\chi_{\vec{k}}'^*\chi_{\vec{k}} - \chi_{\vec{k}}^*\chi_{\vec{k}}' = i. \tag{2.25}$$

As in Ref. 4, the $A_{\vec{k}}$ will ultimately be chosen so that they correspond to physical particles in the adiabatic limit; this has the effect of a boundary condition to complete the determination of $\chi_{\vec{k}}$.

Let $|0_A\rangle$ be a normalized state annihilated by all the $A_{\vec{k}}$. We calculate from Eq. (2.21) the expectation values of $\Lambda_0{}^0$ and $\Lambda_i{}^i$ in that state, recalling that in Eq. (2.19) $\chi\partial_\eta\chi$ really means $\tfrac{1}{2}[\chi(\partial_\eta\chi) + (\partial_\eta\chi)\chi]$:

$$\rho_0 \equiv \langle 0_A|\Lambda_0{}^0|0_A\rangle = (16\pi^3)^{-1}V^{-4/3}\int d^3k[|\chi_{\vec{k}}'|^2 + (\Omega_{\vec{k}}{}^2 - Q)|\chi_{\vec{k}}|^2], \tag{2.26}$$

$$(P_i)_0 \equiv -\langle 0_A|\Lambda_i{}^i|0_A\rangle = \tfrac{1}{3}(16\pi^3)^{-1}V^{-4/3}\int d^3k\{|\chi_{\vec{k}}'|^2 + (d_i - D)\partial_\eta(|\chi_{\vec{k}}|^2)$$
$$+ [6V^{2/3}a_i{}^{-2}k_i{}^2 + \partial_\eta(d_i - D) - \Omega_{\vec{k}}{}^2 - Q]|\chi_{\vec{k}}|^2\}, \tag{2.27}$$

where we have written d_i for a_i'/a_i and D for $\tfrac{1}{3}V'/V$. It is interesting that Q enters Eqs. (2.22) and (2.26) with opposite signs relative to $\Omega_{\vec{k}}{}^2$.

Let us now consider the expectation values of $\Lambda_\mu{}^\nu$ in states other than $|0_A\rangle$. We are interested only in states for which $\langle\Lambda_\mu{}^\nu\rangle$ is consistent with the Einstein equation (2.8), where the $G_\mu{}^\nu$ is that of the Kasner universe under study. We must therefore have

$$\langle\Lambda_\mu{}^\nu\rangle = 0 \text{ for } \mu \neq \nu; \tag{2.28}$$

this will be true for any quantum state which is symmetric under reflections of the three principal space axes. Furthermore, the energy density and the pressures in the three principal directions,

$$\rho \equiv \langle \Lambda_0{}^0 \rangle, \quad P_i \equiv -\langle \Lambda_i{}^i \rangle, \tag{2.29}$$

must be independent of the spatial coordinates $\vec{x}$. Because of the infinite volume of the three-space, a state with the latter property does not belong to the Fock space of the operators $A_{\vec{k}}$ (unless ρ and P_i are zero), and the quantities $A_{\vec{k}}^\dagger A_{\vec{k}}$, etc., have infinite expectation values in such a state. We have outlined in Ref. 4 how to deal with this complication by considering the three-space as a limiting case of a finite space. The results are

$$\rho = \rho_0 + (2\pi)^{-3} V^{-4/3} \int d^3k \{ [|\chi_{\vec{k}}'|^2 + (\Omega_{\vec{k}}{}^2 - Q) |\chi_{\vec{k}}|^2] \langle A_{\vec{k}}^\dagger A_{\vec{k}} \rangle + \mathrm{Re}[((\chi_{\vec{k}}')^2 + (\Omega_{\vec{k}}{}^2 - Q)\chi_{\vec{k}}{}^2) \langle A_{-\vec{k}} A_{\vec{k}} \rangle] \} \tag{2.30}$$

and

$$\begin{aligned} P_i = (P_i)_0 + \tfrac{1}{3}(2\pi)^{-3} V^{-4/3} \int d^3k \{ [|\chi_{\vec{k}}'|^2 + (d_i - D)\partial_\eta(|\chi_{\vec{k}}|^2) + (6V^{2/3} a_i{}^{-2} k_i{}^2 + \partial_\eta(d_i - D) - (\Omega_{\vec{k}}{}^2 + Q)) |\chi_{\vec{k}}|^2] \langle A_{\vec{k}}^\dagger A_{\vec{k}} \rangle \\ + \mathrm{Re}\big[[(\chi_{\vec{k}}')^2 + (d_i - D)\partial_\eta(\chi_{\vec{k}}{}^2) + (6V^{2/3} a_i{}^{-2} k_i{}^2 + \partial_\eta(d_i{}' - D) \\ - (\Omega_{\vec{k}}{}^2 + Q))\chi_{\vec{k}}{}^2] \langle A_{-\vec{k}} A_{\vec{k}} \rangle \big] \}, \end{aligned} \tag{2.31}$$

where the quantities $\langle A_{\vec{k}}^\dagger A_{\vec{k}} \rangle$ and $\langle A_{-\vec{k}} A_{\vec{k}} \rangle$ are "renormalized" so that in the adiabatic limit the physical density[13] of particles is

$$(2\pi)^{-3} V^{-1} \int d^3k \langle A_{\vec{k}}^\dagger A_{\vec{k}} \rangle . \tag{2.32}$$

Formulas (2.30) and (2.31) are valid only for states of the assumed homogeneity. In deriving them one uses the fact that

$$\chi_{-\vec{k}} = \chi_{\vec{k}} . \tag{2.33}$$

B. Adiabatic analysis of vacuum energy and pressure

Our goal is to construct from the formal expressions (2.30) and (2.31) the physical energy density and pressures which should be used in calculations of the effect of the quantized matter on the geometry through Eq. (2.8). The method will be, in this section, precisely analogous to that used in Ref. 4 for a simpler model. There are two important steps in this procedure:

(1) The operators $A_{\vec{k}}$ and $A_{\vec{k}}^\dagger$ must be chosen so that in the adiabatic limit (limit of arbitrarily slow time variation of the metric functions a_i) they become the annihilation and creation operators for physical particles. The identification with physical particles is required to be valid at least to fourth order in a large parameter T, representing the slowness of the change of the metric. Then the terms in Eqs. (2.30) and (2.31) involving $\langle A^\dagger A \rangle$, $\langle AA \rangle$, and $\langle A^\dagger A^\dagger \rangle$ will be finite for physically realizable states.

(2) The infinite quantities ρ_0 and $(P_i)_0$, given by Eqs. (2.26) and (2.27), must be regularized by subtracting inherently unobservable terms associated with empty space. The subtraction is to be made mode by mode (i.e., for each $\vec{k}$) in the integrands. The terms subtracted are the three leading terms in an asymptotic expansion of the integrands in powers of T^{-1}. This is the minimal number of subtractions that suffices to make ρ_0 and $(P_i)_0$ converge to zero in the adiabatic limit, where all the physically relevant energy and pressure are attributable to particles. Also, it is these terms in the series which lead to divergent integrals.

Let

$$Y = \Omega_{\vec{k}}{}^2 + Q . \tag{2.34}$$

We do not introduce the parameter T explicitly, but regard T as set equal to 1 in most formulas. The order of a quantity in T^{-1} can be determined by counting the number of derivatives with respect to η or t which it contains. Q, by virtue of its definition (A13), consists of terms like $\dot{a}_1\dot{a}_2/a_1a_2$ and $(\dot{a}_1/a_1)^2$, and thus it is of order T^{-2}. However, $\Omega_{\vec{k}}{}^2$ is of order T^0. Generally speaking, in a given expression encountered in these calculations, terms which differ by n powers of T will also differ by n powers of $|\vec{k}|$ or m.

For adiabatic regularization, in accordance with Ref. 4, we need an approximate solution of Eq. (2.22) of the positive-frequency generalized WKB form

$$\chi_{\vec{k}}^{\text{approx}}(\eta) = [2V^{1/3}(\eta) W_{\vec{k}}(\eta)]^{-1/2} \times \exp\left[-i \int^\eta V^{1/3}(\eta') W_{\vec{k}}(\eta') d\eta' \right], \tag{2.35}$$

where an appropriate positive function $W_{\vec{k}}$ is given by the method of Chakraborty[14] as

$$W_{\vec{k}} = V^{-1/3} [Y(1 + \underline{\epsilon_2})(1 + \underline{\epsilon_4})]^{1/2}, \tag{2.36}$$

with

$$\underline{\epsilon_2} = -Y^{-3/4} \partial_\eta (Y^{-1/2} \partial_\eta Y^{1/4}), \tag{2.37}$$

$$\underline{\epsilon}_4 = -Y^{-1/2}(1+\underline{\epsilon}_2)^{-3/4}\partial_\eta \times\{[Y(1+\underline{\epsilon}_2)]^{-1/2}\partial_\eta[(1+\underline{\epsilon}_2)^{1/4}]\}. \tag{2.38}$$

Since any function $W_{\vec{k}}$ which agrees with that of Eq. (2.36) up to order T^{-4} is acceptable for our purpose, we may use

$$W_{\vec{k}} = \omega_{\vec{k}}(1+\underline{\epsilon}_2 + Q\Omega_{\vec{k}}^{-2} + \underline{\epsilon}_4 + \underline{\epsilon}_2 Q\Omega_{\vec{k}}^{-2})^{1/2} = \omega_{\vec{k}}(1+\epsilon_2+\epsilon_4)^{1/2}, \tag{2.39}$$

where $\omega_{\vec{k}}$ is defined in Eq. (2.23), and

$$\epsilon_2 = \underline{\epsilon}_2 + Q\Omega_{\vec{k}}^{-2}, \tag{2.40}$$

$$\epsilon_4 = \underline{\epsilon}_4 + \underline{\epsilon}_2 Q\Omega_{\vec{k}}^{-2}. \tag{2.41}$$

We have not indicated the $\vec{k}$ dependence of ϵ_2 and ϵ_4 by a subscript, and in what follows we shall omit the subscript $\vec{k}$ from other quantities as well when there is no chance of confusion. Note that ϵ_2 contains terms of order T^{-2}, T^{-4}, and higher, when the fractional powers of Y are expanded as power series in $Q\Omega^{-2}$. The leading term of ϵ_4 is of order T^{-4}. We write $\epsilon_{n(m)}$ for the term of ϵ_n which is of order T^{-m} (and similarly for other quantities). Thus the quantities of interest to us are $\epsilon_{2(2)}$, $\epsilon_{2(4)}$, and $\epsilon_{4(4)}$ [see Eqs. (B1)]. Of course, a term like $\epsilon_{2(2)}\epsilon_{4(4)}$ is to be dropped, because it is of sixth order.

With $W_{\vec{k}}$ known, a positive-frequency solution $\chi_{\vec{k}}$ is determined up to the proper order by the condition that it be approximated by the $\chi_{\vec{k}}^{\text{approx}}$ of Eq. (2.35). Thus the $A_{\vec{k}}$ in the field expansion (2.21) are defined to that order, and the first step of the program is completed.

