Sign of the real part of the elastic forward-scattering amplitude at high energies*

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(Received 18 March 1974)

For the reactions $a+b\to a+b$ and $a+\bar{b}\to a+\bar{b}$ we obtain some general results about the sign of the real part of the forward-scattering amplitude, taking into account some general experimental features of the total cross sections. These results give numerical bounds on the energy beyond which the real part must remain positive for various reactions, neglecting some low-energy effects. (i) For crossing-symmetric amplitudes, assuming that beyond the resonance region the symmetric total cross section decreases to a minimum, then increases, and remains a nondecreasing function of energy, we get a value of the energy beyond which the real part of the symmetric amplitude must remain positive. (ii) Assuming that the antisymmetric amplitude satisfies an unsubtracted dispersion relation and that the antisymmetric imaginary part does not change sign and is a nondecreasing function of energy, we find that the real part of the antisymmetric amplitude is always positive. (iii) We assume for the determination of the real part of $a+\bar{b}\to a+\bar{b}$ that (1) the antisymmetric amplitude satisfies an unsubtracted dispersion relation, (2) $E^{+1/2}\sigma^{A}$ does not change sign and does not decrease, or decreases under particular conditions to a constant, after some well-defined energy, (3) the total cross section for $a+b$ decreases to a minimum, then increases sufficiently and remains nondecreasing. Then, we determine the energy beyond which the real part of the forward *antiparticle* amplitude must be positive. Using the experimental data for $\sigma_{\text{tot}}^{K^+p}$ and σ_{tot}^{pp} from threshold up to the observed rise and low-energy data for the corresponding antisymmetric parts, we show that the real parts of the forward amplitudes for $K^-p\to K^-p$ and $p\bar{p}\to p\bar{p}$ must be positive at, and remain positive beyond, energies close to the minima of $\sigma_{\text{tot}}^{K^+p}$ and σ_{tot}^{pp}.

I. INTRODUCTION

In this paper we reexamine the known asymptotic theorems concerning the real part of forward elastic amplitudes corresponding to $a+b\to a+b$ and $a+\bar{b}\to a+\bar{b}$, a and b being strongly interacting particles, in light of two recent experimental results: the rise of σ_{tot}^{pp} at the CERN Intersecting Storage Rings (ISR),[1] and the positivity of the real part in $pp\to pp$ experiments at the Fermi National Accelerator Laboratory (FNAL).[2] Since the experimental results of last year indicating the rise of σ_{tot}^{pp}, a number of models calculating R^{pp} and $R^{\bar{p}p}$ have been presented.[3-6] It is our aim to try to generalize some results of these models independently of the particular choice of behavior for σ_{tot}^{pp}, and to clarify the conditions sufficient to have a positive real part.

If we consider the experimental values of σ_{tot}^{pp} and $\sigma_{\text{tot}}^{K^+p}$, they suggest the following picture for the total cross sections beyond the resonance region (see Fig. 1): σ_{tot} decreases from some energy value m_1 to a minimum value at m_2, beyond which σ_{tot} increases at least up to the energy value m. We call this behavior of type A. Whether the increase continues or stops far beyond m is really an asymptotic property that we cannot hope to learn from experiments for many years, but it seems reasonable to assume that the cross section does not decrease beyond m. Keeping in mind this picture for $pp, p\bar{p}, K^{\pm}p, \pi^{\pm}p$ (although neither σ^{π^+p} nor σ^{π^-p} has yet shown such a behavior), it is tempting to see if such a behavior allows us to go beyond the results of asymptotic theorems concerning the forward real parts of the amplitudes. We discuss in Sec. II the present status of such theorems. A general feature of such theorems is[7-9] that they are only true asymptotically; that is, they hold for E higher than some unknown fixed energy. In this paper we give calculable numerical energy values beyond which theorems on the positivity of the real parts hold (modulo some low-energy effects).

In Sec. II we show with some explicit, simple examples that the results concerning the sign of the real part of physical processes can be strongly affected by different hypotheses on the antisymmetric amplitude, which is unknown experimentally above 50–60 GeV.

In Sec. III we study the real part of antiparticle amplitudes, for example $R^{\bar{p}p}$ and R^{K^-p}, by making the usual decomposition into a sum of real parts of the symmetric and antisymmetric amplitudes. In Sec. III A we assume that $\sigma_{\text{tot}}^{\text{sym}}$ has a behavior of type A, and determine the rise of the cross section necessary to ensure that R^{sym} becomes and remains positive. In this way we determine two different kinds of results. First, we neglect the contribution of the cut from the physical threshold

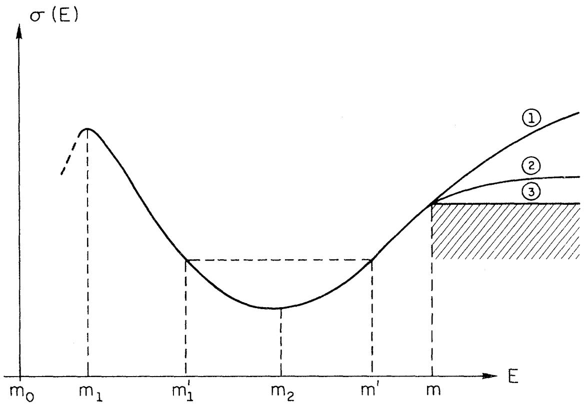

FIG. 1. A general cross section having behavior of type A. σ decreases from energy value m_1 to m_2, and increases from m_2 up to m. Beyond m, σ is nondecreasing. m_1' and m' are intermediate values such that $\sigma(m_1') = \sigma(m')$.

up to the lowest value where the behavior of type A is satisfied and we take into account a limited part of the knowledge of $\sigma_{\rm tot}^{\rm sym}$ up to the maximum rise. Second, we include all the contributions coming from the physical threshold up to the maximum rise and try to take into account full knowledge of the total cross sections in that region. This leads to our main theorem, theorem III, which is valid for the principal-part integral of any function having the same analytic structure as a symmetric subtracted forward amplitude with a positive discontinuity. In Sec. III B we assume that the antiparticle amplitude satisfies an unsubtracted dispersion relation, and that its imaginary part does not change sign and is a nondecreasing function of energy. Then we show that R^A is always positive. In Sec. III C we give numerical bounds for the positivity of $R^{\bar{p}p}$ and R^{K^-p}. Since neither a minimum nor a rise in $\sigma_{\rm tot}^{\rm sym}$ has been experimentally observed, we illustrate our theorems using models which fit the existing experimental data and extrapolate the cross sections to higher energies.

In Sec. IV we again study the positivity of $R^{a\bar{b}}$, making a different decomposition into a term coming from $\sigma_{\rm tot}^{ab}$ plus a term coming from the antisymmetric cross section. The first term has the same analytic structure as the symmetric term in Sec. III, $\sigma_{\rm tot}^{\rm sym}$ being replaced by $\sigma_{\rm tot}^{ab}$. First, if we assume $\sigma_{\rm tot}^{ab}$ satisfies a behavior of type A, then we find the rise of $\sigma_{\rm tot}^{ab}$ sufficient to ensure the positivity of the first term. For the second term, we assume that the antisymmetric amplitude satisfies an unsubtracted dispersion relation and that its imaginary part does not change sign. If $E^{-1/2}I^A$ does not decrease beyond some well-defined energy, then the contribution of the principal-value integral from that energy to infinity is non-negative. For the remaining part of the second term, corresponding to the low energies where $E^{-1/2}I^A$ decreases, we obtain a correction to the calculation of the bound where the first term becomes positive. Even in the case of $E^{-1/2}I^A$ decreasing (under particular conditions) and going to a constant after some well-defined energy, we obtain corrections to the first term such that the sum can be positive. Thus, we find the rise of $\sigma_{\rm tot}^{ab}$ sufficient to ensure the positivity of $R^{a\bar{b}}$, and we report numerical results for $R^{\bar{p}p}$ and R^{K^-p}. The virtue of this new decomposition is that it allows us to obtain bounds on the positivity of $R^{a\bar{b}}$ using as input only experimentally measured cross sections. When experimental measurements show a rise in $\sigma_{\rm tot}^{a\bar{b}}$, a similar decomposition may be useful to obtain results on the positivity of R^{ab} as well.

II. ASYMPTOTIC THEOREMS ON THE SIGN OF THE REAL PART AND IMPORTANCE OF THE ANTISYMMETRIC AMPLITUDE

Khuri and Kinoshita[7] (and, later, others[8,9]) obtained general results for the real part of the symmetric forward-scattering amplitude corresponding to $a+b^{\pm}\to a+b^{\pm}$. It is well known that if $\sigma_{\rm tot}^{\rm sym} > C(\ln E)^{\beta}$, with β positive but arbitrarily small, then some average over the real part of the forward symmetric amplitude with a positive weight function must be positive.[8] If the stronger assumption is made that the real part has only a finite number of sign changes, then the real part of the symmetric amplitude, $R^{\rm sym}$, must be positive asymptotically. It is also well known[9] that if $\sigma_{\rm tot}^{\rm sym}$ decreases not too quickly to a constant, then $R^{\rm sym}$ is the sum of a positive term and a negative term in the dispersion relation, the negative term dominating asymptotically. A general feature of such theorems about the real part is that they are only true asymptotically, that is, they hold for E higher than some unknown fixed energy. Such theorems were proved at a time when most physicists believed that the total cross section would decrease to a constant and that the ratio of real part to imaginary part would remain negative, going to zero as a function of energy.

Although the physical processes are described by a sum or difference of the symmetric and antisymmetric amplitudes, it is frequently assumed that the antisymmetric amplitude is negligible. Then, the results derived for the symmetric case may be applied to the physical processes as well. Nevertheless, the possibility of a non-negligible antisymmetric amplitude cannot be ruled out,[5] and the results for physical processes can be quite different from the symmetric case when such a non-negligible antisymmetric amplitude is taken into account. This is illustrated in the following with some simple examples. First, we recall that even if $\sigma_{\text{tot}}^{\text{sym}} \to \infty$, no general results about R or R/I for physical processes can be obtained without drastic assumptions about the growth of the antisymmetric amplitude. Second, we show that even if σ_{tot}^{pp} decreases to a constant we can have R^{pp} positive by taking an antisymmetric amplitude with imaginary part which changes sign, that is, σ^{pp} and $\sigma^{\bar{p}p}$ cross. If we require that σ^{pp} and $\sigma^{\bar{p}p}$ do not cross, we can still have R^{pp} become positive by choosing an antisymmetric amplitude such that $|\sigma^{\bar{p}p} - \sigma^{pp}| \ln E \not\to 0$, that is, the Pomeranchuk theorem is weakly satisfied or violated.

