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Comment on vector-meson dominance and the $\gamma \to 3\pi$ vertex in the low-energy limit

S. Rudaz*

Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794 and Physics Department, McGill University, Montréal, Québec, Canada

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It is found that a simple derivation of a relation between the $\gamma \to 3\pi$ and $\pi^0 \to 2\gamma$ vertices, previously obtained from current algebra and the partial-conservation-of-axial-vector-current anomaly, can be given on the basis of a pole model together with vector-meson dominance if we require consistency between a ρ-pole description of $\pi\pi$ scattering in the framework of a Yang-Mills theory and a theorem of Weinberg for the same process in the two-soft-pion limit.

The radiative decays of mesons have been the subject of much theoretical activity over the past decade. In fact, several models have been elaborated to explain the features of such decays. Foremost among these are pole models based on vector-meson dominance (VMD),[1] as applied first by Gell-Mann, Sharp, and Wagner,[2] and more recently by Kotlewski, Lee, Suzuki, and Thaler[3]; current algebra[4] and the hypothesis of partial conservation of the axial-vector current (PCAC),[5] and the possible existence of its anomaly[6] for $\pi^0 \to 2\gamma$; and several versions of the quark model.[7–9] All can be made to compare favorably with experiment.

In the present paper, we shall consider the process $\gamma \to \pi^+\pi^-\pi^0$ in the low-energy limit, making use of the well-known vector-meson-pole dominance ideas of Gell-Mann, Sharp, and Wagner.[2] The amplitude for this process is related to that for $\pi^0 \to 2\gamma$ in a very simple way in this framework. Further, using a result of Basdevant and Zinn-Justin, which is essentially a relation between various coupling constants required for consistency of a Yang-Mills theory of $\pi\pi$ scattering with current algebra, we obtain a simple relation between the amplitudes for $\gamma \to \pi^+\pi^-\pi^0$ and $\pi^0 \to 2\gamma$, and the charged-pion decay constant. This relation was first derived by Adler, Lee, Treiman, and Zee,[10] Terent'ev,[11] and others[12] as a low-energy theorem abstracted from the theory of PCAC anomalies. An excellent review of low-energy theorems and chiral anomalies as applied to radiative meson processes is that of Terazawa.[13]

We start by recalling the result of Refs. 10–12. Denote the amplitudes for $\pi^0 \to 2\gamma$ and $\gamma \to \pi^+\pi^-\pi^0$ as follows:

$$M(\pi^0 \to 2\gamma) = i\epsilon_{\mu\nu\lambda\sigma}\epsilon_1^\mu \epsilon_2^\nu k_1^\lambda k_2^\sigma F^\pi(q^2; k_1^2, k_2^2)\,, \qquad (1)$$

$$M(\gamma(K) \to \pi^+(q_+) + \pi^-(q_-) + \pi^0(q_0)) = -i\epsilon_{\mu\nu\lambda\sigma}\epsilon^\mu(K) q_+^\nu q_-^\lambda q_0^\sigma \times F^{3\pi}(q_+^2, q_-^2, q_0^2, K^2; Q_+^2, Q_0^2)\,, \qquad (2)$$

where the particles are not necessarily on-shell. In Eq. (1), q, k_1, and k_2 are the four-momenta of the initial pion and two final photons, respectively. We have defined here, for later convenience, the quantities

$$Q_+^2 = (q_+ + q_0)^2\,, \quad Q_-^2 = (q_- + q_0)^2\,, \quad Q_0^2 = (q_+ + q_-)^2$$

satisfying the constraint

$$Q_+^2 + Q_-^2 + Q_0^2 = q_+^2 + q_-^2 + q_0^2 + K^2\,. \qquad (3)$$

All PCAC results hold for $F^\pi(0;0,0)$ and $F^{3\pi}(0,0,0,0;0,0)$, and it is precisely the content of PCAC that these provide a good approximation to the physical coupling constants $F^\pi(m_\pi^2;0,0)$ and $F^{3\pi}(m_\pi^2, m_\pi^2, m_\pi^2, K^2; Q_+^2, Q_0^2)$, when K^2, Q_+^2, and Q_0^2 are at most of the order of a few m_π^2.