The next task is to use the above formulas of the adiabatic approximation to expand the integrands of Eqs. (2.26) and (2.27) in asymptotic series. We find to fourth order

$$|\chi|^2 \approx (2V^{1/3}W)^{-1} \approx (2\Omega)^{-1}\{1-[\tfrac{1}{2}\epsilon_{2(2)}]+\tfrac{3}{8}\epsilon_{2(2)}{}^2-\tfrac{1}{2}\epsilon_{2(4)}-\tfrac{1}{2}\epsilon_{4(4)}\}, \tag{2.42}$$

$$|\chi'|^2 \approx (2V^{1/3}W)^{-1}\{V^{2/3}W^2+\tfrac{1}{4}[\partial_\eta \ln(V^{1/3}W)]^2\}$$
$$\approx (2\Omega)^{-1}\left\{\Omega^2+\left[\tfrac{1}{2}\Omega^2\epsilon_{2(2)}+\tfrac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2\right]+\tfrac{1}{2}\Omega^2\epsilon_{2(4)}+\tfrac{1}{2}\Omega^2\epsilon_{4(4)}-\tfrac{1}{8}\Omega^2\epsilon_{2(2)}{}^2-\tfrac{1}{8}\left(\frac{\Omega'}{\Omega}\right)^2\epsilon_{2(2)}+\frac{1}{4}\frac{\Omega'}{\Omega}\epsilon'_{2(3)}\right\}. \tag{2.43}$$

Here and in Eqs. (2.45) and (2.46) the bracketed terms are those of order T^{-2} and the terms following them are of order T^{-4}. Also, the expressions

$$\partial_\eta|\chi|^2 \approx -(2V^{1/3}W)^{-1}\partial_\eta \ln(V^{1/3}W) \approx -(2\Omega)^{-1}\left\{\frac{\Omega'}{\Omega}+\tfrac{1}{2}\left[\epsilon'_{2(3)}-\frac{\Omega'}{\Omega}\epsilon_{2(2)}\right]\right\} \tag{2.44}$$

are good through third order.

So from Eq. (2.26) we have

$$\rho_{0\,\text{div}} = (32\pi^3)^{-1}V^{-4/3}\int d^3k\,\Omega^{-1}\left\{2\Omega^2+\left[\tfrac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2-Q\right]-\tfrac{1}{8}\left(\frac{\Omega'}{\Omega}\right)^2\epsilon_{2(2)}+\frac{1}{4}\frac{\Omega'}{\Omega}\epsilon'_{2(3)}+\tfrac{1}{4}\Omega^2\epsilon_{2(2)}{}^2+\tfrac{1}{2}Q\epsilon_{2(2)}\right\}, \tag{2.45}$$

and from Eq. (2.27) we have

$$(P_i)_{0\,\text{div}} = (96\pi^3)^{-1}V^{-4/3}\int d^3k\,\Omega^{-1}\left\{6V^{2/3}\left(\frac{k_i}{a_i}\right)^2+\left[\epsilon_{2(2)}\left(\Omega^2-3V^{2/3}\frac{k_i{}^2}{a_i{}^2}\right)+\tfrac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2-\frac{\Omega'}{\Omega}(d_i-D)+\partial_\eta(d_i-D)-Q\right]\right.$$
$$+\Omega^2(\epsilon_{2(4)}+\epsilon_{4(4)}-\tfrac{1}{2}\epsilon_{2(2)}{}^2)-3V^{2/3}\left(\frac{k_i}{a_i}\right)^2(\epsilon_{2(4)}+\epsilon_{4(4)}-\tfrac{3}{4}\epsilon_{2(2)}{}^2)$$
$$\left.-\tfrac{1}{2}\epsilon_{2(2)}\left[\tfrac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2-\frac{\Omega'}{\Omega}(d_i-D)+\partial_\eta(d_i-D)-Q\right]-\tfrac{1}{2}\epsilon'_{2(3)}\left(d_i-D-\frac{1}{2}\frac{\Omega'}{\Omega}\right)\right\}. \tag{2.46}$$

These quantities are the divergent leading terms of the adiabatic expansions of ρ_0 and $(P_i)_0$. The hypothesis of Ref. 4 is that these are precisely the terms which should be subtracted from ρ and P_i, as given in Eqs. (2.30) and (2.31), in order to obtain the physical energy density and pressure. The prescription of Zel'dovich and Starobinsky (Ref. 10) is equivalent; see Appendix C for further discussion. On the other hand, one would like to make the vacuum subtraction procedure more physical by associating it with a renormalization of coupling constants, as in Ref. 5. Such an interpretation is possible only if the subtracted terms are formally proportional to certain geometrical tensors. This consideration will lead us, after explicit calculation of the terms in Eq. (2.45), to consider a modified vacuum subtraction ansatz.

III. RENORMALIZATION

A. Geometrical interpretation of the leading terms of the energy density

We shall now reduce the expression (2.45) to a more explicit form. The results of Ref. 5 lead one to expect that the three terms in that expression (of orders T^0, T^{-2}, T^{-4}) should be proportional, respectively, to the time-time component of the metric ($g_0{}^0=\delta_0{}^0=1$, independent of time), of the Einstein tensor ($G_0{}^0$), and of some linear combination of two "quadratic" divergenceless[15] curvature tensors, ${}^{(1)}H_\mu{}^\nu$ and ${}^{(2)}H_\mu{}^\nu$ [see Eqs. (3.5), (3.6), and (A28)–(A34)]. [In view of the discussion in Sec. 5 of Ref. 5, the situation with regard to the space-space components of the Einstein equation should be more complicated; we shall not investigate the pressure expressions (2.46) further in this paper.] It will turn out that this expectation is not fulfilled precisely in the present case, but that a statement of the same type can be made after a further manipulation.

Following Ref. 5, we introduce new variables of integration

$$p_i = k_i/a_i \tag{3.1}$$

and pass to polar coordinates in p space [see Eq. (B7)]. The term in ρ_0 of lowest order thus takes the form

$$\rho_{0(0)} = (4\pi^2)^{-1}\int_0^\infty \omega p^2 dp\,, \tag{3.2}$$

where $\omega^2 = p^2+m^2$. This (divergent) expression is indeed a constant, as expected, and has the familiar form of the vacuum energy subtraction for a free scalar field in Minkowski space.

Next we turn to the second-order term in Eq. (2.45),

$$\rho_{0(2)} = (32\pi^3)^{-1}V^{-4/3}\int d^3k\,\Omega^{-1}[\tfrac{1}{4}(\Omega'/\Omega)^2 - Q]$$

$$= (128\pi^3)^{-1}V^{-2/3}\int d^3p\left[\omega^{-1}D^2 - 2\omega^{-3}D\sum p_i{}^2 d_i + \omega^{-5}\left(\sum p_i{}^2 d_i\right)^2\right] - (32\pi^3)^{-1}V^{-4/3}\int d^3k\,\Omega^{-1}Q\,.$$

After integrating over the angles in p space (see Appendix B), we have for the first term on the right-hand side

$$(32\pi^2)^{-1}V^{-2/3}\int_0^\infty p^2dp\left[\omega^{-1}D^2 - 2\omega^{-3}p^2D^2 + \omega^{-5}p^4\left(\tfrac{1}{5}\sum d_i{}^2 + \tfrac{2}{15}\sum_{i<j} d_i d_j\right)\right].$$

Finally, use of Eqs. (A22) to eliminate D and $\sum d_i{}^2$ in favor of $G_0{}^0$ and Q yields

$$\rho_{0(2)} = -(96\pi^2)^{-1}m^4\int_0^\infty \omega^{-5}p^2dp\,G_0{}^0 - (160\pi^2)^{-1}\int_0^\infty \omega^{-5}p^2(16p^4+40m^2p^2+15m^4)dp\,V^{-2/3}Q\,. \tag{3.3}$$

The term proportional to $G_0{}^0$ is convergent. This is consistent with the observation of Zel'dovich and Starobinsky (Ref. 10) that in the isotropic case ($Q=0$) no vacuum subtraction to remove infinities in the conformal energy-momentum tensor is needed beyond that indicated by Eq. (3.2). From a renormalization point of view, however, it appears from Eq. (3.3) that the vacuum fluctuations of a massive scalar field give rise to a *finite* change in the effective value of the gravitational constant G. For further discussion see Sec. V.

The other term in Eq. (3.3) is problematical. Since it is divergent, it must somehow be eliminated from the theory. Its subtraction has no renormalization interpretation, however, since $V^{-2/3}Q$ is not a component of any divergenceless tensor formed covariantly from purely geometrical quantities. This circumstance casts doubt on the correctness of the regularization prescription as presented so far. We propose a remedy in the next subsection.

First, however, we examine the remaining (fourth-order) term in Eq. (2.45). In Appendix B we derive a complicated expression for this term [Eq. (B14)], which simplifies when m is formally set equal to zero:

$$\rho_{0(4)} \sim (480\pi^2)^{-1}V^{-4/3}\int_0^\infty p^{10}\omega^{-11}dp(9U - 3Q'' + 36Q^2) \tag{3.4}$$

[U defined by Eq. (A21)]. The precise significance of Eq. (3.4) depends on whether or not m is zero; the situation in each case is essentially the same as in Ref. 5, which may be consulted for a more extensive discussion.

When $m=0$, the integrand in Eq. (3.4) (with $\omega=p$) is the exact fourth-order term in the adiabatic expansion of the integrand of ρ_0 [Eq. (2.26)]; the nonuniformity of the latter expansion produces a divergence at the lower limit, however. A finite expression for the physical energy density can be obtained by subtracting from the formal expression (2.26) only the two lowest-order terms [the integrands of Eqs. (3.1) and (3.3)] for values of p smaller than an arbitrary positive constant, while subtracting all three leading terms at large values of p.

When $m\neq 0$, there is no infrared divergence in the fourth-order term. By putting all the terms in the integrand of Eq. (B14) over the common denominator ω^{11}, one writes $\rho_{0(4)}$ as the sum of the divergent term (3.4) and convergent terms of order m^2 or higher in the mass.

In either case, the significance of Eq. (3.4) is that the logarithmic ultraviolet divergence associated with the fourth-order term can be removed by renormalization. The physical constants to be renormalized are the coupling constants associated with two new tensors which are to be added to the gravitational field equation[16]:

$$^{(1)}H_{\mu\nu}=2|g|^{-1/2}\delta(|g|^{1/2}R^2)/\delta g^{\mu\nu} \tag{3.5}$$

and

$$^{(2)}H_{\mu\nu}=2|g|^{-1/2}\delta(|g|^{1/2}R_{\alpha\beta}R^{\alpha\beta})/\delta g^{\mu\nu}. \tag{3.6}$$

Explicit expressions for $^{(1)}H_\mu{}^\nu$ and $^{(2)}H_\mu{}^\nu$ in a general metric are given in Eqs. (A28) and (A29), and in the Kasner metric (2.1) they take the form (A31)–(A34). We note from Eq. (A33) that the time-dependent factor $V^{-4/3}(9U-3Q''+36Q^2)$ in Eq. (3.4) is precisely

$$^{(-)}H_0{}^0={}^{(2)}H_0{}^0-\tfrac{1}{3}\,{}^{(1)}H_0{}^0. \tag{3.7}$$

Hence the term (3.4) can be "moved to the other side" of the gravitational field equation—i.e., absorbed into the postulated new term proportional to $^{(-)}H_\mu{}^\nu$ [see Eqs. (1.2) and (1.3)].