Let us call $F^{\pm} = R^{\pm} + iI^{\pm}$ the forward-scattering amplitude corresponding to

$$a + b \to a + b \ ,$$

$$a + \bar{b} \to a + \bar{b} \ .$$

For instance, F^{+} corresponds to the elastic pp, $K^{+}p$, or $\pi^{+}p$ process and F^{-} corresponds to $\bar{p}p$, $K^{-}p$, or $\pi^{-}p$. We define, as usual, the antisymmetric and symmetric amplitudes:

$$F^{A} = \tfrac{1}{2}(F^{-} - F^{+}) = R^{A} + iI^{A} \ ,$$

$$F^{S} = \tfrac{1}{2}(F^{-} + F^{+}) = R^{S} + iI^{S} \ ,$$

$$F^{\mp} = F^{S} \pm F^{A} \ .$$

A. General results

In this section we want to recall and emphasize that the signs and behaviors of $R^{\pm}$ or $\rho^{\pm} = R^{\pm}/I^{\pm}$ depend not only on F^{S}, but also on F^{A}, for which we have very few experimental results and very few theoretical constraints.[10] In particular, even if

$$\sigma_{\text{tot}}^{\pm} \xrightarrow[E\to\infty]{} \infty \ ,$$

the results $\rho^{\pm} \to 0^{+}$ quoted in the literature depend strongly on assumptions about F^{A}. For simplicity we consider a very simple family of examples satisfying the appropriate crossing properties. Let us assume that for E very large

$$F^{S,A}(E) = C^{S,A}(Ee^{-i\pi/2})^{\alpha^{S,A}}[\ln(Ee^{-i\pi/2})]^{\beta^{S,A}} \ ,$$

with $C^{S} = \text{const} > 0$ and $C^{A} = -iD^{A}$, $D^{A} = \text{const}$. We consider $\alpha^{S,A} = 1$. From positivity and unitarity we recall[11] that in the (β^{S}, β^{A}) plane (see Fig. 2) we have the restrictions

$$\beta^{S} \leqslant 2, \quad \beta^{A} \leqslant \tfrac{1}{2}\beta^{S} + 1 \quad \text{if } \beta^{S} \geqslant 0 \ ,$$

and

$$\beta^{A} \leqslant \beta^{S} + 1 \quad \text{if } \beta^{S} \leqslant 0 \ .$$

We get for $\rho^{\pm}$

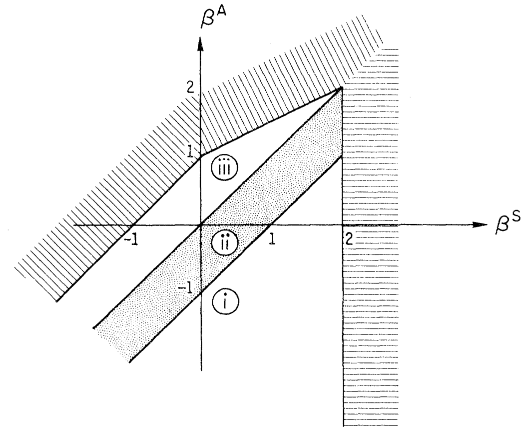

FIG. 2. The (β^{S}, β^{A}) plane. The hatched area is forbidden by unitarity and positivity. The allowed region is divided into three subdomains according to the behavior of ρ.

$$\rho^{\pm} = \frac{C^{S}\pi\beta^{S}(\ln E)^{\beta^{S}-1} \pm 2D^{A}(\ln E)^{\beta^{A}}}{2C^{S}(\ln E)^{\beta^{S}} \mp \pi\beta^{A}D^{A}(\ln E)^{\beta^{A}-1}} \ .$$

In the plane (β^{S}, β^{A}) we have an allowed domain shown in Fig. 2, and three subdomains classifying the possible asymptotic behaviors of $\rho^{\pm}$. Let us call η the sign of D^{A}, $\eta = 1$ if $D^{A} > 0$ and $\eta = -1$ if $D^{A} < 0$. Then, the three subdomains are

(i) $\beta^{S} - 1 > \beta^{A}$, $\rho^{\pm} \simeq \dfrac{\pi}{2}\,\dfrac{\beta^{S}}{\ln E}$,

$\rho^{\pm} \to 0^{+}$ if $\beta^{S} > 0$, $\rho^{\pm} \to 0^{-}$ if $\beta^{S} < 0$;

(ii) $\beta^{S} - 1 < \beta^{A} < \beta^{S}$, $\eta\rho^{\pm} \to \pm 0$;

(iii) $\beta^{S} < \beta^{A}$, $\eta\rho^{\pm} \to \pm\infty$.

Of course, there exist also the boundaries of these domains; for instance, if $\beta^{S} = \beta^{A}$ then $\eta\rho^{\pm} \to \pm\text{const}$, whereas if $\beta_{S} = 0$ in (i) then it is necessary to take into account the nonleading term in F^{S} and compare it with the leading term in F^{A}.

Summarizing, we see that even having $\beta^{S} > 0$ ($\sigma_{\text{tot}}^{\pm} \to \infty$) does not completely constrain the behavior of $\rho^{\pm}$. Roughly, one gets $\rho^{\pm} \to 0^{+}$ if F^{S} is highly dominant [case (i)] with $\beta^{S} > 0$, $\eta\rho^{\pm} \to \pm\infty$ if F^{A} is dominant [case (iii)], and $\rho^{\pm} \to \pm 0$ in the intermediate domain where $\beta^{S} > \beta^{A}$ [case (ii)]. We remark also that in the two last cases the sign of $\rho^{\pm}$ depends on the sign of D^{A} ($\beta^{A}D^{A}$ having the asymptotic sign of I^{A}), which is unknown experimentally as well as theoretically.

Some physicists can object that an antisymmetric amplitude with a behavior near $\text{const} \times |E|$ is not reasonable and that certainly $|\sigma^{+} - \sigma^{-}|$ decreases like a power. But even in this case we shall see that the antisymmetric amplitude can be very important if we take into account the freedom about the sign of I^A, namely, that a crossover of σ^+ and σ^- for some energy is not forbidden. We shall take an example interesting in the context of the recent experimental result $R^{pp}>0$ at FNAL energies (300 GeV and above).[2]

B. A particular example

Let us impose the following requirements.

(i) σ^{pp} or σ^S goes to a constant from above, $\sigma^S-\sigma^S(\infty)>0$, and decreases like a power. This means that we ignore for a moment the ISR rise of σ^{pp}.

(ii) $|\sigma^{p\bar{p}}-\sigma^{pp}|$ decreases like a power.

(iii) We take into account the behavior of σ^{pp} and $\sigma^{p\bar{p}}$ at "low accelerator energy": $\sigma^{p\bar{p}}>\sigma^{pp}$ and $R^{pp}<0$ from a few GeV up to 60 GeV, and $R^{p\bar{p}}>0$ up to 10 GeV.

(iv) We include the FNAL result: R^{pp} has a zero in the FNAL energy range, becoming positive thereafter. Can we construct an example which satisfies these conditions? If the answer is yes, it means that it is theoretically possible to have a zero in R^{pp} as observed at FNAL, with R^{pp} positive thereafter, and at the same time to have no rise in σ^{pp} with $|\sigma^{p\bar{p}}-\sigma^{pp}|$ decreasing like a power.

In fact, we can construct such a model but we have to pay a price. From (i), R^S is always negative or becomes and remains negative after some value. If we want $R^{pp}=R^S-R^A$ to satisfy (iii), then R^A must dominate R^S and be negative at FNAL energies. This requires that the decrease of $|\sigma_{\rm tot}^{p\bar{p}}-\sigma_{\rm tot}^{pp}|$ be less than the decrease of $\sigma^S-\sigma^S(\infty)$ and that $\sigma^{p\bar{p}}-\sigma^{pp}$ change sign. As an output of the model we see that $R^{p\bar{p}}=R^S+R^A$ has to be negative at least at FNAL energies and has to remain negative beyond. Such a model can be tested when $p\bar{p}$ results are obtained at higher energies. From the family of models described in Sec. II A it is easy to construct such a model:

$$F_S=iC_1E-C_2(Ee^{-i\pi/2})^{\alpha^S},\quad 0<\alpha^S<1$$

$$F_A=-iC_3(Ee^{-i\pi/2})^{\alpha_1^A}+iC_4(Ee^{-i\pi/2})^{\alpha_2^A},$$

$$0<\alpha_2^A<\alpha_1^A<1,\ \ C_i>0,\ \ \alpha^S<\alpha_1^A$$

or if $\alpha^S=\alpha_1^A$, then $C_2<C_3\tan(\frac{1}{2}\pi\alpha^S)$. The reader can verify that R^{pp} becomes positive after some energy, $R^{p\bar{p}}$ becomes negative after some energy, and $\sigma^{p\bar{p}}-\sigma^{pp}$ has to change sign with appropriate choices of the parameters C_i, α^S, α_i^A.

If we neglect the last term in F^A, then we get

$$I^{pp}\sim C_1E+C_2\sin(\tfrac{1}{2}\pi\alpha^S)E^{\alpha^S}+C_3\cos(\tfrac{1}{2}\pi\alpha_1^A)E^{\alpha_1^A},$$

$$R^{pp}\sim -C_2\cos(\tfrac{1}{2}\pi\alpha^S)E^{\alpha^S}+C_3\sin(\tfrac{1}{2}\pi\alpha_1^A)E^{\alpha_1^A}>0\ .$$

Introducing the last term in F^A we can modify the model so that for small E we have $I^A>0$ and $R^{pp}<0$. Of course, the decrease of $\sigma^S-\sigma^S(\infty)$ and $\sigma_{\rm tot}^{p\bar{p}}-\sigma_{\rm tot}^{pp}$ like powers is not necessary in order to get the above result. Let us assume $\sigma_{\rm tot}^S-\sigma_{\rm tot}^S(\infty)\simeq(\ln E)^{-\beta^S}$ and $|\sigma_{\rm tot}^{p\bar{p}}-\sigma_{\rm tot}^{pp}|\simeq(\ln E)^{-\beta^A}$ ($\beta^A>1$). Because $|R^S|\simeq E(\ln E)^{-\beta^S-1}$ and $|R^A|\simeq E(\ln E)^{-\beta^A+1}$, in order to have $|R^A|$ dominate $|R^S|$ it is sufficient that $\beta^A<\beta^S+2$. However, it is still necessary to have one sign change in I^A in order to have R^{pp} become and remain positive beyond some energy.

Now another question arises: Can we reconcile R^{pp} becoming positive and σ^S or σ^{pp} going to a constant from above without having this sign change in I^A? The answer is still yes, but we have to pay another price: namely, that F^A cannot satisfy an unsubtracted dispersion relation. Let us assume that for E large

$$I^A\simeq E(C^{p\bar{p}}-C^{pp})(\ln E)^\gamma,\quad \gamma>-1,$$

and $C^{p\bar{p}}>C^{pp}$. Then

$$R^A\simeq-\frac{2(C^{p\bar{p}}-C^{pp})}{\pi(\gamma+1)}E(\ln E)^{\gamma+1}$$

and will dominate R^S. Thus $R^{pp}\simeq(C^{p\bar{p}}-C^{pp})E(\ln E)^{\gamma+1}$ and becomes and remains positive after some energy value. A well-known example is $\gamma=0$, the violation of the Pomeranchuk theorem where $\sigma^{\bar{p}p}-\sigma^{pp}\to C^{\bar{p}p}-C^{pp}\neq 0$. It should be noted that in these models we also have $R^{p\bar{p}}$ becoming and remaining negative after some energy. Concluding, we see that another possibility for reconciling a positive R^{pp} and a σ^{pp} which does not rise is to have the Pomeranchuk theorem violated or weakly satisfied, in particular such that

$$(\ln E)(\sigma^{pp}-\sigma^{pp})\underset{E\to\infty}{\not\to}0\ .$$

The aim of this section is to recall to the reader that in order to get general results about $R^\pm$ one cannot ignore the assumptions about F^A, whether hidden or stated. In conclusion, it appears meaningless to try to obtain general theorems about $R^\pm$ using only the symmetric amplitude.

III. THE REAL PART OF THE ANTIPARTICLE AMPLITUDE AS A SUM OF R^S AND R^A

By making the usual separation $R^-=R^S+R^A$ and seeking the conditions which ensure the positivity of R^S and R^A separately, we arrive at the conditions for the positivity of R^-.