In the presence of the anomaly, for the π^0 case, the "strong PCAC" equation reads

$$\partial_\mu A_3^\mu(x) = f_\pi m_\pi^2 \pi_3(x) + \frac{\alpha S}{4\pi}\epsilon_{\alpha\beta\gamma\delta} : F^{\alpha\beta}(x) F^{\gamma\delta}(x) : .$$

Here, $f_\pi \simeq 0.68 m_\pi$ is the charged-pion decay constant and S is the anomalous constant. Using Eq. (1), the π^0 decay width is given by

$$\Gamma(\pi^0 \to 2\gamma) = \frac{m_\pi^3}{64\pi} |F^\pi(m_\pi^2; 0, 0)|^2\,,$$

with

$$F^\pi(m_\pi^2; 0, 0) \simeq F^\pi(0; 0, 0) = \frac{-2\alpha S}{\pi f_\pi}\,.$$

Present experimental results seem to indicate that $S^{\exp} \approx 0.5$; the colored-quark model,[14] which yields $S = \frac{1}{2}$ exactly, was invented partly to accommodate this result.

On the basis of the picture just described, the result of Adler *et al.*, Terent'ev, and others[10–12] is that

$$eF^{3\pi}(0,0,0,0;0,0) = \frac{F^\pi(0;0,0)}{f_\pi^2}\,. \qquad (4)$$

This was derived using gauge invariance, current algebra, PCAC and its anomaly, and the assumption that the electromagnetic current and the neutral axial charge commute at equal times. This last assumption, it should be noted, predicts that $\eta \to 3\pi$ is forbidden,[15] but on the other hand is not inconsistent with existing photoproduction results.[16] It is to be emphasized that in the framework of PCAC, Eq. (4) is meaningful only if $F^{\pi}(0;0,0)$ and $F^{3\pi}(0,0,0,0;0,0)$ are not suppressed to zero, which requires the presence of the anomaly. The experimental verification of Eq. (4) has thus been hailed as a crucial test of the theory of PCAC anomalies.[13,17]

We will see in what follows that the result of the VMD pole model for the $\gamma \to \pi^+\pi^-\pi^0$ vertex is in agreement with the value given by Eq. (4). Note also that the VMD pole model gives a result for the $\gamma \to 3\pi$ vertex in the physical region of its invariants.

We define the usual effective interactions

$$\mathcal{L}_{\rho\pi\pi} = g_{\rho\pi\pi}\,\epsilon_{abc}\,\rho_\mu^a \pi^b \partial^\mu \pi^c \ , \tag{5}$$

$$\mathcal{L}_{\omega\rho\pi} = G_{\omega\rho\pi}\delta_{ab}\,\epsilon^{\mu\nu\lambda\sigma}\partial_\mu \omega_\nu \partial_\lambda \rho_\sigma^a \pi^b \ , \tag{6}$$

and we write the off-shell amplitude for the process $\gamma \to \pi^+\pi^-\pi^0$ as follows:

$$F^{3\pi}(q_+^2, q_-^2, q_0^2, K^2; Q_+^2, Q_0^2) = \frac{em_\omega^2}{2\gamma_\omega}\frac{G_{\omega\rho\pi}\,g_{\rho\pi\pi}}{K^2 - m_\omega^2}\,2\left(\frac{1}{Q_+^2 - m_\rho^2} + \frac{1}{Q_-^2 - m_\rho^2} + \frac{1}{Q_0^2 - m_\rho^2}\right), \tag{7}$$

corresponding to the mechanism $\gamma \to \omega \to \rho\pi \to \pi\pi\pi$.