That only the particular linear combination (3.7) should arise in the model with conformal coupling is again consistent with the fact (Ref. 10) that no regularization is necessary when the universe is isotropic: $^{(-)}H_\mu{}^\nu$ vanishes in that case (and, more generally, whenever the spacetime is conformally flat).

B. A method of renormalization

We return to the puzzling second term in Eq. (3.3). It can be made to disappear by a formal trick, whose vindication must be sought in the covariance of the resulting theory. We observe that the identification of the terms of a given order in the adiabatic expansion of an integral depends on the variable of integration used. For instance, in the study of the energy density of a minimally coupled scalar field in a spatially curved Robertson-Walker universe (Ref. 5, Sec. 6) one obtains leading terms of tensorial form by employing p, the "physical momentum" which naturally occurs in the adiabatic expansion of the integrand, in preference to another variable, q/a, which is more naturally associated with the eigenfunctions of the Laplacian on the curved Robertson-Walker three-space. In the present case we shall choose a variable of integration by the criterion that no nontensorial divergent terms may appear in the corresponding adiabatic expansion.[17]

Let us introduce the variable r by

$$p^2=r^2+\tfrac{2}{5}V^{-2/3}Q. \tag{3.8}$$

Since $V^{-2/3}Q$ is of order T^{-2}, an integral over p of a given order in T^{-2} will appear to be of mixed order after the transformation (3.8) is carried out. The first term in the energy density, given by Eq. (3.2), can be written as

$$(4\pi^2)^{-1}\int_0^{[r_0^2+(2/5)V^{-2/3}Q]^{1/2}}p^2(p^2+m^2)^{1/2}dp+(4\pi^2)^{-1}\int_{r_0}^\infty r(r^2+\tfrac{2}{5}V^{-2/3}Q)^{1/2}(r^2+m^2+\tfrac{2}{5}V^{-2/3}Q)^{1/2}dr, \tag{3.9}$$

where r_0 is an arbitrary positive number. The first, or infrared, term of Eq. (3.9) is finite. For sufficiently large T (small Q) we may expand to fourth order the integrand of the second (ultraviolet) term, obtaining for that term

$$\begin{aligned}(4\pi^2)^{-1}\int_{r_0}^\infty dr\,r^2&(1+r^{-2}\tfrac{1}{5}V^{-2/3}Q-r^{-4}\tfrac{1}{50}V^{-4/3}Q^2+\cdots)\\&\times(r^2+m^2)^{1/2}[1+(r^2+m^2)^{-1}\tfrac{1}{5}V^{-2/3}Q-(r^2+m^2)^{-2}\tfrac{1}{50}V^{-4/3}Q^2+\cdots]\\&=(4\pi^2)^{-1}\int_{r_0}^\infty dr\,r^2(r^2+m^2)^{1/2}\left[1+\frac{2r^2+m^2}{r^2(r^2+m^2)}\tfrac{1}{5}V^{-2/3}Q-\frac{m^4}{r^4(r^2+m^2)^2}\tfrac{1}{50}V^{-4/3}Q^2+\cdots\right].\end{aligned} \tag{3.10}$$

Only the first two terms in the integrand of Eq. (3.10) give rise to divergences. The second of these terms is of manifest order T^{-2}.

We now treat the second-order expression (3.3) in the same way. The term proportional to $G_0{}^0$ is already convergent. The second term becomes

$$-(160\pi^2)^{-1}\int_0^{[r_0^2+(2/5)V^{-2/3}Q]^{1/2}} \omega^{-5}p^2(16p^4+40m^2p^2+15m^4)dp\, V^{-2/3}Q$$

$$-(160\pi^2)^{-1}\int_{r_0}^{\infty} dr\, r(r^2+\tfrac{2}{5}V^{-2/3}Q)^{1/2}(r^2+m^2+\tfrac{2}{5}V^{-2/3}Q)^{-5/2}$$

$$\times[16(r^4+\tfrac{4}{5}r^2V^{-2/3}Q+\tfrac{4}{25}V^{-4/3}Q^2)+40m^2(r^2+\tfrac{2}{5}V^{-2/3}Q)+15m^4]\,V^{-2/3}Q\,. \quad (3.11)$$

Once again, the first term in Eq. (3.11) is a convergent contribution from the infrared modes. The ultraviolet term, after expansion of the integrand, reduces to

$$-(10\pi^2)^{-1}\int_{r_0}^{\infty} dr\, r^4(r^2+m^2)^{-5/2}(r^2+\tfrac{5}{2}m^2+\tfrac{15}{16}m^4/r^2)V^{-2/3}Q + \text{convergent terms}\,. \quad (3.12)$$

The sum of the divergent ultraviolet terms in Eqs. (3.10) and (3.12) is

$$(4\pi^2)^{-1}\int_{r_0}^{\infty} dr\, r^2(r^2+m^2)^{1/2} + (10\pi^2)^{-1}\int_{r_0}^{\infty} dr(r^2+m^2)^{-5/2}(\tfrac{17}{16}r^2m^4+\tfrac{1}{2}m^6)V^{-2/3}Q\,. \quad (3.13)$$

The second term in Eq. (3.13) is convergent. Thus the first two orders of adiabatic regularization yield a constant vacuum energy,

$$(4\pi^2)^{-1}\int_{r_0}^{\infty} dr\, r^2(r^2+m^2)^{1/2}, \quad (3.14)$$

plus finite terms. The remarkable fact is that the manipulation resulting in cancellation of the quadratic divergence has not given rise to any new logarithmic divergences. Furthermore, the change of variable from p to r in the fourth-order term (3.4) does not change the form of the logarithmic divergence, although, of course, it does engender convergent higher-order corrections and infrared terms. *Therefore, in the theory of a conformally coupled scalar field in a generalized Kasner spacetime, all divergences in the energy density can be removed by renormalization of the cosmological constant* Λ *and the coefficient* σ_- *of* ${}^{(-)}H_\mu{}^\nu$ *in the generalized gravitational field equation (1.2).* [Since there are no divergences in this theory proportional to $G_\mu{}^\nu$ or ${}^{(1)}H_\mu{}^\nu$, renormalization of G and insertion of a term proportional to ${}^{(1)}H_0{}^0$ in Eq. (1.2) are not theoretically necessary. The appearance (when $m\neq 0$) of a finite term proportional to $G_\mu{}^\nu$, however, probably calls for a finite redefinition of the effective value of G to agree with the experimentally determined value. See Sec. V of this paper, and also Sec. 4 of Ref. 5.]

The role of the "anomalous" convergent terms in Eqs. (3.10) and (3.12), which appear only when the mass is not zero, as well as similar terms which would appear if the infrared contributions in Eqs. (3.9) and (3.11) were evaluated, is still unclear. See Ref. 5, Secs. 3, 6, and 10, for discussion of this problem.

In contrast, when $m=0$ the results just obtained become rather clean. In this case the leading contributions (3.9) and (3.11) are simply

$$(4\pi^2)^{-1}\int_0^{[r_0^2+(2/5)V^{-2/3}Q]^{1/2}} (p^3-\tfrac{2}{5}pV^{-2/3}Q)dp$$

$$+(4\pi^2)^{-1}\int_{r_0}^{\infty}(r^2+\tfrac{2}{5}V^{-2/3}Q)r\,dr$$

$$-(10\pi^2)^{-1}\int_{r_0}^{\infty} r\,dr\,V^{-2/3}Q\,. \quad (3.15)$$

The sum of the two ultraviolet terms is exactly

$$(4\pi^2)^{-1}\int_{r_0}^{\infty} r^3dr = (4\pi^2)^{-1}\int_0^{\infty} r^3dr - (16\pi^2)^{-1}r_0^4\,. \quad (3.16)$$

The infrared contribution [first term in Eq. (3.15)] is

$$(16\pi^2)^{-1}r_0^4 - (100\pi^2)^{-1}V^{-4/3}Q^2\,. \quad (3.17)$$

The arbitrary division point r_0 cancels out, as it must, and we are left with a divergent constant vacuum energy,

$$(4\pi^2)^{-1}\int_0^{\infty} r^3dr\,, \quad (3.18)$$

and a convergent term of order T^{-4},

$$-(100\pi^2)^{-1}V^{-4/3}Q^2\,, \quad (3.19)$$

which originates only from the lowest modes, where the adiabatic expansion of the integrand in ρ_0 is not strictly rigorous in the massless case. The ultraviolet part of the original $O(T^{-4})$ term, Eq. (3.4), is proportional to

$$\int_{r_0}^{\infty}(r^2+\tfrac{2}{5}V^{-2/3}Q)^{-1}r\,dr\,{}^{(-)}H_0{}^0\,. \quad (3.20)$$

Expansion of the integrand in Eq. (3.20) yields, besides the divergent term to be removed by renormalization of σ_-, convergent terms of order T^{-6} and higher, proportional to

$$(V^{-2/3}Q)^{2n}\,{}^{(-)}H_0{}^0\,. \quad (3.21)$$

The latter should be retained as part of the effective or physical energy density.

In the massless case, therefore, the renormalized vacuum energy density consists of (1) the integral over $p > (r_0^2 + \frac{2}{5}V^{-2/3}Q)^{1/2}$ of the remainder of the integrand of Eq. (2.26) after subtraction of the three leading terms, as given in Eq. (2.45); (2) the terms of the form (3.21); (3) the integral over $p < (r_0^2 + \frac{2}{5}V^{-2/3}Q)^{1/2}$ of the integrand of Eq. (2.26); (4) a term

$$-(16\pi^2)^{-1} r_0^4 \tag{3.22}$$

to compensate for the inclusion in component (3) of the infrared contribution to the unobservable renormalization of the cosmological constant. If the adiabatic expansion may be trusted in the infrared region so long as it does not yield divergent integrals, then the components (3) and (4) may be replaced by (3′) the integral over $p < (r_0^2 + \frac{2}{5}V^{-2/3}Q)^{1/2}$ of the remainder of the integrand of Eq. (2.26) after subtraction of the *two* leading terms, and (4′) the term (3.19). On the other hand, contributions (1) and (2) may be combined by writing Eq. (2.26) in terms of r as explicit variable of integration, then subtracting from the integrand the three leading terms of manifest orders T^0, T^{-2}, and T^{-4}, and finally integrating over $r > r_0$. The renormalized vacuum pressure must be defined similarly from Eqs. (2.27) and (2.46), in such a way that the condition of vanishing covariant divergence, Eq. (A7), is satisfied; we leave the details for later work.

We believe that the interpretation of this vacuum subtraction procedure in terms of renormalization of physical constants—in other words, basically, the formal tensorial nature of the subtracted quantities—argues strongly for its correctness, in preference to the method of regularization which was arrived at in Sec. I and in Ref. 10.

IV. MINKOWSKI SPACE AS A KASNER UNIVERSE

The methods introduced in Refs. 4 and 5 and the present paper apply, at their present stage of development, to special classes of metrics, with respect to which the scalar field equation can be solved by separation of variables. The procedure is thus tied to a particular coordinate system in which the equation separates. There are a few Riemannian spacetimes, however, with such a high degree of symmetry that they admit, at least locally, two or more of such privileged coordinate systems. If our methods are correct, they must yield physically equivalent results when carried out from the point of view of each of these coordinate systems. This vestige of the principle of general covariance survives in the situation at hand.