A. Real part of symmetric amplitudes

We define $F_{\rm HE}^S$, the high-energy part of the symmetric amplitude $F^S(E)$, as

$$F_{\rm HE}^S=\tfrac{1}{2}\pi[F^S(E)-F_{\rm LE}^S(E)]$$

$$=E^2\int_{m_0}^\infty\frac{dx\,I^S(x)}{x(x^2-E^2)}\ ,\tag{1}$$

where F_{LE}^{S} contains the low-energy contributions below the physical threshold m_0: the pole terms, the subtraction constants, and the unphysical cuts. We define m_1 such that $m_1 > m_0$. Let us define $F_{\mathrm{HE}}^{S} = R^S + iI^S$. Our aim is to obtain general results on $R^S(E)$ deduced from assumptions about $I^S(E) = (E^2 - m_0{}^2)^{1/2}\sigma^S(E)$ for $E \geq m_1$. We consider now a function $\sigma(E)$ which could be σ^S, σ^A, $\sigma^{\pm}$, or another function linked to the total cross section. We state now what we call assumption A concerning the behavior of $\sigma(E)$ for $E > m_1$.

Assumption A (see Fig. 1).

(i) There exists m_2 such that $\sigma(E)$ is decreasing for $E \in [m_1, m_2]$;

(ii) There exists m such that $\sigma(E)$ increases for $E \in [m_2, m]$;

(iii) $\sigma(E)$ is nondecreasing for $E > m$; i.e., $\sigma(E_2) \geq \sigma(E_1)$ if $E_2 \geq E_1 \geq m$.

We define m_1 as the smallest energy value beyond which assumption A is satisfied by $\sigma(E)$. We remark that a number of recent models which fit the present experimental cross-section data and extrapolate to higher energies for pp and $p\bar{p}$, $K^{\pm}p$, and $\pi^{\pm}p$ are of this type, but with $m = \infty$, i.e., an infinite rise. Here we want to give results applicable to these models as well as to models with only a finite rise: for example, models having a Pomeron with cuts, where σ^S goes to a constant after some rise. We have included condition (iii) to accommodate this additional type of behavior. For each type of reaction we will determine an upper bound (if it is possible) for the smallest E value where R^S becomes and remains positive, i.e., to find E_0 such that we can ensure that $R^S(E) > 0$ for any $E > E_0$. However, there does not exist such a general feature of σ_{tot} when E belongs to the interval (m_0, m_1). This is the region where the contributions of the resonances are very important and differ for different reactions. In general, models do not take into account these contributions and it is generally thought that they do not affect results about the positivity of the real part. However, we notice that for $E > m_1$ the contributions to the real part coming from the cut between m_0 and m_1 lead to a negative quantity. Thus, theoretically they could affect the determination of our upper bound E_0 beyond which R^S remains positive. It is interesting to verify numerically whether this happens.

In the following we shall obtain two different kinds of results. First, we obtain general theorems valid for $E > m_1$, neglecting the contribution of the cut between m_0 and m_1. Further, we take into account very little of the information about σ for $m_1 < E < m$. This leads to crude theorems (I and II), but they have the advantage of explaining simply the main features of the bounds, such as the decrease of E_0 when m increases. Moreover, these theorems have to be applied when experimental knowledge is insufficient to determine σ up to $E = m$, a fact generally true for $\sigma_{\mathrm{tot}}^{\mathrm{sym}}$. Second, we try to include in our bounds as much as we can of the knowledge of σ for $m_0 < E < m$. The application of our bounds becomes more convenient when σ is known up to $E = m$, as we shall see in the following for $\sigma_{\mathrm{tot}}^{K^+p}$ and $\sigma_{\mathrm{tot}}^{pp}$.

1. Theorems taking into account only limited information on σ for $m_1 < E < m$

Let us define

$$R_0^S(E) = E^2 \int_{m_0}^{m_1} \frac{dx\, I^S(x)}{x(x^2 - E^2)}$$

and consider in Eq. (1) the principal part $R^S - R_0^S$ for $E > m_1$. It is a sum of two terms, the first one corresponding to the integral along the path $[m_1, E - \epsilon]$ is negative, whereas the second corresponding to $[E + \epsilon, \infty]$ is positive. We use a trick which appears convenient here because it leads to the same interval for both terms. In the first term we use the change of variable $x = E\lambda$, and in the second we use $x = E/\lambda$. We get

$$\frac{R^S - R_0^S}{E} = \lim_{\epsilon \to 0} \int_{m_1/E}^{1-\epsilon/E} df_\lambda [f\sigma^S(E/\lambda) - g\sigma^S(E\lambda)] + \int_0^{m_1/E} df_\lambda f\sigma^S(E/\lambda) + \lim_{\epsilon \to 0} G^S(\epsilon, E)\,, \tag{2a}$$

where

$$G^S(\epsilon, E) = \int_{1-\epsilon/E}^{1/(1+\epsilon/E)} d\lambda\, \frac{\lambda I^S(E/\lambda)}{1-\lambda^2} \underset{\epsilon\ \mathrm{small}}{\simeq} \frac{\epsilon I^S(E)}{2E} \to 0 \ \text{ for fixed } E, \quad \epsilon \to 0$$

and (2b)

$$df_\lambda = \frac{d\lambda}{1-\lambda^2}\,,$$

$$f = \left(1 - \frac{\lambda^2 m^2}{E^2}\right)^{1/2}, \quad g = \left(1 - \frac{m^2}{\lambda^2 E^2}\right)^{1/2},$$

$$f \geq g \text{ for } \lambda \leq 1\,.$$

Let us define $\tilde{\sigma}^S(E) = I^S(E)/E$. With the same trick as we used for (2a) we get instead

$$\frac{R^S - R_0^S}{E} = \lim_{\epsilon \to 0} \int_{m_1/E}^{1-\epsilon/E} df_\lambda [\tilde{\sigma}(E/\lambda) - \tilde{\sigma}(E\lambda)] + \int_0^{m_1/E} df_\lambda \tilde{\sigma}(E/\lambda) + \lim_{\epsilon \to 0} G^S(\epsilon, E)\,. \tag{2c}$$

We shall get a first theorem when the amount of increase after m_2 is higher than, or equal to, the amount of decrease between m_1 and m_2. Let us call m_3 (if it exists) the value of m such that $\sigma^S(m_1)=\sigma^S(m_3)$ [or $\tilde{\sigma}^S(m_1)=\tilde{\sigma}^S(m_3)$].

Theorem I. Let σ be σ^S or $\tilde{\sigma}^S$. If σ satisfies assumption A and $m=m_3$ is such that $\sigma(m_3)=\sigma(m_1)$, then $R^S-R_0{}^S>0$ for $E>E_0=(m'm_2)^{1/2}$, where $m_2<m'<m$, $m'/m_1'=m_3/m_2$, $m_1<m_1'<m_2$, $\sigma(m')=\sigma(m_1')$.

As a first comment we note that if $m\geq m_3$ then the theorem still applies, while if $m<m_3$ then the theorem cannot be applied. As a second comment we note that (m', m_1') always exists. In fact, $m_3/m_2>1$, $m'/m_1'=1$ for $m'=m_2=m_1'$ and increases when m' increases, and m' cannot be at m_3 because $m_3/m_1>m_3/m_2$.

Now we prove the theorem. From (2) we write $R^S-R_0^S$ as a sum of four terms [let us define (m_1', m'), any pair such that $\sigma(m_1')=\sigma(m')$ and $m_1<m_1'<m_2<m'<m_3$]:

$$\frac{R^S-R_0^S}{E}=R_1^S+R_2^S+R_3^S+R_4^S\ ,$$

$$R_1^S=\int_{m_1/E}^{m_1'/E}df_\lambda[f\sigma^S(E/\lambda)-g\sigma^S(\lambda E)]$$
$$=\int_{m_1/E}^{m_1'/E}df_\lambda[(\tilde{\sigma}^S(E/\lambda)-\tilde{\sigma}^S(E\lambda)]\ ,$$
$$R_2^S=\int_0^{m_1/E}f\sigma^S(E/\lambda)df_\lambda$$
$$=\int_0^{m_1/E}\tilde{\sigma}^S(E/\lambda)df_\lambda>0\ , \tag{3}$$
$$R_3^S=\int_{m_1'/E}^{m_2/E}df_\lambda[f\sigma^S(E/\lambda)-g\sigma^S(\lambda E)]$$
$$=\int_{m_1'/E}^{m_2/E}df_\lambda[\tilde{\sigma}^S(E/\lambda)-\tilde{\sigma}^S(E\lambda)]\ ,$$
$$R_4^S=\lim_{\epsilon\to 0}\int_{m_2/E}^{1-\epsilon/E}df_\lambda[f\sigma^S(E/\lambda)-g\sigma^S(\lambda E)]$$
$$=\lim_{\epsilon\to 0}\int_{m_2/E}^{1-\epsilon/E}df_\lambda[\tilde{\sigma}^S(E/\lambda)-\tilde{\sigma}^S(E\lambda)]\ .$$

We recall from Eq. (2b) that $f\geq g$ for $\lambda\leq 1$. For R_4^S we have $\lambda E\in[m_2, E-\epsilon]$, $E/\lambda\in[E+\epsilon, E^2/m_2]$, and thus $R_4^S>0$ for $E>m_2$. For R_3^S we have $\lambda E\in[m_1', m_2]$, $E/\lambda\in[E^2/m_2, E^2/m_1']$, and thus $R_3^S\geq 0$ if $E^2\geq E_{\min}{}^2=m_2m'$. R_2^S is always positive. For R_1^S we have $E\lambda\in[m_1, m_1']$ and $E/\lambda\in[E^2/m_1', E^2/m_1]$, and $R_1^S\geq 0$ if $E^2/m_1'\geq E_{\min}{}^2/m_1'=m_3$. Finally, $R^S-R_0^S>0$ if $E^2>E_{\min}{}^2=E_0{}^2=m_2m'=m_3m_1'$. As a first application of this theorem we note that because $m'<m_3$ then $E_0<(m_2m_3)^{1/2}$, where $\sigma(m_3)=\sigma(m_1)$ always. Although this result is not as good as the result of theorem I it is simpler to visualize.

Corollary I. If σ (σ^S or $\tilde{\sigma}^S$) satisfies assumption A with $\sigma(m)=\sigma(m_3)=\sigma(m_1)$, and even if we do not know the shape of σ exactly between m_1 and m_2 and between m_2 and m_3, then $R^S-R_0^S\geq 0$ for $E>(m_2m_3)^{1/2}$.

This result, as we have said, is an application of theorem I because the interval (m_1', m') always exists and $m'<m_3$. It can be proved directly without introducing (m_1', m'), using explicitly only the values m_1, m_2, and m_3. Let us write Eqs. (3) with $m_1'=m_1$. Then $R_1^S\equiv 0$, $R_2^S>0$, and $R_3^S\geq 0$ if $E^2\geq E_{\min}{}^2=m_2m_3$, and $R_4^S\geq 0$ for $E>m_2$. As a second application, we consider the case where σ is always nondecreasing.

Corollary II. If σ (σ^S or $\tilde{\sigma}^S$) is nondecreasing for $E\geq m_1$, then $R^S-R_0^S>0$ for $E\geq m_1$.

This can be seen intuitively as an application of theorem I where we put $m_1\equiv m_2$, and any $m>m_1$ is such that $\sigma(m)\geq\sigma(m_1)$, and thus $(mm_2)^{1/2}$ is as close as we want to m_2. This result can also be proved directly. If we put $m_1'\equiv m_1\equiv m_2$ in Eqs. (3), then $R_1\equiv R_3\equiv 0$, $R_2>0$, and $R_4>0$ for $E>m_2$.

It is clear that theorem I does not give the best E_0 value because R_2^S is always positive and has not been taken into account in this game of cancelling negative terms with positive ones. On the other hand, this theorem applies only if the rise is at least up to a $\sigma(m_3)$, but does not say anything if the rise is less than $\sigma(m_3)$. The following theorem takes these two facts into account and improves theorem I in the sense that it gives smaller E_0 values.