The dependence on the width of the ρ meson is neglected, as we will deal with a low-energy limit in which the values of the Q_i^2 $(i = +, -, 0)$ are far from the vector-meson pole. We will later set the photon and pions on their mass shells. Further, we assume a weak variation of the $\omega\rho\pi$ and $\rho\pi\pi$ couplings with Q_i^2; this is borne out by the success of the pole model in the description of such decays as $\omega \to 3\pi$, $\pi^0 \to 2\gamma$, $\omega \to \pi^0\gamma$, etc. We will comment on this shortly. Notice that we use the Gell-Mann–Zachariasen[1] definition of the γ-V coupling constant, i.e., $em_V^2/2\gamma_V$; we neglect a possible γ-ϕ contribution, in view of the strong suppression of the $\phi\rho\pi$ coupling.

The connection with $\pi^0 \to 2\gamma$ decay is now easy to see: In the pole model,[2] this process is dominated by ρ and ω, through the mechanism $\pi^0 \to \rho\omega \to \gamma\gamma$; the $\phi\rho\pi$ coupling is again neglected.

It is a recurring misconception that vector dominance (or vector dominance alone) cannot account for the decay of the neutral pion into two photons.[18] In fact, with the above conventions, VMD gives the result

$$F^{\pi}(m_\pi^2; 0, 0) = \frac{2\pi\alpha G_{\omega\rho\pi}}{\gamma_\rho \gamma_\omega} \ , \tag{8}$$

where α is, as before, the fine-structure constant. The value of $G_{\omega\rho\pi}/\gamma_\rho$ can be found from a similar model for $\omega \to \pi^0\gamma$, as the partial width for this decay is fairly well known: $\Gamma^{\exp}(\omega \to \pi^0\gamma) = 890 \pm 90$ keV.[19] In the pole model, one obtains

$$\Gamma(\omega \to \pi^0\gamma) = \frac{\alpha}{4}\left(\frac{G_{\omega\rho\pi}}{\gamma_\rho}\right)^2 \frac{(m_\omega^2 - m_\pi^2)^3}{24 m_\omega^3} \ ,$$

and therefore $(m_\omega^2 \gg m_\pi^2)$

$$\Gamma(\pi^0 \to 2\gamma) = \frac{2\alpha}{3}\left(\frac{m_\pi}{m_\omega}\right)^3 \left(\frac{4\pi}{g_{\rho\pi\pi}^2}\right)\Gamma(\omega \to \pi^0\gamma) \ ,$$

where we have used SU(3) and ρ universality to write $\gamma_\omega^2 = \frac{9}{4} g_{\rho\pi\pi}^2$. Taking $g_{\rho\pi\pi}^2/4\pi \simeq 2.4$ (this corresponds to a ρ width of 125 MeV, consistent with the results of a recent and very detailed analysis of Spital and Yennie[20]), we have

$$\Gamma(\pi^0 \to 2\gamma) = 9.6 \pm 1.0 \text{ eV} \ ,$$

which is the vector-dominance prediction, in excellent agreement with experiment, $\Gamma^{\exp}(\pi^0 \to 2\gamma) = 7.84 \pm 0.92$ eV. Note that taking a ρ width larger than 125 MeV improves this agreement, but this is not significant in our opinion in view of the ambiguities involved in the experimental determination of this quantity.[20]

We further note that the decay $\omega \to \pi^+\pi^-\pi^0$ is amenable to a similar treatment,[2] with good success.

After this digression, we now come back to the $\gamma \to 3\pi$ vertex. With the photon and pions on their mass shells, and in the low-energy limit $|Q_i^2| \lesssim 6m_\pi^2$, we obtain from Eq. (7)

$$F^{3\pi}(m_\pi^2, m_\pi^2, m_\pi^2, 0; Q_+^2, Q_0^2) = \frac{e\,G_{\omega\rho\pi}\,g_{\rho\pi\pi}}{\gamma_\omega m_\rho^2}\left(3 + \frac{Q_+^2 + Q_-^2 + Q_0^2}{m_\rho^2}\right), \tag{9}$$

where we have neglected terms of order Q_i^4/m_ρ^4 and higher. In view of Eq. (3), Eq. (9) becomes

$$F^{3\pi}(m_\pi^2, m_\pi^2, m_\pi^2, 0; Q_+^2, Q_0^2) = \frac{3e\,G_{\omega\rho\pi}\,g_{\rho\pi\pi}}{\gamma_\omega m_\rho^2}(1 + m_\pi^2/m_\rho^2) \ . \tag{10}$$