One such manifold is flat Minkowski space, regions of which can be cast into the guise of a Kasner universe or an open (hyperbolic) Robertson-Walker universe by a proper choice of coordinates. Our investigation of these special cases of those two classes of metrics confirms the adiabatic concept of physical particles introduced in Refs. 9 and 4 and used in Sec. II of this paper. That is, the generalized notion of a positive-frequency solution, defined in terms of a higher-order WKB approximation [see Eq. (2.35) and the extensive motivation in Ref. 4], coincides in all three ways of looking at the space. If follows that the adiabatic annihilation and creation operators and vacuum state, and also the one-particle states, etc., are the same from all three points of view, to within the fuzziness inherent in things which are defined only by their asymptotic behavior.

We demonstrate here the Kasner part of this proposition. In a Minkowski space with Cartesian coordinates (y^0, y^1, y^2, y^3), introduce new coordinates (t, x^1, x^2, x^3) by

$$y^0 = t\cosh x^1,\ \ y^1 = t\sinh x^1,\ \ y^2 = x^2,\ \ y^3 = x^3\,. \tag{4.1}$$

The metric in the region where $y^0 > |y^1|$ takes the form

$$ds^2 = dt^2 - t^2(dx^1)^2 - (dx^2)^2 - (dx^3)^2\,, \tag{4.2}$$

where $0 < t < \infty$, $-\infty < x^i < \infty$. This is a Kasner metric; in fact, it is one of the Kasner vacuum solutions represented in Eq. (2.2), with $p_1 = 1$, $p_2 = p_3 = 0$. Some of the quantities employed earlier in the paper are

$$\eta = \int_0^t (t')^{-1/3}\,dt' = \tfrac{3}{2}\,t^{2/3}\,, \tag{4.3}$$

$$t = V = a_1 = (\tfrac{2}{3})^{3/2}\eta^{3/2}\,, \tag{4.4}$$

$$D = \tfrac{1}{3}d_1 = \tfrac{1}{2}\eta^{-1}\,,\ \ d_2 = d_3 = 0\,,\ \ Q = \tfrac{1}{4}\eta^{-2}\,. \tag{4.5}$$

The Klein-Gordon equation (2.4) of a scalar field becomes

$$\frac{d^2\phi}{dt^2} + \frac{1}{t}\frac{d\phi}{dt} - \frac{1}{t^2}\frac{d^2\phi}{d(x^1)^2} - \frac{d^2\phi}{d(x^2)^2} - \frac{d^2\phi}{d(x^3)^2} + m^2\phi = 0\,. \tag{4.6}$$

It is easy to see that the separated equation for the time dependence can be solved in terms of Bessel functions; indeed, one finds that the solutions of Eq. (2.22) are

$$\chi_{\vec k} = N\eta^{1/2} Z_{ik_1}(\beta\eta^{3/2})\,,\ \ \beta = (\tfrac{2}{3})^{3/2}(k_2^{\,2} + k_3^{\,2} + m^2)^{1/2}\,, \tag{4.7}$$

where $Z_{ik_1}(z)$ is a Bessel function of imaginary index, ik_1, and the normalization constant N is determined to be $i\pi^{-1/2}/2$ by the Wronskian condition (2.25). This solution is valid even when $m=0$, except for the modes with $k_2=k_3=0$, which form a set of measure zero in $\vec{k}$ space.

We now ask what the positive-frequency solutions, in the ordinary sense of special-relativistic quantum theory, look like in this new picture. It is convenient to revert to the Gaussian time variable t and to consider a basis of solutions of Eq. (4.6) with elements

$$\phi_{\vec{k}}^{\pm}=H_{ik_1}^{(j)}((k_2{}^2+k_3{}^2+m^2)^{1/2}\,t)e^{i\vec{k}\cdot\vec{x}}\,, \tag{4.8}$$

where $H_{ik_1}^{(j)}(z)$ ($j=2$ for ϕ^+, $j=1$ for ϕ^-) are the Hankel functions. We shall show that $\phi_{\vec{k}}^{\pm}$ is a superposition of positive-frequency plane waves in Minkowski space. There is an integral representation[18]

$$H_{ik_1}^{(2)}(\alpha t)=i\pi^{-1}e^{-\pi k_1/2}\times\int_{-\infty}^{\infty}dz'\,e^{-i\alpha t\cosh z'}\,e^{-ik_1z'}\,. \tag{4.9}$$

So, using

$$t=[(y^0)^2-(y^1)^2]^{1/2}\,,\qquad x^1=\tanh^{-1}(y^1/y^0)=\sinh^{-1}(y^1/t)=\cosh^{-1}(y^0/t)\,, \tag{4.10}$$

and setting $z=z'-x^1$, we find that

$$\begin{aligned}H_{ik_1}^{(2)}((k_2{}^2+k_3{}^2+m^2)^{1/2}\,t)e^{ik_1x^1}&=i\pi^{-1}e^{-\pi k_1/2}\int_{-\infty}^{\infty}dz\,\exp\{-i(k_2{}^2+k_3{}^2+m^2)^{1/2}[(y^0)^2-(y^1)^2]^{1/2}\\&\qquad\times(\cosh z\cosh x^1+\sinh z\sinh x^1)\}\,e^{-ik_1z}\\&=i\pi^{-1}e^{-\pi k_1/2}\int_{-\infty}^{\infty}dz\,\exp[-i(k_2{}^2+k_3{}^2+m^2)^{1/2}\cosh z\,y^0]\\&\qquad\times\exp[-i(k_2{}^2+k_3{}^2+m^2)^{1/2}\sinh z\,y^1]\,e^{-ik_1z}\,.\end{aligned}$$

Let

$$p_1=-(k_2{}^2+k_3{}^2+m^2)^{1/2}\sinh z\,,\quad p_2=k_2\,,\quad p_3=k_3\,, \tag{4.11}$$

$$\omega_{\vec{p}}^{+}=(k_2{}^2+k_3{}^2+m^2)^{1/2}\cosh z=(p_1{}^2+p_2{}^2+p_3{}^2+m^2)^{1/2}\,.$$

Then we have

$$\phi_{\vec{k}}^{+}=i\pi^{-1}e^{-ik_1/2}\int_{-\infty}^{\infty}dz\,e^{-i\omega_{\vec{p}}^{+}y^0}\,e^{i\vec{p}\cdot\vec{y}}\,e^{-ik_1\phi}\,, \tag{4.12}$$

where p_2 and p_3 are fixed and p_1 and $\omega_{\vec{p}}^{+}$ are parametrized by z through Eqs. (4.11). Thus the solutions in the Kasner coordinates involving only $H^{(2)}$ functions ($\phi_{\vec{k}}^{+}$) are superpositions of purely positive-frequency plane waves. Similarly, $\phi_{\vec{k}}^{-}$ contains only negative frequencies.

In other words, in this model there is a natural *exact* definition of "positive-frequency solution," carried over from the Minkowski universe of which our Kasner universe is a part. The adiabatic method, on the other hand, defines "positive frequency" to an arbitrary finite order by requiring that the solution behave asymptotically like the expression (2.35). Are these definitions consistent? To verify this, we exploit the fact that here the limit of large t is effectively an adiabatic limit. As $t\to\infty$ (with k_1 fixed) we have

$$W\sim\omega\sim(k_2{}^2+k_3{}^2+m^2)^{1/2}\,,$$

and hence the $\chi_{\vec{k}}^{\text{approx}}$ of Eq. (2.35) goes like

$$t^{-1/6}(k_2{}^2+k_3{}^2+m^2)^{-1/4}\exp[-i(k_2{}^2+k_3{}^2+m^2)^{1/2}\,t]\,. \tag{4.13}$$

On the other hand, the familiar asymptotic expansions for the Hankel functions[19] show that $\chi_{\vec{k}}^{\pm}\equiv\phi_{\vec{k}}^{\pm}V^{1/3}e^{-i\vec{k}\cdot\vec{x}}$ have the behavior

$$\begin{aligned}\chi^{+}&\sim t^{1/3}H^{(2)}\sim t^{-1/6}(k_2{}^2+k_3{}^2+m^2)^{-1/4}\times\exp[-i(k_2{}^2+k_3{}^2+m^2)^{1/2}\,t]\,,\\ \chi^{-}&\sim t^{1/3}H^{(1)}\sim t^{-1/6}(k_2{}^2+k_3{}^2+m^2)^{-1/4}\times\exp[+i(k_2{}^2+k_3{}^2+m^2)^{1/2}\,t]\,,\end{aligned} \tag{4.14}$$

where irrelevant constant factors have been dropped. Thus a solution proportional to $H^{(2)}$ is consistent with the adiabatic requirement, and a solution involving a linear combination of $H^{(1)}$ and $H^{(2)}$ is inconsistent with the requirement unless the coefficient of $H^{(1)}$ vanishes to the required order in the adiabatic limit (i.e., in the present context, in the limit of large $|\vec{k}|$).

Thus the adiabatic concept of particles in the Kasner spacetime is consistent with the exact concept induced from Minkowski space. The adiabatic analysis has passed a crucial test.

For the two-dimensional analog of this model (coordinates y^2 and y^3 omitted), the connection between solutions involving the $H^{(2)}$ function and the conventional concept of particles has been found by one of us[20] and by Sommerfield[21] and diSessa.[22] A related four-dimensional model is the isotropic one defined by

$$y^0 = t\cosh\chi\,,\quad r = t\sinh\chi\,, \tag{4.15}$$

where r and χ are radial coordinates in the Minkowski and the open Robertson-Walker pictures, respectively. Sommerfield gives the analog of Eq. (4.12) for that case [Ref. 21, Eqs. (6.26) and (6.27)], so our conclusion applies also to it. diSessa offers a nonconstructive argument which applies to all three cases: A positive-frequency function is one which vanishes as the time coordinate approaches negative imaginary infinity, and $H^{(2)}$ has this property. These considerations (in Refs. 20–22) apply, however, only when $m\neq 0$. diSessa (Ref. 22) gives a separate treatment of the massless case.

V. CONFORMAL ENERGY-MOMENTUM TENSOR IN ROBERTSON-WALKER METRICS

We now turn to the Robertson-Walker universes, and show that, even when the three-space is curved, there is no infinite renormalization of the Newtonian constant G, or of any coupling constants associated with the higher-order curvature tensors like ${}^{(1)}H_\mu{}^\nu$. That result is to be expected because of the conformal flatness of the Robertson-Walker metrics (see Refs. 6, 9, and 10).

The metric is

$$ds^2 = dt^2 - a^2(t)h_{ij}\,dx^i\,dx^j\,, \tag{5.1}$$

with $h_{ij}\,dx^i\,dx^j$ the line element of a space of constant curvature. The Lagrangian of the scalar field is given by Eq. (2.3), with the scalar curvature

$$R = 6\left[\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{\epsilon}{a^2}\right]\,, \tag{5.2}$$

where ϵ can take the values $+1$, 0, -1 corresponding to positive, vanishing, or negative spatial curvature, respectively. The field equation (2.4) takes the form

$$\partial_0{}^2\phi + 3(\dot a/a)\partial_0\phi - a^{-2}\Delta^{(3)}\phi + (m^2 + \tfrac{1}{6}R)\phi = 0\,, \tag{5.3}$$

where

$$\Delta^{(3)}\phi = h^{-1/2}\partial_j(h^{1/2}h^{jk}\partial_k\phi)\,. \tag{5.4}$$

The field is quantized as in Eqs. (2.6) and (2.7), and is coupled to the metric through Eqs. (2.8) and (2.9).