Theorem II. Let σ (σ^S or $\tilde{\sigma}^S$) satisfy assumption A with $m\leq m_3$ and let there exist (m_1', m'), $m_1\leq m_1'<m_2<m'\leq m$, such that $\sigma(m_1')=\sigma(m')$ and satisfying (i)

$$m'\leq m\leq\frac{m'm_2}{m_1'}\ , \tag{4}$$

and (ii) the condition B (or the more general condition C) of Appendix A. [*Condition B for* σ^S *is*

$$\frac{\sigma(m)}{\sigma(m_1)}>1-\frac{\ln\{[1+(\alpha_{m_0})^{1/2}]/[1-(\alpha_{m_0})^{1/2}]\}}{\ln\{[1+(\beta_{m_0})^{1/2}]/[1-(\beta_{m_0})^{1/2}]\}}\ ,$$
$$\alpha_{m_0}=(m_1{}^2-m_0{}^2)/(E_0{}^2-m_0{}^2)\ ,$$
$$\beta_{m_0}=m_1'^2(E_0{}^2-m_0{}^2)/(E_0{}^4-m_0{}^2m_1'^2)\ ,$$
$$E_0=(m'm_2)^{1/2},\quad m_1'^2\geq m_1{}^2+3m_0{}^2\ . \tag{5}$$

Condition B for $\tilde{\sigma}^S$ is the same as (5) but with $m_0=0$, α_{m_0} and β_{m_0} being replaced by $\alpha_{m_0=0}$, $\beta_{m_0=0}$.] Then $R^S-R_0^S\geq 0$ for $E>E_0$.

We give the method of the proof; the details are given in Appendixes A and B. We do not distinguish between σ^S and $\tilde{\sigma}^S$. From Eq. (3) we have $R_3^S+R_4^S\geq 0$ if $E\geq(m_2m')^{1/2}$ and $m'\leq m$; however, in this section we neglect the contribution of these terms in the determination of E_0.

In Sec. III A 2, in our main theorem III, we take into account these terms and obtain improved results. Here we consider only the positivity of $R_1^S + R_2^S = R$, which can be written in the form

$$R = R_a + R_b\,, \quad h = (x^2 - m_0{}^2)^{1/2}/x(x^2 - E^2)$$

$$R_a = \int_{E^2/m_1'}^{\infty} \sigma(x) h(x, m_0, E)dx\,,$$

$$R_b = \int_{m_1}^{m_1'} \sigma(x) h(x, m_0, E)dx\,. \qquad (6)$$

For σ^S we have to replace σ by σ^S in (6), whereas for $\tilde{\sigma}^S$ we have to replace σ by $\tilde{\sigma}^S$ in (6) and use $h(x, m_0 \equiv 0, E)$.

We consider $E > E_0 = (m'm_2)^{1/2}$; thus R_a is positive and R_b is negative. Our aim is by a combination of the two terms to find a condition such that R remains positive for any $E \geq E_0$. Because in R_a, $x \geq E^2/m_1' \geq E_0{}^2/m_1' \geq m'm_2/m_1' \geq m_2$, $\sigma(x)$ has a lower bound $\sigma(E_0{}^2/m_1') \geq \sigma(m)$ if $m \leq m'm_2/m_1'$. It follows that R_a has a lower bound:

$$R_a \geq \sigma(m) \int_{E^2/m_1'}^{\infty} h(x, m_0, E)dx\,. \qquad (7)$$

Similarly, we have to find an upper bound for $|R_b|$ or a lower bound for R_b. This is done easily if we recall that σ is decreasing from m_1 to m_1':

$$R_b \geq \sigma(m_1) \int_{m_1}^{m_1'} h(x, m_0, E)dx\,. \qquad (8)$$

In (7) and (8) the integrands are elementary functions which can be integrated explicitly and so we will arrive at an explicit condition on (m_1', m') ensuring that R remains positive for $E \geq E_0$. This is done in Appendix A and leads to condition B written down there.

Now we shall explain how the inequality (5) expressing condition B works in order to give E_0, an upper bound of E where R^S becomes positive. First, let us remark that $\alpha < \beta < 1$ and so the right-hand side of (5) is never negative. Second, notice (as is easily seen) that the right-hand side of (5) decreases when m' increases, which means that if the inequality is satisfied for some m' value it is always satisfied for higher m' values, $m' \leq m$.

From (4) we see that, independently of (5) being satisfied, m' has an upper bound m and a geometrical lower bound m_L', defined by

$$m_L' = \frac{m m_{1,L}'}{m_2}\,, \qquad (4')$$

which of course also depends on m.

Now we discuss how (4) and (5) may both be satisfied.

(i) If we choose $m = m_3$, then $\sigma(m) = \sigma(m_1)$. It follows that inequality (5) is satisfied for any m', but condition (4′) requires that $m_L' \leq m' \leq m$. In this case we recover the theorem I.

(ii) Next we let m decrease from m_3. From the continuity properties of the functions entering into the inequality [we assume, of course, $\sigma^S(m)$ continuous] it is clear, first, that there always exists a range of values of m where (5) is satisfied for $m' \geq m_L'$. Second, there exists another range of lower m values where the inequality is satisfied only for some $m' > m_L'$. Finally, there exists another m value higher than m_2 for which the only m' value where Eq. (5) can be satisfied is for $m' = m = m_4$, whereas for $m < m_4$ any $m' \leq m$ violates Eq. (5). m_4 corresponds to an interesting physical quantity; it is the minimum rise for $m > m_2$ sufficient in order to ensure that R becomes >0 somewhere. Thus, in this approximation $E_0(m_4)$ is the highest E_0, and when m increases between m_3 up to m_4, then the corresponding $E_0(m)$ is a nondecreasing function.

We want to improve this approximation. Coming back to the inequality (5), we recall that it comes from (7) and (8), and we see that the best value for E_0 has not been found. For instance, instead of (7) and (8) we can divide the intervals of integration in R_a and R_b, taking great care in each subinterval that σ decreases between m_1 and m_1' and increases (or at least does not decrease) for $E > m_2$. This is done in the Appendixes and leads to condition C of Appendix A. The condition C is not reproduced here because of its cumbersome expression.

2. *Theorems taking into account full knowledge of σ for $m_0 \leq E \leq m$*

The rough previous theorems (I and II) were useful from the pedagogical point of view. First, they show us very simply why an assumption about σ, such as behavior of type A, can be a sufficient condition to have R become and remain positive beyond some E_0. Second, they show why in general E_0 decreases when the rise m increases. However, they are certainly not the best bounds for E_0 we can hope for and there certainly exists a possibility of improving the determination of E_0. For instance, we have entirely ignored the information coming from $R_3^S + R_4^S \geq 0$ for $E \geq E_0$. It is clear that this sum remains higher than some function depending on E_0 which, if introduced in our positivity game, will improve the E_0 value. Further, in our determination of $R_1^S + R_2^S \geq 0$ we take into account only discrete values of σ for $m_1 \geq E \geq m$, thus ignoring part of the information if σ is known experimentally up to m. On the other hand, we have not included in our formalism the part of the physical cut from m_0 to m_1, and if we can improve E_0 we must also verify whether the resonance region (or more generally the region above the physical threshold where the behavior of type A is not satisfied) modifies the result.

For all these reasons we now develop a more general formalism. Consider a function having the same analytic structure as a subtracted symmetric equation without subtraction constants, pole terms, and unphysical cuts (although all these terms can be easily introduced). The imaginary part, $I(E)$, is assumed positive along the cut and we still define $\tilde{\sigma}=I/E$. Although the formalism can also be defined for $\sigma=I(E^2-M^2)^{-1/2}$, for simplicity we consider only $\tilde{\sigma}$. Instead of Eq. (1) we define

$$F=E^2\int_{m_0}^{\infty}\frac{\tilde{\sigma}(x)dx}{E^2-x^2} \qquad (1')$$

and call R the principal part for E real. As in Sec. III A 1 we assume that $\tilde{\sigma}(E)$ satisfies the assumption A for $E\geq m_1>m_0$ and we define (m', m_1') such that $\tilde{\sigma}(m')=\tilde{\sigma}(m_1')$, with $m_1<m_1'<m_2<m'\leq m$. We consider $E\geq E_0=(m_2m')^{1/2}$, and in the same manner as that which led to Eq. (3) we get the real part as a sum of five terms:

$$\begin{aligned}
&R=R_0+E(R_1+R_2+R_3+R_4)\,,\\
&R_0=-\int_{m_0}^{m_1}\frac{E^2}{E^2-x^2}\,\tilde{\sigma}(x)dx<0 \quad \text{for } E\geq E_0\,,\\
&ER_1=\int_{m_1}^{m_1'}\frac{E^2}{E^2-x^2}\left[\tilde{\sigma}\left(\frac{E^2}{x}\right)-\tilde{\sigma}(x)\right]dx\,,\\
&ER_2=\int_{0}^{m_1}\frac{E^2}{E^2-x^2}\,\tilde{\sigma}\left(\frac{E^2}{x}\right)dx\geq 0 \quad \text{for } E\geq E_0\,,\\
&ER_3=\int_{m_1'}^{m_2}\frac{E^2}{E^2-x^2}\left[\tilde{\sigma}\left(\frac{E^2}{x}\right)-\tilde{\sigma}(x)\right]dx\geq 0 \quad \text{for } E\geq E_0\,,\\
&ER_4=\lim_{\epsilon\to 0}\int_{m_2}^{E-\epsilon}dx\,\frac{E^2}{E^2-x^2}\left[\tilde{\sigma}\left(\frac{E^2}{x}\right)-\tilde{\sigma}(x)\right]\geq 0 \quad \text{for } E\geq E_0\,.
\end{aligned} \qquad (3')$$

Our aim is to find for $R_0(E)$ and each $ER_i(E)$ a lower bound for $E\geq E_0$. R_0 is negative and we get easily

$$R_0(E)\geq\bar{R}_0(E_0) \quad \text{for } E\geq E_0\,,$$

$$\bar{R}_0(E_0)=-\int_{m_0}^{m_1}\frac{E_0^2}{E_0^2-x^2}\,\tilde{\sigma}(x)dx\,.$$

We still have to get lower bounds for ER_i, $i=1,\ldots,4$. First, because the integrand in R_4 is always positive, if we replace the upper limit of integration E by E_0, then R_4 is lowered. Second, we remark that in R_1, R_2, R_3, R_4 the coefficient of $\tilde{\sigma}(E^2/x)$ is positive and $E^2/x>m_2$ so that E^2/x is always in the domain where σ is nondecreasing. It follows that all these quantities are lowered if we replace $\tilde{\sigma}(E^2/x)$ by $\tilde{\sigma}(E_0^2/x)$. Third, $\tilde{\sigma}(E_0^2/x)-\tilde{\sigma}(x)>0$ for $x\in[m_2, E_0]$ and $x\in[m_1', m_2]$. It follows that

$$ER_4(E)\geq\bar{R}_4(E_0),\quad ER_3(E)\geq\bar{R}_3(E_0)\,,$$

$$\bar{R}_4(E_0)=\int_{m_2}^{E_0}\frac{dx(2E_0+x)}{2(x+E_0)}\left[\tilde{\sigma}\left(\frac{E_0^2}{x}\right)-\tilde{\sigma}(x)\right]\,,$$

$$\bar{R}_3(E_0)=\int_{m_1'}^{m_2}\frac{1}{2}\left(\frac{2E_0+x}{E_0+x}\right)\left[\tilde{\sigma}\left(\frac{E_0^2}{x}\right)-\tilde{\sigma}(x)\right]dx\,.$$