Comparing Eq. (10) with Eq. (8) yields immediately

$$eF^{3\pi}(m_\pi^2, m_\pi^2, m_\pi^2, 0; Q_+^2, Q_0^2) = \frac{3F^{\pi}(m_\pi^2; 0, 0)}{m_\rho^2}\,2\gamma_\rho g_{\rho\pi\pi} \ , \tag{11}$$

up to a small correction term of order $m_\pi^2/m_\rho^2 \simeq 3\%$. If we also consider the uncertainties involved in the extrapolations in Q_i^2 of the couplings, we may quote a typical margin of error of about

10–15% in the application of Eq. (11) [note, however, that the quantities F^{π} and $F^{3\pi}$ related by this equation now correspond to physical processes, in contrast to those appearing in Eq. (4)]. If we make use of the ρ-universality relation $2\gamma_\rho = g_{\rho\pi\pi}$ already invoked above, Eq. (11) becomes

$$eF^{3\pi}(m_\pi^2, m_\pi^2, m_\pi^2, 0; Q_+^2, Q_0^2) = F^{\pi}(m_\pi^2; 0, 0)\,\frac{3g_{\rho\pi\pi}^2}{m_\rho^2}. \tag{12}$$

We emphasize again that we expect this to be valid for $|Q_i^2| \lesssim 6m_\pi^2$. At this point, it may prove instructive to compare numerical values obtained from the right-hand sides of Eq. (4) and Eq. (12) (all necessary values of coupling constants have been given above in the text):

PCAC anomaly ($S = \frac{1}{2}$)

$$|eF^{3\pi}(0,0,0,0;0,0)| = 7.5\times 10^{-3} m_\pi^{-3};$$

VMD pole model (13)

$$|eF^{3\pi}(m_\pi^2, m_\pi^2, m_\pi^2, 0; Q_+^2, Q_0^2)| = (11.0 \pm 2)\times 10^{-3} m_\pi^{-3}.$$

The error quoted in Eq. (13) is only approximate, and probably slightly overestimated. It is difficult to draw a clear-cut conclusion from Eq. (13) because of the uncertainties introduced by the various extrapolations required.

Nevertheless, it seems to us that predictions based on vector-meson dominance and on PCAC (with anomalous constant $S \sim 0.5 \neq 0$) are in agreement, to within errors. We will now see that the similarity between the predictions of VMD and PCAC for $\gamma \to 3\pi$ is not fortuitous; indeed, we will show that Eq. (12) reduces to Eq. (4) if we require that the predictions of a pole model for $\pi\pi$ *scattering* in a non-Abelian gauge theory of the Yang-Mills type, on the one hand, and of current algebra and PCAC for the same process on the other, be *equivalent* in the limit where two pion momenta go to zero. This was done by Basdevant and Zinn-Justin,[21] and we now briefly review their argument.

Consider the process $\pi\pi \to \pi\pi$ using a Lagrangian where the ρ meson is treated as a Yang-Mills field; such a theory forms the basis of the phenomenological idea of vector dominance and universality,[1] both concepts used above. In the zero-loop approximation, the $I = 1$ Born term is given by[21]

$$B^{(1)}(s,t,u) = g\left(2\frac{t-u}{m_\rho^2 - s} + \frac{s-u}{m_\rho^2 - t} - \frac{s-t}{m_\rho^2 - u}\right), \tag{14}$$

with $g = 3m_\rho^2 \Gamma_\rho / \pi(m_\rho^2 - 4m_\pi^2)^{3/2}$. Here, s, t, and u are the usual Mandelstam invariants.