The time-time component of the conformal energy-momentum tensor (2.10) takes the form (operator symmetrization understood)

$$\begin{aligned}\Lambda_{00} = {} & \tfrac{2}{3}(\partial_0\phi)^2 - \tfrac{1}{3}\phi\partial_0{}^2\phi - \tfrac{1}{6}g^{\lambda\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi) \\ & + \tfrac{1}{3}g^{\lambda\sigma}\phi(\partial_\lambda\partial_\sigma\phi) + \tfrac{1}{2}m^2\phi^2 + \frac{\dot a}{a}\phi\partial_0\phi \\ & + \tfrac{1}{3}a^{-2}h^{ij}\,{}^*\Gamma_{ij}\,\phi\partial_l\phi - \tfrac{1}{6}\phi^2 G_{00}\,,\end{aligned} \tag{5.5}$$

where we have used

$$\Gamma^0_{ij} = a\dot a h_{ij}\,,\quad \Gamma^i_{0j} = \frac{\dot a}{a}\delta^i_j\,,\quad \Gamma^i_{jk} = {}^*\Gamma^i_{jk}\,, \tag{5.6}$$

with ${}^*\Gamma^i_{jk}$ representing the Christoffel symbol formed from the spatial metric h_{ij}.

Since $\langle\Lambda_{00}\rangle$ must have no spatial dependence when evaluated in a state of the proper symmetry, we can write

$$\langle\Lambda_{00}\rangle = \left(\int d^3x\sqrt{h}\right)^{-1}\left\langle\int d^3x\sqrt{h}\,\Lambda_{00}\right\rangle\,, \tag{5.7}$$

where the right-hand side is to be understood in a limiting sense for the cases in which the volume $\int d^3x\sqrt{h}$ is infinite. Then, for example, the next to last term in Eq. (5.5) gives rise to a term of the form

$$\begin{aligned}&\int d^3x\sqrt{h}\,a^{-2}h^{ij}\,{}^*\Gamma^l_{ij}(\partial_l\phi)\phi \\ &\quad = -a^{-2}\int d^3x\,\partial_k(\sqrt{h}\,h^{lk})(\partial_l\phi)\phi \\ &\quad = -a^{-2}\int d^3x\sqrt{h}\,\phi\Delta^{(3)}\phi + a^{-2}\int d^3x\sqrt{h}\,h^{lk}(\partial_l\partial_k\phi)\phi\,,\end{aligned} \tag{5.8}$$

where we have used the identity

$$h^{ij}\,{}^*\Gamma^l_{ij} = -\frac{1}{\sqrt{h}}\partial_k(\sqrt{h}\,h^{lk})\,. \tag{5.9}$$

Treating the other terms in analogous fashion and making use of the spatial homogeneity to cancel the spatial integration, one finally obtains for a state of the proper symmetry

$$\begin{aligned}\langle\Lambda_{00}\rangle = \Big\langle & \tfrac{1}{2}[(\partial_0\phi)^2 - a^{-2}\phi\Delta^{(3)}\phi + m^2\phi^2] \\ & + \frac{\dot a}{a}\phi\partial_0\phi - \tfrac{1}{6}\phi^2 G_{00}\Big\rangle\,.\end{aligned} \tag{5.10}$$

When we let

$$\phi = a^{-1}\chi$$

and

$$\frac{d}{dt} = a^{-1}\frac{d}{d\eta}\,, \tag{5.11}$$

as in Eqs. (2.14) and (2.15), and use

$$G_{00} = -3a^{-2}\dot a^2 - 3\epsilon a^{-2}\,, \tag{5.12}$$

the expression for the energy-momentum tensor simplifies to

$$\langle\Lambda_{00}\rangle = \langle\tfrac{1}{2}a^{-4}[\chi'^2 - \chi\Delta^{(3)}\chi + (a^2m^2+\epsilon)\chi^2]\rangle\,, \tag{5.13}$$

where a prime denotes differentiation with respect to η.

We decompose the field into modes as in Ref. 4:

$$\chi = \int d\tilde{\mu}(k)[A_{\underline{k}} \mathcal{Y}_{\underline{k}}(\underline{x})\chi_k(t) + \text{H.c.}] , \tag{5.14}$$

where H.c. denotes the Hermitian conjugate, and

$$\Delta^{(3)}\mathcal{Y}_{\underline{k}}(\underline{x}) = -k^2\mathcal{Y}_{\underline{k}}(\underline{x}) , \tag{5.15}$$

with

$$k = (q^2 - \epsilon)^{1/2} \quad (q = 1, 2, \ldots \text{ if } \epsilon = 1 , \quad 0 < q < \infty \text{ if } \epsilon = 0 \text{ or } -1) , \tag{5.16}$$

and

$$\int d\tilde{\mu}(\underline{k}) = \begin{cases} \int d^3k & \text{if } \epsilon = 0 , \\ \sum\limits_{l,m,n} \text{ or } \sum\limits_{l,J,M} & \text{if } \epsilon = 1 , \\ \int_0^\infty dq \sum\limits_{J,M} & \text{if } \epsilon = -1 . \end{cases} \tag{5.17}$$

The properties of the functions $\mathcal{Y}_{\underline{k}}$ are described in Ref. 4, Appendix A. As a consequence of the canonical commutation relations and the Wronskian condition discussed below, the operators $A_{\underline{k}}$ obey the relations

$$\begin{aligned} &[A_{\underline{k}}, A_{\underline{k}'}] = [A^\dagger_{\underline{k}}, A^\dagger_{\underline{k}'}] = 0 , \\ &[A_{\underline{k}}, A^\dagger_{\underline{k}'}] = \delta(\underline{k}, \underline{k}') , \end{aligned} \tag{5.18}$$

where

$$\int d\tilde{\mu}(\underline{k}) f(\underline{k}) \delta(\underline{k}, \underline{k}') = f(\underline{k}') .$$

The equation of motion (5.3) now separates, and one finds after some calculation that $\chi_k(t)$ defined through Eq. (5.14) satisfy the very simple equation

$$\chi_k'' + \Omega_q{}^2 \chi_k = 0 , \tag{5.19}$$

where k and q are related through Eq. (5.16), and

$$\Omega_q = (q^2 + m^2a^2)^{1/2} . \tag{5.20}$$

(Note that Ω_q is real for all k even when m vanishes.[23]) The Wronskian condition has the same form as Eq. (2.25). As discussed in Sec. II B and described more fully in Ref. 4, the functions χ_k will be chosen such that the operators $A_{\underline{k}}$ correspond to physical particles at least to fourth order in an adiabatic parameter.

In this section we will confine our considerations to the expectation value of $\Lambda_\mu{}^\nu$ in the adiabatic vacuum state $|0_A\rangle$ annihilated by the $A_{\underline{k}}$. The various expectation values appearing in Eq. (5.13) are readily calculated with the aid of the identity

$$\int d\tilde{\mu}(\underline{k}) |\mathcal{Y}_{\underline{k}}(\underline{x})|^2 f(k) = (2\pi^2)^{-1} \int d\mu(k) f(k) , \tag{5.21}$$

where $f(k)$ is a function of k alone, and

$$\int d\mu(k) = \begin{cases} \int_0^\infty dq\, q^2 & \text{if } \epsilon = 0 \text{ or } -1 , \\ \sum\limits_{q=1}^\infty q^2 & \text{if } \epsilon = 1 . \end{cases} \tag{5.22}$$

One finds that

$$\langle \Lambda_{00} \rangle = (4\pi^2 a^4)^{-1} \int d\mu(k) (|\chi_k'|^2 + \Omega_q{}^2 |\chi_k|^2) . \tag{5.23}$$

Note that Eq. (5.19) is identical to Eq. (2.22) with $a_1 = a_2 = a_3 = a$ (so $Q = 0$), and k replaced by q in Ω. Therefore, as in Eqs. (2.35)–(2.38) we require that χ_k agree with the extended WKB approximation at least to order T^{-4} (see Sec. II B). Thus,

$$\chi_k \approx (2aW_q)^{-1/2} \exp\left(-i \int^\eta aW_q \, d\eta' \right) , \tag{5.24}$$

where

$$W_q = \omega_q (1 + \epsilon_2)^{1/2} (1 + \epsilon_4)^{1/2} , \tag{5.25}$$

with

$$\omega_q = a^{-1}\Omega_q , \tag{5.26}$$

$$\epsilon_2 = -\Omega_q{}^{-3/2} \frac{d}{d\eta} \left[\Omega_q{}^{-1} \frac{d}{d\eta} (\Omega_q{}^{1/2}) \right] , \tag{5.27}$$

and

$$\epsilon_4 = -\Omega_q{}^{-1}(1 + \epsilon_2)^{-3/4} \times \frac{d}{d\eta} \left\{ \Omega_q{}^{-1} (1 + \epsilon_2)^{-1/2} \frac{d}{d\eta} [(1 + \epsilon_2)^{-1/4}] \right\} . \tag{5.28}$$

Because $\Omega_q{}^2 = q^2 + m^2a^2$, the differentiations reduce the power of q appearing in Eqs. (5.27) and (5.28), in contrast with the anisotropic expansion considered earlier and the minimally coupled field considered in Ref. 4 and Ref. 5. This will lead to convergence of all but the term of order T^0 in $\langle \Lambda_\mu{}^\nu \rangle$.

In the case when $m = 0$, Eq. (5.19) reduces to the equation of a simple harmonic oscillator of constant frequency q, so that χ_k in Eq. (5.24) can be chosen as the exact positive-frequency solution, with $\omega_q = q/a$ and the frequency corrections (ϵ_2, ϵ_4) identically zero. Therefore, for $m = 0$ one finds

$$\langle \Lambda_{00} \rangle = (4\pi^2 a^2)^{-1} \int d\mu(k) \omega_q , \tag{5.29}$$

so that upon transformation to the momentum variable $p = q/a$ it is evident that the entire vacuum energy density corresponds to a renormalization of the cosmological constant Λ, exactly as in the flat spacetime. There are no further divergent terms, and even the finite terms of higher order in the adiabatic series for $\langle \Lambda_{00}\rangle$ vanish; that is, there is no vacuum polarization. By choosing χ_k to be the solution of precisely positive frequency, rather than one which merely becomes such in the adiabatic limit, we have in fact assured that the renormalized $\langle \Lambda_{00}\rangle$ [obtained by subtracting the right-hand side of Eq. (5.29)] is identically zero. Since $\langle \Lambda_\mu{}^\mu\rangle = 0$ when $m=0$ [Eq. (5.31) below], it is clear that no anomalous terms appear in the pressure. [As explained following Eq. (5.10) of Ref. 5, one should not expect $\langle \Lambda_\mu{}^\nu\rangle$ to be proportional to $\delta_\mu{}^\nu$ here.]