Now we consider R_1, where $\tilde{\sigma}(E^2/x)$ is replaced by $\tilde{\sigma}(E_0^2/x)$ and $\tilde{\sigma}(E_0^2/x)=\tilde{\sigma}(m)$, if $x\leq E_0^2/m$. It has an unknown sign because $[\tilde{\sigma}(E_0^2/x)-\tilde{\sigma}(x)]$ can change sign in the range of integration. Let us define $\mu(E_0, x, \tilde{\sigma})$ such that

$$\mu(E_0,x,\tilde{\sigma})=\begin{cases}-\left(\dfrac{E_0^2}{E_0^2-x^2}\right) & \text{if } \tilde{\sigma}\left(\dfrac{E_0^2}{x}\right)-\tilde{\sigma}(x)<0\,,\\[2ex] \dfrac{1}{2}\left(\dfrac{2E_0+x}{E_0+x}\right) & \text{if } \tilde{\sigma}\left(\dfrac{E_0^2}{x}\right)-\tilde{\sigma}(x)>0\,.\end{cases}$$

Then we get

$$ER_1(E)\geq\bar{R}_1(E_0) \quad \text{for } E\geq E_0\,,$$

$$\bar{R}_1(E_0)=\int_{m_1}^{m_1'}\mu(E_0,x,\tilde{\sigma})\left[\tilde{\sigma}\left(\frac{E_0^2}{x}\right)-\tilde{\sigma}(x)\right]dx\,,$$

$$\tilde{\sigma}\left(\frac{E_0^2}{x}\right)=\tilde{\sigma}(m) \quad \text{if } \frac{E_0^2}{x}\geq m\,.$$

The last term is $ER_2(E)$ with $\tilde{\sigma}(E^2/x)$ replaced by $\tilde{\sigma}(E_0^2/x)$. It follows that

$$ER_2(E)\geq\bar{R}_2(E_0) \quad \text{for } E\geq E_0\,,$$

$$\bar{R}_2(E_0)=\int_0^{m_1}\tilde{\sigma}\left(\frac{E_0^2}{x}\right)dx\,,$$

$$\tilde{\sigma}\left(\frac{E_0^2}{x}\right)=\tilde{\sigma}(m) \quad \text{if } x\leq\frac{E_0^2}{m}\,.$$

Taking together our results about R_i we see that $R(E)$ for $E\geq E_0$ is higher than the sum $\sum_{i=0}^{i=4}\bar{R}_i(E)$, and we obtain our main theorem.

Theorem III. If $\tilde{\sigma}>0$ satisfies assumption A for $E\geq m_1$ and R is the real part of the symmetric-like subtracted integral equation (1′), then

$$R(E)>0 \quad \text{for } E\geq E_0\,,$$

where E_0 is determined such that $\sum_{i=0}^{i=4}\bar{R}_i(E_0)>0$.

Although theorem III includes the contribution of the cut between m_0 and m_1, it can also be useful if we do not consider this part. As a trivial application it is easy to obtain a sufficient condition (not *necessary*) in order that $R(E)-R_0(E)\geq 0$ for E higher than the value m_2 corresponding to the minimum of $\tilde{\sigma}$. We remark that for the quantities $\bar{R}_i(E_0)$, $m_2<E_0<m$, $i=1, 2, 3, 4$, only $\bar{R}_1(E_0)$ could be negative. However, this cannot happen if $\tilde{\sigma}(E_0^2/x)\geq\tilde{\sigma}(x)$ for $x\in[m_1, m_2]$. Thus, we get the following:

Corollary. If $\tilde{\sigma}(x)$ satisfies assumption A for

$x \geq m_1$ and if $\tilde{\sigma}(m_2^2/x) \geq \tilde{\sigma}(x)$ for $x \in [m_1, m_2]$ [with $\tilde{\sigma}(m_2^2/x) = \tilde{\sigma}(m)$ if $m_2^2/x \geq m$], then $R(E) - R_0(E) \geq 0$ for $E > m_2$.

An example satisfying these assumptions is

$$\tilde{\sigma}(x) = c_0 + \sum_1^P c_i \left| \ln\left(\frac{x}{m_2}\right)\right|^{\alpha_i} ,$$

$$c_0 > 0, \ c_i > 0, \ m \geq m_2^2/m_1, \ \alpha_i > 0 .$$

B. The real part of antisymmetric amplitudes

We consider first the case where the antisymmetric amplitude satisfies an unsubtracted dispersion relation. In this case there is no subtraction constant, but we still define the high-energy and low-energy parts of the amplitude,

$$F_{\rm HE}^A = \tfrac{1}{2}\pi(F^A - F_{\rm LE}^A)$$
$$= E\int_{m_0}^{\infty} \frac{dx\, I^A(x)}{x^2 - E^2} . \qquad (9)$$

As for the symmetric case, we define

$$R_0{}^A = E\int_{m_0}^{m_1} \frac{dx\, I^A(x)}{x^2 - E^2} ,$$

with m_1 higher than or equal to the physical threshold m_0.

With the same trick as we used for the symmetric amplitude, we get for $E > m_1$

$$R^A - R_0{}^A = \lim_{\epsilon\to 0} \int_{m_1/E}^{1-\epsilon/E} df_\lambda [I^A(E/\lambda) - I^A(\lambda E)] + \int_0^{m_1/E} df_\lambda I^A(E/\lambda) + \lim_{\epsilon\to 0} G^A(\epsilon, E) , \qquad (10)$$

where df_λ has been defined in (2) and

$$G^A(\epsilon, E) = \int_{1-\epsilon/E}^{(1+\epsilon/E)^{-1}} \frac{I^A(E/\lambda)d\lambda}{1-\lambda^2} ,$$

$$G^A \underset{\epsilon \text{ small}}{\simeq} \epsilon\left(\frac{I^A(E)}{2E}\right) .$$

Assumption D, for $E \geq m_1$. F^A satisfies an unsubtracted equation, I^A does not change sign, and $I^A(E)$ is a nondecreasing function. m_1 is defined as the smallest energy value beyond which assumption D holds for I^A.

Theorem IV. If F^A and I^A satisfy assumption D, then $R^A - R_0{}^A > 0$ for $E \geq m_1$.

We give the proof now. In Eq. (10) the second term is always positive and so we need only to look at the first term. We have $\lambda E \in [m_1, E-\epsilon]$ and $E/\lambda \in [E+\epsilon, E^2/m_1]$ and so the first term is positive for $E > m_1$. Now let us consider the case where F^A requires a subtracted equation. If we write a subtracted dispersion relation, then the subtraction term is of the form const$\times E$. In this case we can also find general assumptions about I^A or σ^A in order to exhibit positivity (or negativity) properties for E higher than some computable E_0. However, we see that the term const$\times E$ can always modify the sign of R^A for E finite (unless we know more about the subtraction constant), and so we do not think the result will have a great significance as long as E remains finite. In other words, if F^A requires a subtracted dispersion relation, all we can get are the usual asymptotic theorems, valid for $E > E_0$ with E_0 unknown. As for the symmetric case, we can also have a formalism where the corrective term $R_0{}^A$ is included.

C. Calculations

In this section we present numerical bounds where $R^- - R_0^S - R_0^A$ becomes positive. In Figs. 3 and 4 we present I^A for pp, $p\bar{p}$, and $K^\pm p$ from some models quoted in the literature. We see that the part of the curve corresponding to experimental data satisfies assumption D very well,[12] and thus theorem IV ensures that $R^A - R_0^A > 0$ for $E > m_1$.

Second, we discuss the term $R^S - R_0^S$ defined in Sec. III A and give the numerical values for which it must be positive. We use as input the experimental values of $\sigma^S(x)$ from m_1 up to the highest experimental value and a portion of the extrapolation of σ^S given by models (see Figs. 5 and 6). Since the minimum and rise are not given by the experimental data alone, our results for this case are model-dependent and so we do not include the contribution to the integral from m_0 to m_1. Thus it is sufficient to use our crude theorem II for the determination of E_0 and the results should be considered only as an illustration of theorem II. The precise values of m_1 used, as well as the numerical bounds we have found, are given in Figs. 5 and 6 for pp and Kp. As we take into account the rise of $\sigma_{\rm sym}$ to higher energies (i.e., increase m), E_0 decreases. Since $R^A - R_0^A > 0$ for $E > m_1$, the E_0 values for $R^S - R_0^S > 0$ are also the energy values beyond which $R^- - R_0^S - R_0^A > 0$.

IV. A DECOMPOSITION OF $R^\mp$ AS A SUM OF TWO TERMS WITH ONE DEPENDING ONLY ON $\sigma^\pm$

In Sec. III we have seen that if σ^A satisfies assumption D, then $R^A(E) > 0$ for $E > m_1$, and if σ^S (or $\tilde{\sigma}^S$) satisfies assumption A, then we can compute the value beyond which R^S becomes and remains positive. Thus for the three reactions pp and $p\bar{p}$, $K^\pm p$, and $\pi^\pm p$ (A and D being satisfied) we are able to predict upper bounds on the energy where $R^-(E) > 0$. However, for $R^{p\bar{p}}$, for instance, σ^S depends not only on σ^{pp} (known up to ISR energies) but also on $\sigma^{p\bar{p}}$ (unknown experimentally be-

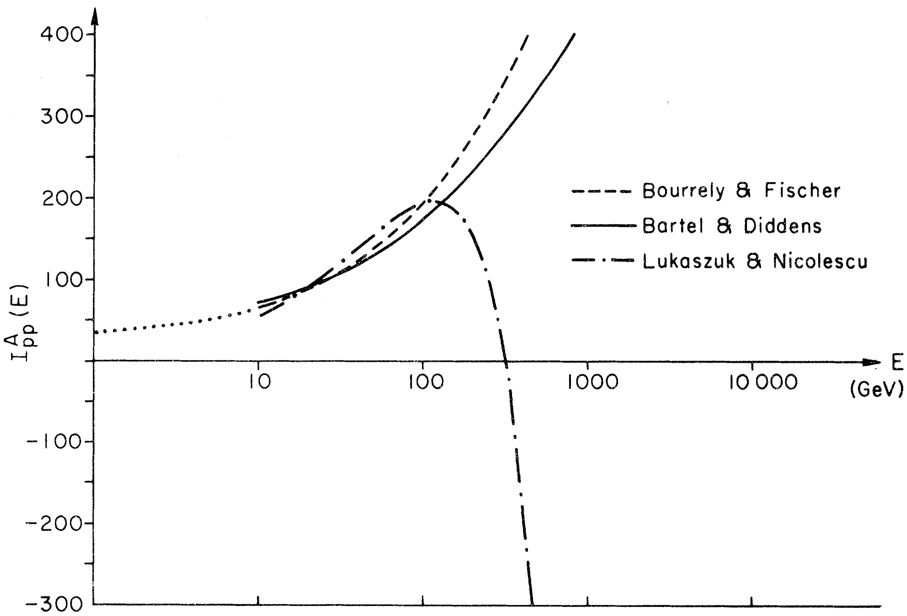

FIG. 3. $I^A_{pp}(E)$ is plotted from three models for $E > 10$ GeV. The points below 10 GeV are taken directly from the data.

yond 60 GeV). Therefore, E_0 given by the application of theorem III depends strongly on assumptions about $\sigma^{p\bar{p}}$. On the other hand, we lose the positivity included in R^A, which we do not take into account in the bound determined from R^S. If we use a formulation for R^- where both σ^S and σ^A appear explicitly, then we can perhaps rearrange terms to make better use of the positivity of R^A and obtain better results than in theorem III. This is our purpose in this section.