On the other hand, in the two-soft-pion limit and in the framework of current algebra and PCAC, the $\pi\pi$ amplitude satisfies the following constraint, first derived by Weinberg[22] (the anomaly does not contribute here):

$$\lim_{p_1^\mu, p_2^\mu \to 0} T_{\alpha\beta\gamma\delta}(p_1, p_2; p_3, p_4) \simeq M^{(0)}_{\alpha\beta\gamma\delta} - \frac{2p_1\cdot p_3}{8\pi^2 f_\pi^2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}) + O(p_1^2, p_2^2, p_1\cdot p_2). \tag{15}$$

Basdevant and Zinn-Justin find consistency of the off-shell extension of Eq. (14) with Eq. (15) provided the following condition on the coupling constants holds[21]:

$$16\pi^2 g f_\pi^2 = \tfrac{1}{3} m_\rho^2. \tag{16}$$

Recalling the definition of g given above, and the expression for the ρ-meson width

$$\Gamma_\rho = \frac{2}{3}\left(\frac{g_{\rho\pi\pi}^2}{4\pi}\right)\frac{(m_\rho^2 - 4m_\pi^2)^{3/2}}{8m_\rho^2},$$

Eq. (16) becomes

$$g_{\rho\pi\pi}^2 f_\pi^2 = \tfrac{1}{3} m_\rho^2. \tag{17}$$

Equation (17) differs from the controversial Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation[23] by a factor of $\frac{2}{3}$.[24] No convincing derivation of the KSRF relation seems to exist, and in fact arguments have been given against it[25]; furthermore, it appears that the KSRF relation may be incompatible with crossing symmetry, a point made by Kamal.[26] The presence of the factor of $\frac{2}{3}$ can be understood here as arising from the fact that in usual derivations of the KSRF relation, the ρ-dominance assumption is only applied to direct-channel exchanges, whereas Basdevant and Zinn-Justin also consider crossed ρ poles, and these contribute one third of the total scattering amplitude in the low-energy limit.[21]

Upon comparison with Eq. (17), Eq. (12) yields immediately

$$eF^{3\pi}(m_\pi^2, m_\pi^2, m_\pi^2, 0; Q_+^2, Q_0^2) = \frac{F^{\pi}(m_\pi^2; 0, 0)}{f_\pi^2} \tag{18}$$

in the low-energy limit $|Q_i^2| \lesssim 6m_\pi^2$. Equation (18) is just Eq. (4) above, together with the PCAC smoothness assumption. However, Eq. (4) was derived on the basis of current algebra and the existence of the PCAC anomaly, while we have obtained Eq. (18) from the following assumptions:

(1) vector-meson dominance of the isoscalar part of the electromagnetic current, and universality of vector-meson couplings, as abstracted from a Yang-Mills theory[1];

(2) dominance of $\pi^0 \to 2\gamma$ and $\omega \to 3\pi$ by vector-meson pole diagrams, along the lines of the Gell-Mann–Sharp–Wagner model[2];

(3) dynamical equivalence of a Yang-Mills theory with ρ-pole dominance, and of current algebra and PCAC, in the description of $\pi\pi$ scattering in the two-soft-pion limit.[21,22]

We note that assumption (1) has been fairly successful in many applications in high-energy physics[27]; results based on assumption (2) are found to be in good agreement with experiment, as outlined above. Assumption (3) is a natural one, in view of the successes of both approaches in their respective applications.

A few comments are in order: The approach leading to Eq. (12) should be contrasted with a point of view adopted by Terent'ev,[11] who considers vector-meson pole diagrams as corrections to the value of $F^{3\pi}(0,0,0,0;0,0)$ which he calculates from the PCAC anomaly; that is, he assumes that the amplitude for the process $\gamma \to 3\pi$ satisfies a once-subtracted dispersion relation, choosing $F^{3\pi}(0,0,0,0;0,0)$ to be the subtraction point, fixed by current algebra.

We, on the other hand, consider VMD and PCAC (and its anomaly) to provide *equivalent* descriptions of radiative meson processes. Such a point of view has had many proponents in the past; a recent contribution is that of Freund and Nandi.[28] In a very interesting paper, these authors require consistency between VMD and PCAC descriptions of $\pi^0 \to 2\gamma$, and make use of $SU(6)_W$ symmetry and the KSRF relation to obtain a value of 2.96 for the coupling constant $g_{\rho\pi\pi}{}^2/4\pi$ (see Ref. 28); we merely observe here that this value is modified to 2.42 when one uses Eq. (17) instead of the KSRF relation, which is an eminently reasonable value, used in our calculations above.