When the mass does not vanish, there continues to be no infinite renormalization of constants other than Λ. Substituting Eqs. (5.24)–(5.26) into Eq. (5.23), one obtains the analog of Eq. (2.45):

$$\langle \Lambda_{00}\rangle = (8\pi^2 a^4)^{-1}\int d\mu(k)\Omega_q{}^{-1}\left[2\Omega_q{}^2 + \tfrac{1}{4}\left(\frac{\Omega_q{}'}{\Omega_q}\right)^2 - \tfrac{1}{8}\left(\frac{\Omega_q{}'}{\Omega_q}\right)^2\epsilon_2 + \tfrac{1}{4}\frac{\Omega_q{}'}{\Omega_q}\epsilon_2{}' + \tfrac{1}{4}\Omega_q{}^2\epsilon_2{}^2\right] + O(T^{-6}) \tag{5.30}$$

However, in the present case only the first term diverges, corresponding to the vacuum energy of Eq. (5.29), while the remaining terms are finite. This follows because $(\Omega_q{}'/\Omega_q)^2$ goes as q^{-4} and ϵ_2 goes as q^{-4} for large q.

Since $\langle \Lambda_1{}^1\rangle = \langle \Lambda_2{}^2\rangle = \langle \Lambda_3{}^3\rangle$, we can obtain the pressure by evaluating the trace $\langle \Lambda_\mu{}^\mu\rangle$ and combining the result with Eq. (5.30). The trace of the conformal energy-momentum tensor has the particularly simple form (Ref. 6)

$$\Lambda_\mu{}^\mu = m^2\phi^2 . \tag{5.31}$$

Using Eqs. (5.11), (5.14), and (5.21) we find that

$$\langle \Lambda_\mu{}^\mu\rangle = (2\pi^2 a^2)^{-1} m^2 \int d\mu(k)|\chi_k|^2 . \tag{5.32}$$

Then Eqs. (5.24)–(5.26) yield

$$\langle \Lambda_\mu{}^\mu\rangle = m^2(4\pi^2 a^2)^{-1} \times \int d\mu(k)\Omega_q{}^{-1}(1 - \tfrac{1}{2}\epsilon_2 + \tfrac{3}{8}\epsilon_2{}^2 - \tfrac{1}{2}\epsilon_4) + O(T^{-6}). \tag{5.33}$$

The term of order T^0 in Eq. (5.33) diverges as ∞^2 and is dropped as part of the ever present vacuum energy along with the other divergent (∞^4) term of order T^0 in Eq. (5.30). These divergent terms evidently can be interpreted as a renormalization of the cosmological constant, but we will not discuss them further here. The remaining terms in Eq. (5.33) are finite, so that there is no further infinite renormalization.

In closing, let us consider the terms of order T^{-2} in the spatially flat case ($\epsilon = 0$). One finds by direct calculation that

$$\begin{aligned}\langle \Lambda_0{}^0\rangle_{(2)} &= (32\pi^2 a^4)^{-1}\int_0^\infty dk\, k^2\Omega_k{}^{-1}\left(\frac{\Omega_k{}'}{\Omega_k}\right)^2 \\ &= (32\pi^2)^{-1}a^{-4}(a')^2 m^4 \int_0^\infty dp\, p^2\omega_k{}^{-5} \\ &= -\frac{m^2}{288\pi^2}G_0{}^0 ,\end{aligned} \tag{5.34}$$

where we have used $p = k/a$, $G_0{}^0 = -3a^{-4}(a')^2$, and $\int_0^\infty dp\, p^2\omega_k{}^{-5} = (3m^2)^{-1}$. This result for $\langle \Lambda_0{}^0\rangle_{(2)}$ agrees with that in the generalized Kasner universe, Eq. (3.3), when $a_1 = a_2 = a_3$. Since $G_0{}^0$ is negative definite, it follows that $\langle \Lambda_0{}^0\rangle_{(2)}$ is a positive definite vacuum energy density.

Similarly, one has for the T^{-2} contribution to $\langle \Lambda_\mu{}^\mu\rangle$ in the spatially flat case

$$\begin{aligned}\langle \Lambda_\mu{}^\mu\rangle_{(2)} &= -m^2(8\pi^2 a^2)^{-1}\int_0^\infty dk\, k^2\Omega_k{}^{-1}\epsilon_2 \\ &= m^2(16\pi^2 a^2)^{-1}\int_0^\infty dk\, k^2\Omega_k{}^{-1} \times\left[\tfrac{1}{2}\left(\frac{\Omega_k{}'}{\Omega_k}\right)^2 + \Omega_k{}^{-1}\left(\frac{\Omega_k{}'}{\Omega_k}\right)'\right] \\ &= m^4(16\pi^2)\int_0^\infty dp\, p^2\,\omega_k{}^{-7} \times\left[(4p^2 - m^2)\frac{\dot a^2}{a^2} + 2\omega_k{}^2\frac{\ddot a}{a}\right] \\ &= -\frac{m^2}{288\pi^2}G_\mu{}^\mu ,\end{aligned} \tag{5.35}$$

where we have used $G_\mu{}^\mu = -R = -6(\dot a^2/a^2 + \ddot a/a) = -6a^{-3}a''$, and

$$\int_0^\infty dp\, p^2\,\omega^{-7}(4p^2 - m^2) = 2\int_0^\infty dp\, p^2\omega^{-5} = 2(3m^2)^{-1} .$$

It follows from Eq. (5.35), Eq. (5.34), and isotropy that in the spatially flat Robertson-Walker universe

$$\langle \Lambda_\mu{}^\nu\rangle_{(2)} = -\frac{m^2}{288\pi^2}G_\mu{}^\nu . \tag{5.36}$$

This term thus constitutes a finite renormalization of the Newtonian constant G. That is, the Einstein equation

$$\begin{aligned}G_\mu{}^\nu + \Lambda g_\mu{}^\nu &= -8\pi G\langle \Lambda_\mu{}^\nu\rangle \\ &= -8\pi G(\lambda_1 g_\mu{}^\nu - \lambda_2 G_\mu{}^\nu + \rho_{\text{ren}})\end{aligned} \tag{5.37}$$

can be written as

$$G_\mu{}^\nu + \Lambda_{\text{ren}}\, g_\mu{}^\nu = -8\pi G_{\text{ren}}\,\rho_{\text{ren}} , \tag{5.38}$$

where

$$\Lambda_{\text{ren}} = (\Lambda + 8\pi G\lambda_1)(1 - 8\pi G\lambda_2)^{-1},$$
$$G_{\text{ren}} = G(1 - 8\pi G\lambda_2)^{-1} \qquad (5.39)$$

are renormalized physical constants. G_{ren} is presumably the gravitational constant measured in any classical gravitational experiment, where the energy density of empty space is taken to be zero.

Note added in proof. We have received a report by Ya. B. Zel'dovich, V. N. Lukash, and A. A. Starobinsky in which it is shown, in the context of certain approximations, that quantum field theory predicts the creation of matter near an anisotropic cosmological singularity in sufficient quantity to bring about the observed isotropy by a classical mechanism. This result increases the interest in a more precise treatment of the problem, including an improved understanding of the initial conditions to be imposed on the quantum state.

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APPENDIX A: GEOMETRIC IDENTITIES FOR A KASNER UNIVERSE

We consider a manifold with the metric[24]

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i(t)^2 (dx^i)^2 . \qquad (A1)$$

The nonvanishing Christoffel symbols and Riemann-tensor components are

$$\Gamma^0_{ii} = \dot{a}_i a_i , \quad \Gamma^i_{i0} = \dot{a}_i / a_i , \qquad (A2)$$

$$R_{0i0i} = -\ddot{a}_i a_i ,$$
$$R_{ikik} = +\dot{a}_i \dot{a}_k a_i a_k \quad (i \neq k), \qquad (A3)$$

and those which follow from these by symmetry. Hence the Ricci tensor and curvature scalar are given by

$$R_0{}^0 = \sum \ddot{a}_i / a_i , \qquad (A4)$$

$$R_i{}^j = \delta_i{}^j \left[\ddot{a}_i / a_i + (\dot{a}_i / a_i) \sum_{k \neq i} \dot{a}_k / a_k \right], \qquad (A5)$$

$$R = 2\sum \ddot{a}_i / a_i + 2 \sum_{i<j} \dot{a}_i \dot{a}_j / a_i a_j . \qquad (A6)$$

The condition of the vanishing of the covariant divergence of a tensor,

$$\nabla_\nu \Theta_\mu{}^\nu = 0, \qquad (A7a)$$

reduces to

$$\partial_0 (V\Theta_0{}^0) = V \sum \Theta_i{}^i \dot{a}_i / a_i \qquad (A7b)$$

if $\Theta_\mu{}^\nu$ is independent of the space coordinates and its off-diagonal components vanish identically. When applied to the energy-momentum tensor, this equation generalizes the familiar relation between energy density and pressure in an expanding universe.

Define the "volume of the universe"

$$V = (-g)^{1/2} = a_1 a_2 a_3 , \qquad (A8)$$

and introduce η by

$$d\eta = V^{-1/3} dt , \qquad (A9)$$

so that

$$\partial_0 = \frac{d}{dt} = V^{-1/3} \frac{d}{d\eta} = V^{-1/3} \partial_\eta . \qquad (A10)$$

Denote ∂_η by a prime. Let

$$d_i = a_i{}' / a_i , \qquad (A11)$$

$$D = \tfrac{1}{3} \sum d_i = \tfrac{1}{3} V'/V , \qquad (A12)$$

$$Q = \tfrac{1}{18} \sum_{i<j} (d_i - d_j)^2 . \qquad (A13)$$

Then one has

$$\ddot{a}_i / a_i = V^{-2/3} (d_i{}' + d_i{}^2 - d_i D), \qquad (A14)$$

$$R_0{}^0 = 3V^{-2/3} (D' + 2Q), \qquad (A15)$$

$$R_i{}^j = \delta_i{}^j V^{-2/3} (d_i{}' + 2 d_i D), \qquad (A16)$$

$$R = 6V^{-2/3} (D' + D^2 + Q)$$
$$= -2G_0{}^0 + 6V^{-2/3} (D' + 2Q), \qquad (A17)$$

and the Einstein tensor, $G_\mu{}^\nu = R_\mu{}^\nu - \frac{1}{2} \delta_\mu{}^\nu R$, is

$$G_0{}^0 = -V^{-2/3} \sum_{i<j} d_i d_j = 3V^{-2/3} (-D^2 + Q), \qquad (A18)$$

$$G_i{}^j = \delta_i{}^j V^{-2/3} [d_i{}' + 2d_i D - 3D' - 3D^2 - 3Q]$$
$$= \delta_i{}^j V^{-2/3} [d_i{}' + 2d_i D - 3D' - 6Q + V^{2/3} G_0{}^0]. \qquad (A19)$$