We assume an unsubtracted equation for F^A and, as above, define the high-energy part of $F^{\mp}$ as

$$F^{\mp}_{\text{HE}} = E \int_{m_0}^{\infty} \frac{(EI^S \pm xI^A)}{x(x^2 - E^2)}\, dx \,. \tag{11}$$

Taking into account $I^{\mp} = I^S \pm I^A$ we get

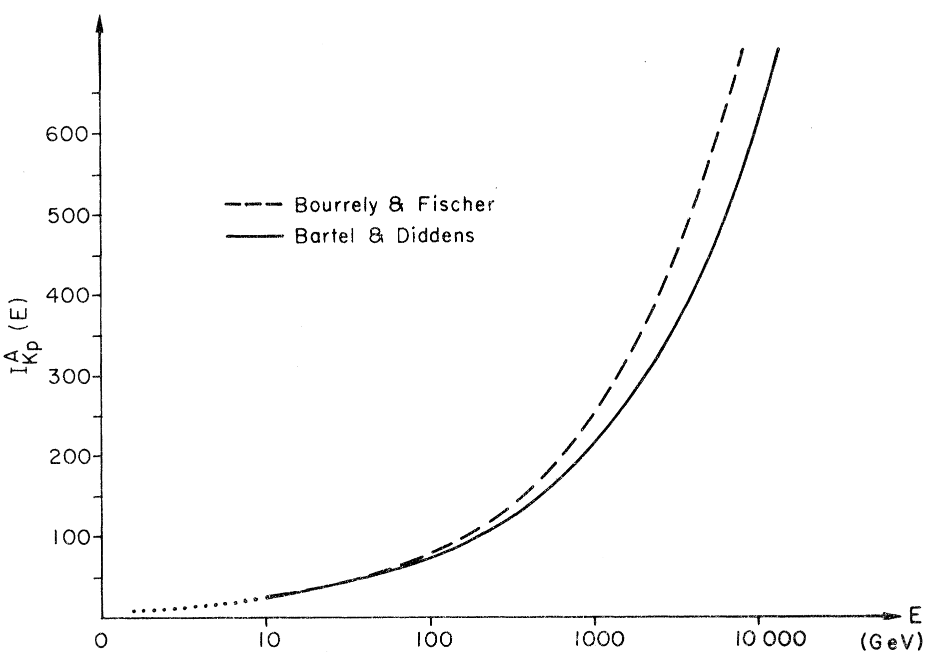

FIG. 4. $I^A_{Kp}(E)$ is plotted from two models for $E > 10$ GeV, the points below 10 GeV coming directly from the data.

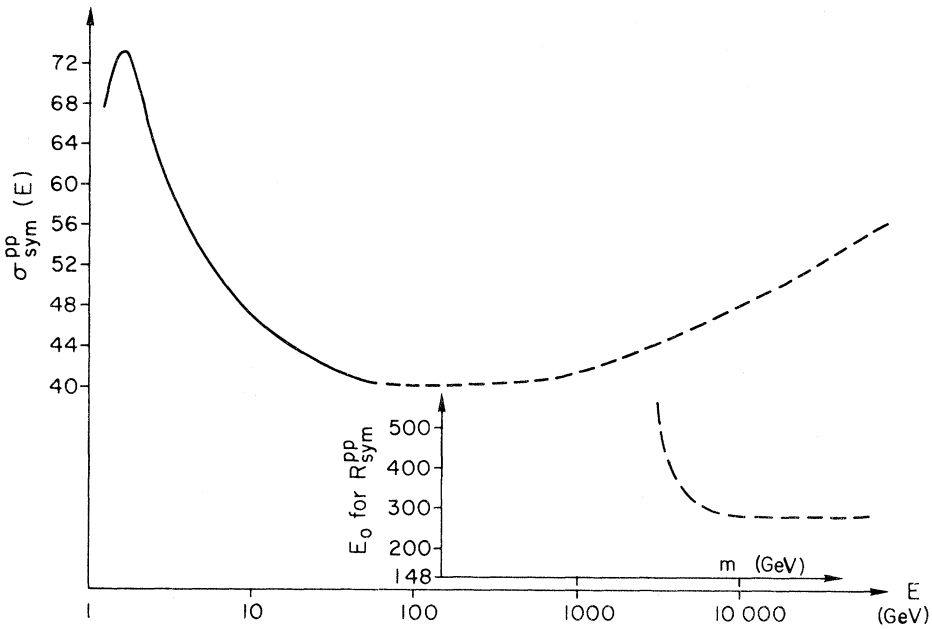

FIG. 5. $\sigma^{pp}_{sym}(E)$ from the model of Bartel and Diddens. The solid line is a fit to existing data, the dashed line an extrapolation to higher energies. $m_1 = 1.8$ GeV, $m_2 = 148$ GeV. The inset plots E_0, the energy such that $R^{pp}_{sym}(E) > 0$ for all $E > E_0$, vs m, the energy marking the amount of rise of σ^{pp}_{sym} taken into account in computing E_0.

$$R^{\mp} = r^{\mp} \pm r^{A}\ , \quad r^{\mp} = \mathrm{PP}\left(E^2 \int_{m_0}^{\infty} \frac{I^{\pm} dx}{x(x^2 - E^2)}\right)\ ,$$

$$r^{A} = \mathrm{PP}\left(E \int_{m_0}^{\infty} \frac{I^{A} dx}{x(x - E)}\right)\ . \qquad (12)$$

In order to see the advantages of this new decomposition, let us compare it with the previous one $R^{\mp} = R^{S} \pm R^{A}$. For the first term, considering $R^{p\bar{p}}$ for instance, we see that instead of having $\sigma^{p\bar{p}}$ (unknown beyond 60 GeV) in the integral defining

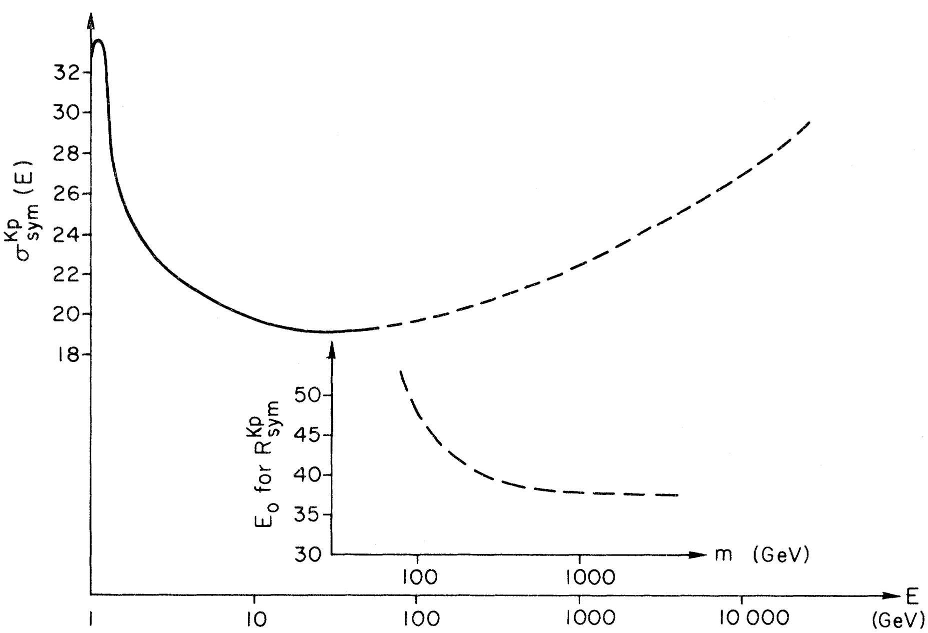

FIG. 6. $\sigma^{Kp}_{sym}(E)$ from the model of Bartel and Diddens, the solid line being a fit to existing data, the dashed line an extrapolation to higher energies. $m_1 = 1.0$ GeV, $m_2 = 30$ GeV. The inset graph plots E_0, the energy such that $R^{Kp}_{sym}(E) > 0$ for all $E > E_0$, vs m, which marks the amount of rise of σ^{Kp}_{sym} included in the computation of E_0.

R^S, we have only σ^{pp} in r^-. Second, for the second term let us assume for instance that there exists a Regge trajectory with intercept $\frac{1}{2}$. This would lead in R^A to an asymptotic behavior like const$\times E^{1/2}$, whereas r^A behaves asymptotically like a constant and thus is negligible. In the following we investigate these two terms more carefully.

A. $r^{\mp}$

Equation (12) has a convenient form. This first term $r^{\mp}$ has the same structure as a symmetric equation with σ^S replaced by $\sigma^{\pm}$ (for instance σ^{pp} for $r^{p\bar{p}}$, σ^{K^+p} for r^{K^-p}, σ^{π^+p} for r^{π^-p}). Thus we can use the results of Sec. III where the symmetric principal part has been studied in great detail.

If $\bar{\sigma}^{\pm}$ satisfies assumption A, then we can apply theorem III to calculate an upper bound for $r^{\mp}$ becoming positive. On the other hand, we know that σ^{pp} (or σ^{K^+p}) rises, and so for $r^{p\bar{p}}$ (or r^{K^-p}) we can see if this rise is sufficient to predict an upper bound for the energy value beyond which $r^{p\bar{p}}(E)>0$ [or $r^{K^-p}(E)>0$].

B. r^A

At first sight this term seems more obscure. We would like to give a rough argument to estimate the behavior of I^A necessary to give a positive real part r^A. Let us put $E=Z$. In fact, this second term has the structure of a symmetric subtracted equation written in a variable $(s-u)^2=Z$ with I^S replaced by I^A. When x is large, then $I^A\approx x^{1/2}\bar{\sigma}^A$, where $\bar{\sigma}^A=\sigma^A x^{1/2}$. If $\bar{\sigma}^A$ is positive and nondecreasing, we can make a rough guess in analogy to the case $\sigma_{\text{tot}}^{\text{sym}}$ nondecreasing for a symmetric equation that the corresponding real part r^A becomes positive. Let us make this argument more rigorous, although we ignore, for the moment, the contribution of the cut from m_0 to some $m_1>m_0$, m_1 being the value beyond which $\sigma^A E^{1/2}$ could exhibit some general feature. We get for the real part r^A, using the same trick as in Sec. III,

$$\lim_{\epsilon\to 0} E\int_{m_1/E}^{1-\epsilon/E}\left(\frac{\lambda+1}{\lambda}\right) df_\lambda\left[f\sigma^A\left(\frac{E}{\lambda}\right)-g\lambda\sigma^A(E\lambda)\right] + E\int_0^{m_1/E}\left(\frac{\lambda+1}{\lambda}\right) f\sigma^A\left(\frac{E}{\lambda}\right)df_\lambda\,, \quad (13)$$

where f, g, and df_λ are defined in Eq. (2b), $f>g$. If $\sigma^A>0$, then the second term of Eq. (13) is positive. Let us define

$$\sigma^A E^{1/2}=\bar{\sigma}^A\,,\qquad E^{-1/2}I^A(E)=\hat{\sigma}^A\,. \quad (14)$$

The first term on the right-hand side of Eq. (13) can be written as either

$$\lim_{\epsilon\to 0} E^{1/2}\int_{m_1/E}^{1-\epsilon/E}\frac{d\lambda}{\lambda^{1/2}(1-\lambda)}\left[f\bar{\sigma}^A\left(\frac{E}{\lambda}\right)-g\bar{\sigma}^A(E\lambda)\right] \quad (15)$$

or

$$\lim_{\epsilon\to 0} E^{1/2}\int_{m_1/E}^{1-\epsilon/E}\frac{d\lambda}{\lambda^{1/2}(1-\lambda)}\left[\hat{\sigma}^A\left(\frac{E}{\lambda}\right)-\hat{\sigma}^A(E\lambda)\right]. \quad (16)$$

It remains positive if $\bar{\sigma}^A$ (or if $\hat{\sigma}^A$) is nondecreasing for $E>m_1$. Finally, r^A (minus the correction coming from the contribution of the cut between m_0 and m_1) remains positive for $E>m_1$.