We note that the existence of such an equivalence could provide stringent constraints on future dynamical models of hadrons. In this connection, it is interesting to note that Dias de Deus,[29] working in the framework of an explicit relativistic quark model (originally devised by Llewellyn Smith[30] to resolve the Van-Royen–Weisskopf paradox[7]), has succeeded in demonstrating that the quark-quark-meson vertex functions are consistent with vector-meson dominance and PCAC at the quark level. It is gratifying that such a result is obtained, as it hints at the validity of assumption (3) above.

Finally, we remark that a comparison of the above predictions for the $\gamma \to 3\pi$ vertex with experiment would be most welcome. Two experiments may be promising in this connection: Coulomb production off nuclei, $\pi Z \to \pi\pi Z$ (Primakoff effect)[17], and pion production off atomic electrons,[31] $\pi e \to \pi\pi e$, for which the threshold is $E_\pi^{\text{lab}} \simeq 56$ GeV, i.e., in the range of energies available at FNAL. Such experiments are unfortunately difficult to perform, because of the large amounts of strong-interaction backgrounds with which the processes of interest must compete.

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*Present address: Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850.

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Finite-width effects in resonance decay near threshold*

Eugene Golowich†
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
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The decay of a resonance into a final state containing two particles, the sum of whose mean masses exceeds that of the parent particle, is investigated. Alternative methods for calculating the transition rate are compared. Two specific decays, $Y_0(1518) \to Y_1(1385)\pi$ and $A_2(1310) \to B(1237)\pi$, are studied numerically.

I. INTRODUCTION

Recently, separate measurements of the transition rate for $Y_0(1518) \to Y_1(1385)\pi$ were performed by groups from Berkeley[1] and the University of Massachusetts.[2] A noteworthy feature of this decay process is that the sum of the pion mass and the mean mass of $Y_1(1385)$ exceeds the mean mass of $Y_0(1518)$. Thus, the physical transition takes place only because of the finite resonance widths.

The purpose of this communication is to comment upon certain questions[3] which arose in the course of the analysis of this system due to its somewhat delicate kinematics. Let us phrase the situation as follows. Suppose we are given the probability amplitude for $Y_0(1518) \to Y_1(1385)\pi$ and wish to calculate the transition rate. Clearly, in the course of integrating over phase space, some averaging over the baryon mass is called for. However, there is more than one way to proceed. One may either fix the initial baryon mass at its central value and average over the mass of the final baryon or, alternatively, average over the masses of both initial and final baryons. What is the relation between the rates calculated these two ways? Can the difference ever be significant?

For any individual situation, one can, of course, use a computer to answer all the above questions numerically. However, an analytic treatment of the problem is more instructive in revealing the basic parameters occurring in the analysis, and in determining the way in which they interrelate to give the final result.

In the following, we shall define and then analyze a model appropriate for dealing with these questions. Two specific resonance decays, $Y_0(1518) \to Y_1(1385)\pi$ and $A_2(1310) \to B(1237)\pi$, will be studied numerically. Finally, we shall comment on theoretical aspects of these transitions.

II. THE MODEL

The physical situation under consideration here is that of an unstable particle of central mass $\bar{M}_R$, width Γ_R, decaying into a zero-width meson of mass μ and a second unstable particle of mean mass $\bar{M}$, width Γ. The mass of each unstable particle is described in terms of some distribution function ρ, which for definiteness, we shall take in the numerical part of our analysis as Lorentzian. For simplicity, we shall assume both unstable particles to have the same mass distribution function.[4] Thus we describe the mass spectrum of the parent and daughter resonances in terms of $\rho(M_R)$ and $\rho(M)$, respectively. The effect of this assumption on our numerical work is expected to be slight.