It is convenient to use as a basis (in the sense of linear algebra) for the symmetric polynomials in the d_i and their derivatives a set of quantities which behave homogeneously as the anisotropy of the metric varies. [That is, we seek to generalize the splitting of $G_0{}^0$ in Eq. (A18) into a term, $3V^{-2/3} Q$, which vanishes in the isotropic limit, and a term, $-3V^{-2/3} D^2$, which depends (see Eq. (A12)) on the overall volume of the universe.] Let us define

$$S = \tfrac{1}{18} [(d_1 - D)(d_2 - d_3)^2 + (d_2 - D)(d_3 - d_1)^2$$
$$+ (d_3 - D)(d_1 - d_2)^2], \qquad (A20)$$

$$U = \tfrac{1}{18} \sum_{i<j} (d_i{}' - d_j{}')^2 . \qquad (A21)$$

If all the quantities $d_i - D$ and their derivatives are regarded as of first order in the anisotropy, and D of order 0, then Q, S, and U are of respective orders 2, 3, and 2. (This anisotropy order is not to be confused with the order of a quantity in the adiabatic parameter T^{-1}, which is 1, 2, 3, and 4, respectively, for D, Q, S, and U.) One then has the useful formulas

$$\tfrac{1}{3}\sum d_i{}^2 = D^2 + 2Q = 3D^2 + \tfrac{2}{3}V^{2/3}G_0{}^0 = 3Q - \tfrac{1}{3}V^{2/3}G_0{}^0, \qquad (A22a)$$

$$\tfrac{1}{3}\sum_{i<j} d_i d_j = D^2 - Q = -\tfrac{1}{3}V^{2/3}G_0{}^0, \qquad (A22b)$$

$$\tfrac{1}{3}\sum d_i{}' d_i = D'D + Q', \qquad (A23a)$$

$$\tfrac{1}{6}\sum_{i\neq j} d_i{}' d_j = D'D - \tfrac{1}{2}Q', \qquad (A23b)$$

$$\tfrac{1}{3}\sum d_i{}^3 = D^3 + 6QD - 2S, \qquad (A24a)$$

$$\tfrac{1}{6}\sum_{i\neq j} d_i{}^2 d_j = D^3 + S, \qquad (A24b)$$

$$d_1 d_2 d_3 = D^3 - 3QD - 2S, \qquad (A24c)$$

$$\tfrac{1}{3}\sum d_i{}'' d_i = D''D + Q'' - 2U, \qquad (A25a)$$

$$\tfrac{1}{6}\sum_{i\neq j} d_i{}'' d_j = D''D - \tfrac{1}{2}Q'' + U, \qquad (A25b)$$

$$\tfrac{1}{3}\sum (d_i{}')^2 = (D')^2 + 2U, \qquad (A25c)$$

$$\tfrac{1}{3}\sum_{i<j} d_i{}' d_j{}' = (D')^2 - U, \qquad (A25d)$$

$$\tfrac{1}{3}\sum d_i{}' d_i{}^2 = D'D^2 + 2Q'D + 2QD' - \tfrac{2}{3}S', \qquad (A26a)$$

$$\tfrac{1}{6}\sum_{i\neq j} d_i{}' d_i d_j = D'D^2 + \tfrac{1}{2}Q'D - QD' + \tfrac{1}{3}S', \qquad (A26b)$$

$$\tfrac{1}{6}\sum_{i\neq j} d_i{}' d_j{}^2 = D'D^2 - Q'D + 2QD' + \tfrac{1}{3}S', \qquad (A26c)$$

$$\tfrac{1}{3}\sum_{\substack{j<k\\ j\neq i\neq k}} d_i{}' d_j d_k = D'D^2 - Q'D - QD' - \tfrac{2}{3}S', \qquad (A26d)$$

$$\tfrac{1}{3}\sum d_i{}^4 = D^4 + 12QD^2 - 8SD + 6Q^2, \qquad (A27a)$$

$$\tfrac{1}{6}\sum_{i\neq j} d_i{}^3 d_j = D^4 + 3QD^2 + SD - 3Q^2, \qquad (A27b)$$

$$\tfrac{1}{3}\sum_{i<j} d_i{}^2 d_j{}^2 = D^4 + 4SD + 3Q^2, \qquad (A27c)$$

$$\tfrac{1}{3}\sum_{\substack{j<k\\ j\neq i\neq k}} d_i{}^2 d_j d_k = D^4 - 3QD^2 - 2SD. \qquad (A27d)$$

In the renormalization theory (see Sec. III) an important role is played by the divergenceless tensors

$$^{(1)}H_\mu{}^\nu = 2R(R_\mu{}^\nu - \tfrac{1}{4}R\delta_\mu{}^\nu) + 2(\nabla^\nu\nabla_\mu R - \nabla^\alpha\nabla_\alpha R\delta_\mu{}^\nu) \qquad (A28)$$

and

$$^{(2)}H_\mu{}^\nu = \nabla^\nu\nabla_\mu R - \nabla^\alpha\nabla_\alpha R_\mu{}^\nu - \tfrac{1}{2}(\nabla^\alpha\nabla_\alpha R + R_\alpha{}^\beta R^\alpha{}_\beta)\delta_\mu{}^\nu + 2R_\alpha{}^\beta R^\alpha{}_{\mu\beta}{}^\nu. \qquad (A29)$$

We define

$$^{(-)}H_\mu{}^\nu = {}^{(2)}H_\mu{}^\nu - \tfrac{1}{3}\,{}^{(1)}H_\mu{}^\nu. \qquad (A30)$$

A lengthy calculation, including use of Eqs. (A22)–(A27), yields for the case of the metric (A1) the expressions

$$^{(1)}H_0{}^0 = 18V^{-4/3}\{[-2D''D + (D')^2 + 3D^4] + 2[-Q'D + 2QD' + 3QD^2] + 3Q^2\}, \qquad (A31)$$

$$^{(1)}H_i{}^j = 6\delta_i{}^j V^{-4/3}\{[-2D^{(3)} + 2D''D - (D')^2 + 12D'D^2 - 3D^4] + 2[(D' + D^2)(d_i - D)' + (D'' + 2D'D)(d_i - D)] + 2[-Q'' + 3Q'D - 3QD^2] + 2[Q'(d_i - D) + Q(d_i - D)'] - 3Q^2\}, \qquad (A32)$$

$$^{(-)}H_0{}^0 = 3V^{-4/3}\{[3U - Q''] + 12Q^2\}, \tag{A33}$$

$$^{(-)}H_i{}^j = \delta_i{}^j V^{-4/3}\{-(d_i - D)^{(3)} - [3U - Q''] + 8[Q'(d_i - D) + Q(d_i - D)'] - 12Q^2\}. \tag{A34}$$

The terms have been grouped in ascending order in the anisotropy. The most laborious part of this calculation is evaluating $\nabla^\alpha\nabla_\alpha R_\mu{}^\nu$; we obtain

$$\nabla^\alpha\nabla_\alpha R_0{}^0 = 3V^{-4/3}[D^{(3)} - 2D''D - 2(D')^2 - 4D'D^2 + 4D^4 + 2Q'' - 12QD' + 12QD^2 - \tfrac{4}{3}S' - 8SD - 24Q^2], \tag{A35a}$$

$$\nabla^\alpha\nabla_\alpha R_i{}^j = \delta_i{}^j V^{-4/3}[d_i^{(3)} + 2d_i{}'D' + 2d_i D'' - 2d_i{}'d_i{}^2 - 4d_i{}'D^2 + 6d_i{}^2D' - 8d_i D'D - 4d_i{}^3 D + 12d_i{}^2 Q]. \tag{A35b}$$

APPENDIX B: CALCULATION OF THE FOURTH-ORDER TERM IN THE ENERGY DENSITY

We are concerned here with the last four terms in the integrand of Eq. (2.45). From Eqs. (2.37) and (2.40) we have

$$\epsilon_{2(2)} = -\tfrac{1}{4}X^{-2}X'' + \tfrac{5}{16}X^{-3}(X')^2 + X^{-1}Q, \tag{B1a}$$

$$\epsilon'_{2(3)} \equiv (\epsilon_{2(2)})' = -\tfrac{1}{4}X^{-2}X^{(3)} + \tfrac{9}{8}X^{-3}X''X' - \tfrac{15}{16}X^{-4}(X')^3 + X^{-1}Q' - X^{-2}X'Q, \tag{B1b}$$

in terms of

$$X = \Omega^2 = V^{2/3}\omega^2. \tag{B2}$$

Hence we obtain

$$\rho_{0(4)} = (1024\pi^3)^{-1}V^{-2/3}\int d^3p\, \omega^{-1}[-X^{-3}X^{(3)}X' + \tfrac{1}{2}X^{-3}(X'')^2 + \tfrac{7}{2}X^{-4}X''(X')^2 - \tfrac{105}{32}X^{-5}(X')^4 + 4X^{-2}X'Q' - 8X^{-2}X''Q + 5X^{-3}(X')^2Q + 24X^{-1}Q^2]. \tag{B3}$$

Into this expression we substitute

$$X' = 2V^{2/3}[D\omega^2 + \tfrac{1}{2}\partial_\eta\omega^2], \tag{B4a}$$

$$X'' = 2V^{2/3}[(D' + 2D^2)\omega^2 + 2D\partial_\eta\omega^2 + \tfrac{1}{2}\partial_\eta{}^2\omega^2], \tag{B4b}$$

$$X^{(3)} = 2V^{2/3}[(D'' + 6D'D + 4D^3)\omega^2 + 3(D' + 2D^2)\partial_\eta\omega^2 + 3D\partial_\eta{}^2\omega^2 + \tfrac{1}{2}\partial_\eta{}^3\omega^2], \tag{B4c}$$

in which

$$\partial_\eta\omega^2 = -2\sum p_i{}^2 d_i, \tag{B5a}$$

$$\partial_\eta{}^2\omega^2 = -2\sum p_i{}^2(d_i{}' - 2d_i{}^2), \tag{B5b}$$

$$\partial_\eta{}^3\omega^2 = -2\sum p_i{}^2(d_i{}'' - 6d_i{}'d_i + 4d_i{}^3). \tag{B5c}$$

[At this stage ω and p_i are still functions of η through Eqs. (2.23) and (3.1).] The result is

$$\rho_{0(4)}=(1024\pi^3)^{-1}V^{-4/3}\int d^3p\,\{\omega^{-3}[-4D''D+2(D')^2+12D'D^2-\tfrac{9}{2}D^4+8Q'D-16QD'-12QD^2+24Q^2]$$
$$+\omega^{-5}(-2D''+12D'D-9D^3+4Q'-12QD)(\partial_\eta\omega^2)$$
$$+\omega^{-7}(D'-\tfrac{51}{4}D^2+5Q)(\partial_\eta\omega^2)^2-\tfrac{49}{4}\omega^{-9}D(\partial_\eta\omega^2)^3$$
$$-\tfrac{105}{32}\omega^{-11}(\partial_\eta\omega^2)^4+\omega^{-5}(2D'+6D^2-8Q)(\partial_\eta{}^2\omega^2)$$
$$+12\omega^{-7}D(\partial_\eta{}^2\omega^2)(\partial_\eta\omega^2)+\tfrac{7}{2}\omega^{-9}(\partial_\eta{}^2\omega^2)(\partial_\eta\omega^2)^2$$
$$+\tfrac{1}{2}\omega^{-7}(\partial_\eta{}^2\omega^2)^2-2\omega^{-5}D(\partial_\eta{}^3\omega^2)-\omega^{-7}(\partial_\eta{}^3\omega^2)(\partial_\eta\omega^2)\}\,. \tag{B6}$$