At low energies, however, the experimental results (we discuss the experimental situation below) show that $\bar{\sigma}^A$ or $\hat{\sigma}^A$ is decreasing to a well-defined energy. We thus modify our formalism in order to take into account this experimental fact. We would like to introduce a finite energy $M>m_0$ such that the interval $[m_0, M]$ contains a subinterval where $\bar{\sigma}^A$ or $\hat{\sigma}^A$ decreases, whereas beyond M these quantities could be nondecreasing, flat, or decreasing, depending on the particular process.

Let us now define I^A and r^A as a sum of two terms,

$$I^A=I_1^A(x)+I_2^A(x),\quad r^A=r_1^A+r_2^A\,, \quad (17)$$

where

$$r_1^A=E\int_{m_0}^{M}\frac{I_1^A dx}{x(x-E)},\quad E>M$$

$$r_2^A=\text{PP}\int_{m_1^A}^{\infty}\frac{I_2^A dx}{x(x-E)}\,,$$

$$I_2^A(x)\equiv I^A\quad\text{for } x>M\,,$$

$$I_1^A(x)\equiv I^A-I_2^A\quad\text{for } m_1^A\leq x\leq M\,,$$

$$I_1^A(x)\equiv I^A\quad\text{for } m_0\leq x\leq m_1^A\,.$$

At this stage m_1^A is a free value that we could choose in the following at our convenience. However, we assume, as suggested by the data,

$$I^A(x)-I_2^A(x)>0\quad\text{for } x\in[m_1^A, M]\,, \quad (17')$$

although the formalism that we shall develop can easily be generalized without this restriction. Taking into account (17′) we define m_1^A as the smallest energy value beyond which the inequality is satisfied. We can, of course, take m_1^A larger than this value. Now with this definition $[m_1^A, M]$ could be longer than the interval where $\bar{\sigma}$ or $\hat{\sigma}$ decreases.

Let us consider $R^{\mp}=r^{\mp}\pm r_1^A\pm r_2^A$, $E>M$, $\bar{\sigma}^{\pm}$ satisfying assumption A, and see if it is possible to obtain conditions ensuring $R^{\mp}>0$ for $E>E_0$. First, it is clear that the problem $r^{\pm}(E)>\text{const}$ can be treated using a slight modification of theorem III. Second, it is also clear that $|r_1^A|$ has a constant upper bound and so the problem $r^{\mp}\pm r_1^A>\text{const}$ for $E>E_0$ can be investigated. Third, we introduce r_2^A into the game. In the cases where we can ensure that r_2^A becomes either always positive or always negative after some computable energy, then at least one of the two problems $R^{\pm}>0$ for $E>E_0$ can be investigated. From the rough discussion above it is clear that the transition between these two cases is provided by $\hat{\sigma}^A$ going to a constant asymptotically. However, in general r_2^A is not bounded in modulus by a constant and we cannot hope with our method to investigate both problems $R^{\pm}>0$ for $E>E_0$.

1. Study of r_1^A

Let us assume $I^A(x)>0$ for $x\in[m_0, m_1^A]$ and $I^A(x)-I_2^A(x)>0$ for $x\in[m_1^A, M]$. It follows that r_1^A is negative for $E>M$ and we can evaluate both a lower and an upper bound for $M<E_0\leq E$. We get easily

$$\bar{r}_{11}^A<r_1^A<\bar{r}_1^A \quad \text{for } E\geq E_0>M \;,$$

$$\bar{r}_1^A=-\left[\int_{m_0}^{m_1^A}\frac{E_0}{E_0-x}I^A dx+\int_{m_1^A}^{M}\frac{E_0}{E_0-x}(I^A-I_2^A)dx\right],$$
$$\bar{r}_{11}^A=-\left[\int_{m_0}^{m_1^A}I^A dx+\int_{m_1^A}^{M}(I^A-I_2^A)dx\right]. \tag{18}$$

2. Study of r_2^A

We distinguish between three cases.

Case one. We can find M such that $E^{-1/2}I^A$ is nondecreasing for $E>M$. Let us define assumption E_1:

Assumption E_1. (1) F^A satisfies an unsubtracted equation, $\sigma^A>0$ for any E. (2) Either $E^{1/2}\sigma_2^A$ or $E^{-1/2}I_2^A$ is nondecreasing for $E>m_1^A$.

It is obvious from Eqs. (15) and (16) that if assumption E_1 holds, then $r_2^A(E)>0$ for $E>m_1^A$. In particular, this is the case if $E^{-1/2}I_2^A$ is a constant for $E>m_1^A$. Let us remark that we can weaken assumption E_1. Instead of point (2) let us assume either $E^{1/2}\sigma_2^A=\bar{\sigma}_2^A$ or $E^{-1/2}I_2^A=\hat{\sigma}_2^A$ satisfies assumption A for $E>m_1^A$ and there exists at least $m=m_3$ such that $\bar{\sigma}^A(m_1^A)=\bar{\sigma}^A(m)$ [or $\hat{\sigma}^A(m_1^A)=\hat{\sigma}^A(m)$]. We will prove that if this last condition holds, then $r_2^A(E)\geq 0$ for $E>(m_2m')^{1/2}$, where m' is such that $\bar{\sigma}^A(m')=\bar{\sigma}^A(m_1')$ [or $\hat{\sigma}^A(m')=\hat{\sigma}^A(m_1')$], $m'/m_1'=m_3/m_2$, $m_1^A<m_1'<m_2<m'<m_3$.

The proof is identical to the proof of theorem I or corollary I: First in Eq. (15) [or (16)], as in corollary I, let us separate the integral path in two parts $\int_{m_1^A/E}^{m_2/E}$ and $\int_{m_2/E}^{1}$. The first term is positive for $E>(m_3m_2)^{1/2}$ and the second is positive for $E>m_2$. Second, in Eq. (15) [or (16)], as in theorem I, we separate the integral into three parts, $\int_{m_1/E}^{m_1'/E}$, $\int_{m_1'/E}^{m_2/E}$, and $\int_{m_2/E}^{1}$, and we again find that each term is positive for $E>(m'm_2)^{1/2}$. Let us notice that we can even obtain finer theorems, like theorem II or III, if we take into account the second term of Eq. (13) which has not been utilized above. The extension of these theorems is straightforward.

In each of these cases we can always find E_0 such that $r_2^A>0$ for $E>E_0$.

Case two. For $E>M$, $E^{-1/2}I^A$ is decreasing and does not go to a constant. Let us define assumption E_2.

Assumption E_2. (1) F^A satisfies an unsubtracted equation, $\sigma^A>0$ for any E. (2) For $E>M$, either $\bar{\sigma}^A=\sigma^A E^{1/2}$ or $\hat{\sigma}^A=I^A E^{-1/2}$ is decreasing and is bounded above by CE^{-n}, $n>0$.

We consider directly the sum $r_1^A+r_2^A$:

$$-r^A(E)=\lim_{\epsilon\to 0}\left\{E^{1/2}\int_{M/E}^{1-\epsilon/E}\frac{d\lambda}{\lambda^{1/2}(1-\lambda)}[\hat{\sigma}^A(E\lambda)-\hat{\sigma}^A(E/\lambda)]\right\}$$
$$+E\left[\int_{m_0}^{M}\frac{I^A dx}{x(E-x)}-\int_{E^2/M}^{\infty}\frac{\hat{\sigma}^A dx}{x^{1/2}(x-E)}\right]. \tag{19}$$

In the right-hand side of Eq. (19), for $E>M$, the first and second terms are positive, the third term is negative. Taking into account the bound on $\hat{\sigma}^A$, we get

$$-r^A(E)>\int_{m_0}^{M}\frac{EI^A dx}{x(E-x)}$$
$$-\frac{c}{n+\frac{1}{2}}\left(1+\frac{1}{E/M-1}\right)\left(\frac{M^{1/2+n}}{E^{2n}}\right) \;. \tag{20}$$

The first term on the right-hand side of Eq. (20) behaves like a constant for large E, whereas for $E>M\alpha$, $\alpha>1$, the second term behaves like a constant multiplied by E^{-2n}. It follows that we can always find a finite E_0 such that the right-hand side of (20) is positive for $E>E_0$ or such that $-r^A(E)>0$ for $E>E_0$.

Case three. For $E>M$, $E^{-1/2}I^A$ is decreasing slowly and goes to a constant.

Assumption E_3. (1) F^A satisfies an unsubtracted equation, $\sigma^A>0$ for any E. (2) $\hat{\sigma}_2^A\to C_1$, $\hat{\sigma}_2^A>C_1$, $\int_{\text{const}}^{\infty}[(\hat{\sigma}_2^A-C_1)/x^{1/2}]dx<\infty$, $x(\hat{\sigma}_2^A-C_1)$ is nondecreasing for $E>m_1$.

r_2^A can be written

$$r_2^A=\int_{m_1^A}^{\infty}\frac{EC_1 dx}{x^{1/2}(x-E)}+\int_{m_1^A}^{\infty}\frac{(\hat{\sigma}_2^A-C_1)x dx}{x^{1/2}(x-E)}+\bar{r}_2^A\;, \tag{21}$$

$$\bar{r}_2^A=-\int_{m_1^A}^{\infty}\frac{(\hat{\sigma}_2^A-C_1)}{x^{1/2}}dx<0\;. \tag{22}$$

At the right-hand side of Eq. (21) the first and the second terms are positive. It follows that if assumption E_3 is satisfied, then

$$r_2^A > \bar{r}_2^A \ .$$

C. R^-

We summarize our results. Let us assume:

(i) $\tilde{\sigma}_{tot}^+$ satisfies assumption A, and

(ii) I^A satisfies either assumption E_1 or assumption E_3.

If (i) and (ii) are satisfied, then our method gives the possibility of computing E_0 such that $R^-(E) > 0$, for $E > E_0$. For instance, if assumption E_3 is satisfied and if

$$\sum_0^4 \bar{R}_i(E_0) + \bar{r}_1^A + \bar{r}_2^A > 0$$

for a particular E_0 value, then $R^-(E) > 0$ for $E > E_0$. Consider now the case where assumption E_1 is satisfied (or one of the alternatives given in case one in Sec. IV B 2). Then the method is slightly different. On one hand, we determine $E_{0,1}$ such that $r_2^A > 0$ for $E > E_{0,1}$ using the method explained in Sec. IV B 1. On the other hand, we determine $E_{0,2}$ such that $\bar{r} + r_1^A > 0$ for $E > E_{0,2}$ by investigating the equation

$$\sum_0^4 \bar{R}_i(E_{0,2}) + \bar{r}_1^A(E_{0,2}) > 0 \ .$$

Finally, we get $R^- > 0$ for $E > \sup(E_{0,1}, E_{0,2})$. We recall from the experimental data that assumption (i) holds for pp and K^+p, but does not hold yet for π^+p. It should be stressed that the method reported here assumes that $\hat{\sigma}^A$ does not decrease after some finite energy (as shown in Figs. 9 and 10). If $\hat{\sigma}^A$ decreases for all energies as in assumption E_2, for example, then we cannot expect to obtain a bound for $R^- > 0$ using the decomposition given here. In that case, however, it may be possible to obtain a bound on R^+, as we outline in Sec. IV D.

D. R^+

In the usual decomposition $R^+ = R^S - R^A$, we have not tried to find conditions giving $R^+ > 0$. Actual experimental information is compatible with $R^A > 0$, and if $R^S > 0$ beyond some energy, we are left with the sum of two terms of opposite sign. Let us now consider $R^+ = r^+ - r^A$.

(1) For the determination of the energy beyond which r^+ becomes positive, it is necessary (applying theorem III) to know that there exists a sufficient rise in σ^-, i.e., in $\sigma^{\bar{p}p}$, σ^{K^-p}, or σ^{π^-p}. This result has not yet been observed experimentally, and thus such a calculation cannot be done independently of models.