Let $p_i=p\lambda_i$, where

$$\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2=1\,. \tag{B7}$$

In conventional spherical coordinates, λ_1 is $\sin\theta\cos\phi$, etc. Symmetry considerations allow the angular integrations in Eq. (B6) to be carried out easily. The integrals $\int d\Omega\,\lambda_1{}^{2j}\lambda_2{}^{2k}\lambda_3{}^{2l}$ (where $d\Omega\equiv\sin\theta d\theta d\phi$) are invariant under permutations of $\{j,k,l\}$. One can evaluate

$$\int d\Omega\,\lambda_3{}^{2j}=\frac{4\pi}{2j+1} \tag{B8}$$

directly, and repeated use of Eq. (B7) then yields relations which can be solved for all the other integrals of interest:

$$\int d\Omega\,\lambda_1{}^2\lambda_2{}^2=\frac{4\pi}{5\times3}\,, \tag{B9a}$$

$$\int d\Omega\,\lambda_1{}^4\lambda_2{}^2=\frac{4\pi}{7\times5}\,,\quad \int d\Omega\,\lambda_1{}^2\lambda_2{}^2\lambda_3{}^2=\frac{4\pi}{7\times5\times3}\,, \tag{B9b}$$

$$\int d\Omega\,\lambda_1{}^6\lambda_2{}^2=\frac{4\pi}{9\times7}\,,\quad \int d\Omega\,\lambda_1{}^4\lambda_2{}^4=\frac{4\pi}{7\times5\times3}\,,\quad \int d\Omega\,\lambda_1{}^4\lambda_2{}^2\lambda_3{}^2=\frac{4\pi}{9\times7\times5}\,. \tag{B9c}$$

It follows, with the aid of Eqs. (A22)–(A27) and (A12), that

$$\int d\Omega\,\partial_\eta\omega^2=-8\pi p^2D\,, \tag{B10a}$$

$$\int d\Omega\,(\partial_\eta\omega^2)^2=16\pi p^4(D^2+\tfrac{4}{5}Q)\,, \tag{B10b}$$

$$\int d\Omega\,(\partial_\eta\omega^2)^3=32\pi p^6(-D^3-\tfrac{12}{5}QD+\tfrac{16}{35}S)\,, \tag{B10c}$$

$$\int d\Omega\,(\partial_\eta\omega^2)^4=64\pi p^8(D^4+\tfrac{24}{5}QD^2-\tfrac{64}{35}SD+\tfrac{48}{35}Q^2)\,, \tag{B10d}$$

$$\int d\Omega\,\partial_\eta{}^2\omega^2=8\pi p^2(-D'+2D^2+4Q) \tag{B11a}$$

$$\int d\Omega\,(\partial_\eta{}^2\omega^2)(\partial_\eta\omega^2)=16\pi p^4(D'D-2D^3+\tfrac{2}{5}Q'-\tfrac{36}{5}QD+\tfrac{8}{5}S)\,, \tag{B11b}$$

$$\int d\Omega\,(\partial_\eta{}^2\omega^2)(\partial_\eta\omega^2)^2=32\pi p^6(-D'D^2+2D^4-\tfrac{4}{5}Q'D-\tfrac{4}{5}QD'+12QD^2+\tfrac{16}{105}S'-\tfrac{176}{35}SD+\tfrac{144}{35}Q^2)\,. \tag{B11c}$$

$$\int d\Omega\,(\partial_\eta{}^2\omega^2)^2=16\pi p^4[(D')^2-4D'D^2+4D^4-\tfrac{16}{5}Q'D-8QD'+\tfrac{144}{5}QD^2+\tfrac{4}{5}U+\tfrac{16}{15}S'-\tfrac{64}{5}SD+\tfrac{96}{5}Q^2]\,, \tag{B12}$$

$$\int d\Omega\,\partial_\eta{}^3\omega^2=8\pi p^2(-D''+6D'D-4D^3+6Q'-24QD+8S)\,, \tag{B13a}$$

$$\int d\Omega\,(\partial_\eta{}^3\omega^2)(\partial_\eta\omega^2)=16\pi p^4(D''D-6D'D^2+4D^4+\tfrac{2}{5}Q''-\tfrac{42}{5}Q'D-\tfrac{24}{5}QD'+\tfrac{168}{5}QD^2-\tfrac{4}{5}U+\tfrac{8}{5}S'-\tfrac{88}{5}SD+\tfrac{48}{5}Q^2)\,. \tag{B13b}$$

Substitution of these expressions into Eq. (B6) yields

$$\rho_{0(4)} = (256\pi^2)^{-1} V^{-4/3} \int_0^\infty dp \{ p^2\omega^{-3}[-4D''D + 2(D')^2 + 12D'D^2 - \tfrac{9}{2}D^4 + 8Q'D - 16QD' - 12QD^2 + 24Q^2]$$
$$+ p^4\omega^{-5}[8D''D - 4(D')^2 - 52D'D^2 + 58D^4 - 32Q'D + 32QD' + 136QD^2 - 32SD - 64Q^2]$$
$$+ p^6\omega^{-7}[-4D''D + 2(D')^2 + 68D'D^2 - 155D^4$$
$$+ \tfrac{1}{5}(-8Q'' + 232Q'D + 32QD' - 2216QD^2 + 24U - \tfrac{64}{3}S' + 608SD) + 16Q^2]$$
$$+ p^8\omega^{-9}[-28D'D^2 + 154D^4 + \tfrac{1}{5}(-112Q'D - 112QD' + 2856QD^2 + \tfrac{64}{3}S'$$
$$- 928SD + 576Q^2)]$$
$$+ p^{10}\omega^{-11}(-\tfrac{105}{2}D^4 - 252QD^2 + 96SD - 72Q^2)\} . \quad \text{(B14)}$$

In studying the ultraviolet divergence it suffices to consider the asymptotic behavior of the integrand of Eq. (B14) at large p; one obtains the expression (3.4), proportional to ${}^{(-)}H_0{}^0$.

APPENDIX C: COMPARISON WITH THE METHOD OF ZEL'DOVICH AND STAROBINSKY

The conformally coupled scalar field in a Kasner universe has been studied by Zel'dovich and Starobinsky (Ref. 10). In Ref. 4 it was shown that their results must be essentially identical to those of the method of adiabatic regularization. We believe that the present treatment has advantages both in conceptual clarity (see Ref. 4, Sec. V) and in the ease with which explicit expressions [Eqs. (2.45) and (2.46)] for the vacuum subtractions are obtained.

Although both Ref. 10 and the present paper prescribe regularization by subtraction of the first three terms of an asymptotic series, comparison of the ensuing explicit calculations is complicated by the use of different techniques for performing the expansion. Zel'dovich and Starobinsky define quantities $s_k^\pm$, $\tau_k^\pm$, and $u_k^\pm$ from $\chi_k^\pm$ and $\chi_k^{\pm\prime}$ and expand them in power series in T^{-1}. The vacuum subtractions $\rho_{0\text{div}}$ and $(P_i)_{0\text{div}}$ are expressed [see Ref. 10, Eqs. (22)] in terms of the coefficients of these series. This method does not explicitly identify the modified effective frequency $W_k^\pm$, and hence the proper concept of "positive-frequency solutions" $\chi_k^\pm$ [cf. Eq. (2.35)] to which the physical particle operators $A_k^\pm$ and the vacuum state are associated. However, the authors of Ref. 10 tacitly recover the effect of that consideration by assuming that the universe is static around some time t_0 and choosing the $A_k^\pm$ to be the particle operators at t_0, when W reduces to ω.

Their method yields for the coefficient of T^{-4} in s

$$s_{(4)}(t) = \tfrac{1}{2} \int_{\eta(t_0)}^{\eta(t)} d\eta' \Omega^{-1}(\Omega' u_{(4)} + Q\tau_{(3)}) .$$

However, it is clear from our method that $s_{(4)}$ ought to be a local function of the a_i and their derivatives. The integrand of $s_{(4)}$, therefore, must be an exact differential. Indeed, after much manipulation we find

$$s_{(4)} = \tfrac{1}{16}\left\{\frac{Q^2}{\Omega^4} + \frac{Q'}{\Omega^3}\frac{\Omega'}{\Omega^2} - \frac{Q}{\Omega^3}\left(\frac{\Omega'}{\Omega^2}\right)' - \frac{3Q}{\Omega^2}\left(\frac{\Omega'}{\Omega^2}\right)^2 - \frac{1}{2\Omega^2}\left(\frac{\Omega'}{\Omega^2}\right)''\frac{\Omega'}{\Omega^2} + \frac{1}{4\Omega^2}\left[\left(\frac{\Omega'}{\Omega^2}\right)'\right]^2 + \frac{1}{2\Omega}\left(\frac{\Omega'}{\Omega^2}\right)'\left(\frac{\Omega'}{\Omega^2}\right)^2 + \tfrac{3}{16}\left(\frac{\Omega'}{\Omega^2}\right)^4\right\} .$$

Similarly, their method gives $s_{(6)}$, $s_{(8)}, \ldots$ as integrals, while ours directly yields a local expression for each term in the asymptotic expansion of the integrands in Eqs. (2.26) and (2.27).

By expressing $\epsilon_{2(2)}$, etc., in terms of Ω'/Ω^2 and its derivatives, we have completed the verification that the vacuum energy density obtained by the method of Ref. 10 agrees with our Eq. (2.45). (But note that in Sec. III we ultimately decide on a different identification of the vacuum subtraction terms.)

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[1]Since the particle concept becomes imprecise in regions of strong spacetime curvature, such statements are physically intuitive descriptions of basically field-theoretic effects.

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[9]See Ref. 6. The absence of particle creation in conformally invariant theories of arbitrary spin was shown in L. Parker, Phys. Rev. 183, 1057 (1969), and L. Parker, Ph.D. thesis, Harvard University, 1966 (unpublished).

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[11]Since a rescaling of the spatial coordinates can make all the a_i equal at a particular time, the anisotropy resides in the differing rates of change of the a_i. Hence the scale-invariant quantities $(\partial_0 a_i)/a_i$ appear prominently in the following analysis.

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[13]I.e., the expectation value of the number of particles in a rectangular parallelepiped with sides a_1^{-1}, a_2^{-1}, and a_3^{-1} in $\vec{x}$ space, whose physical volume is unity.

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[15]I.e., Eq. (A7) is satisfied.

[16]The explicit forms (A28) and (A29) are given [with sign conventions opposite to our (2) and (3) (end of Sec. I)] by H. Nariai, Prog. Theor. Phys. 46, 433 (1971). Other references on renormalization in the gravitational field equations, including the role of these tensors, are R. Utiyama and B. S. DeWitt, J. Math. Phys. 3, 608 (1962); B. S. DeWitt, Phys. Rev. 162, 1239 (1967); A. D. Sakharov, Dokl. Akad. Nauk SSSR 177, 70 (1967) [Sov. Phys.—Dokl. 12, 1040 (1968)]; T. V. Ruzmaikina and A. A. Ruzmaikin, Zh. Eksp. Teor. Fiz. 57, 680 (1969) [Sov. Phys.—JETP 30, 372 (1970)]; V. L. Ginzburg, D. A. Kirzhnits, and A. A. Lyubushin, *ibid.* 60, 451 (1971) [33, 242 (1971)].

[17]A more rigorous treatment (cf. Ref. 5) would involve a cutoff function in the integrals, so as to make all quantities finite, until the final stage of the renormalization. The preferred variable of integration must appear as the argument of the cutoff function.

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[24]For notational conventions see the end of Sec. I.