(2) If it is experimentally confirmed at high energies that $\bar{\sigma}^A$ or $\hat{\sigma}^A$ increases sufficiently, then r^A may lead to a negative contribution to R^+. On the contrary, if future experiments show that $\bar{\sigma}^A$ or $\hat{\sigma}^A$ decreases at high energies, then it is possible to give numerical bounds beyond which r^A remains negative, giving a positive contribution to R^+. However, there exists also a third possibility. We recall that we have introduced M because it appears as a general experimental fact that there exists an interval of energy between m_1 and M where $\hat{\sigma}^A$ decreases. But for $E > M$ it is possible that there exists no general feature for $\hat{\sigma}^A$, i.e., that $\hat{\sigma}^A$ increases for some reactions and decreases for others. Now we come back to R^+ and consider (i) some models such that $\tilde{\sigma}^-$ satisfies assumption A, and (ii) that assumption E_2 is satisfied.

In this case we first determined $E_{0,1}$ such that $r^+ > 0$ for $E > E_{0,1}$ (applying our theorem III), second, we determined $E_{0,2}$ such that the right-hand side of Eq. (20) is positive, i.e., $-r^A(E) > 0$, $E > E_{0,2}$, and finally we get that $R^+ > 0$ for $E > \sup(E_{0,1}, E_{0,2})$.

E. Calculation for $R^{\bar{p}p}$

In Fig. 7 is plotted a parametrization of the experimental data of

$$\tilde{\sigma}_{tot}^{pp} = [(E^2 - m^2)^{1/2}/E]\,\sigma_{tot}^{pp}$$

from the threshold m_0 up to the recent ISR results. We see that assumption A is satisfied and $r^- > \sum_0^4 \bar{R}_i(E_0)$ for $E > E_0$ (theorem III). In Fig. 8

$$\hat{\sigma}^A = \tfrac{1}{2}E^{-1/2}(E^2 - m^2)^{1/2}(\sigma_{tot}^{\bar{p}p} - \sigma_{tot}^{pp}) \ ,$$

is plotted, and we see that assumption E_3 is satisfied if we consider $E > M \simeq 28$ GeV. We have chosen $\hat{\sigma}_2^A = b(1 + E_1/E)^{1/2}$, a parametrization given in Ref. 4, and we have taken $m_1^A = 1$ GeV. We have thus $r_1^A > \bar{r}_1^A$ given by Eq. (18) and $r_2^A > \bar{r}_2^A$ given by Eq. (22). We have computed numerically E_0 such that $\sum \bar{R}_i(E_0) + \bar{r}_1^A + \bar{r}_2^A > 0$, which provides an E_0 value such that $R^{\bar{p}p} > 0$ for $E > E_0$. The results are the following.

(i) Taking into account the rise up to ISR energies or even a smaller rise, E_0 for $R^{\bar{p}p}$ is found to be close to the minimum of σ_{tot}^{pp}.

(ii) The contribution of the term $\bar{R}_4$ is negligible.

(iii) The negative contribution of the cut between $[m_0, m_1]$, i.e., between $\bar{R}_0$ and

$$-\int_{m_0}^{m_1} [I^A E_0/x(E_0 - x)]dx \ ,$$

although important numerically, does not affect the result very much if we include the rise of σ_{tot}^{pp} up to ISR energies.

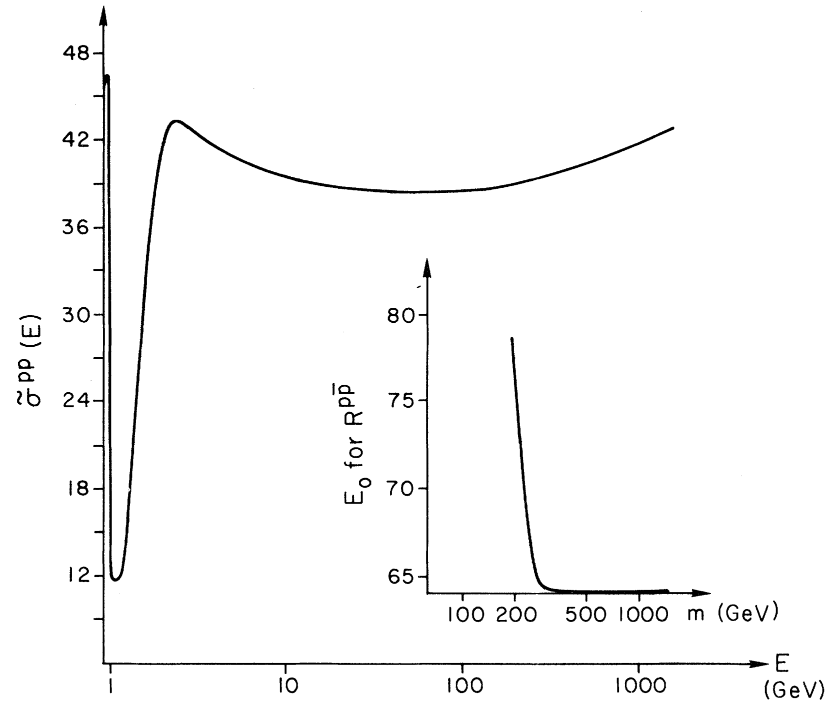

FIG. 7. The upper graph shows experimental values for $\tilde{\sigma}^{pp}(E)=I^{pp}(E)/E$ from threshold up to 1500 GeV, using the parameterization of Ref. 4 for $E>10$ GeV. The inset graph plots E_0 vs m, E_0 being the energy such that $R^{\bar{p}p}>0$ for all $E>E_0$, m marking the amount of rise of $\tilde{\sigma}^{pp}$ taken into account in the computation of E_0.

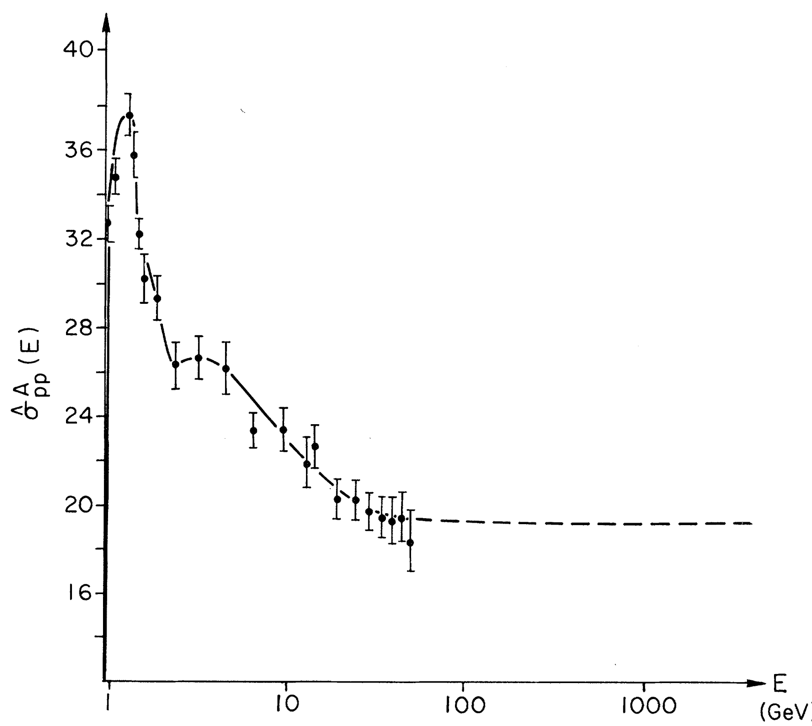

FIG. 8. $\hat{\sigma}^{A}_{\bar{p}p}$ data, the solid curve representing the fit to the data used in calculating E_0 for $R^{\bar{p}p}$ given in Fig. 7. The dashed curve is an extrapolation to higher energies from the model of Bourrely and Fischer.

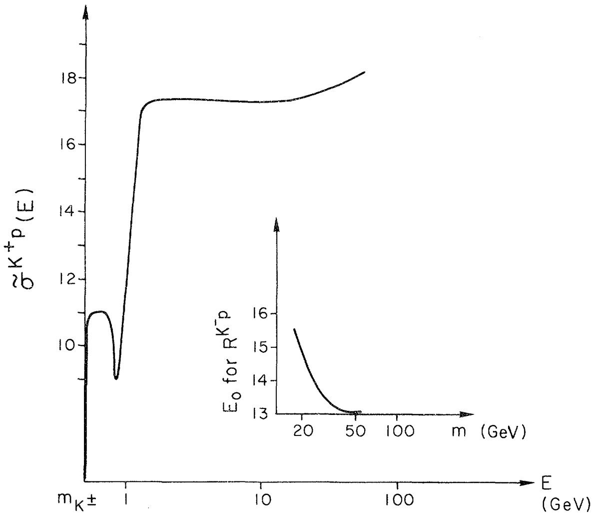

FIG. 9. The upper graph shows the experimental values for $\tilde{\sigma}^{K^+p} = I^{K^+p}/E$ from threshold up to 55 GeV, using the parametrization of Ref. 4 for $E > 10$ GeV. The inset graph plots E_0 vs m, E_0 being the energy that $R^{K^-p} > 0$ for all $E > E_0$, m marking the rise of $\tilde{\sigma}^{K^+p}$ taken into account in computing E_0.

(iv) Taking into account the maximum rise up to ISR energies, we find for E_0 values relatively close to the minimum of σ_{tot}^{pp} that $\sum \overline{R}_i(E_0) + \overline{r}_1^A + \overline{r}_2^A$ is higher than some well-defined non-negligible positive numerical constant. This suggests that if we include in our formalism what has been neglected, i.e., the subtraction constant, the pole terms, and the unphysical cut, then the general features of our result will not change very much.

F. Calculations for R^{K^-p}

In Fig. 9 is plotted a parametrization[4] of the experimental data of $\tilde{\sigma}^{K^+p} = [(E^2 - M^2)^{1/2}/E]\sigma_{\text{tot}}^{K^+p}$ from the threshold m_0 up to the maximum available data, 55 GeV. We see that assumption A is very well satisfied. However, between m_1 and the minimum, $m_2 \simeq 13$ GeV, the data are compatible with a flat cross section. Thus, we have to modify theorem III slightly. It is now unnecessary to introduce the intermediate values (m', m_1').

Consider

$$r^-(E) = \text{PP} \int_{m_0}^{\infty} \frac{\tilde{\sigma}^{K^+p}(x)dx}{x(x^2 - E^2)} .$$

We introduce the maximum rise $m > m_2$ and still assume $\tilde{\sigma}(E) \geq \tilde{\sigma}(m)$ for $E > m$. With a slight modification of the formulation leading to theorem III, it is straightforward to get for $E > E_0 > m_2$

$$r^-(E) > \overline{R}_0(E_0) + \overline{R}_2(E_0) + \overline{R}_4(E_0) + \int_{m_1}^{m_2} dx \frac{1}{2}\left(\frac{2E_0 + x}{E_0 + x}\right)\left[\tilde{\sigma}\left(\frac{E_0^2}{x}\right) - \tilde{\sigma}(x)\right] , \quad (23)$$

where $\overline{R}_0$, $\overline{R}_2$, and $\overline{R}_4$ are still given below Eq. (3′). In Fig. 10 $\hat{\sigma}^A$ is plotted for the reactions $K^\pm p \to K^\pm p$ and we see that assumption E_3 is satisfied if we consider $E > M$, $m_2 < M \simeq 13$ GeV. We have chosen $\hat{\sigma}_2^A = \hat{\sigma}^A(E = M)$ such that $\hat{\sigma}^A$ is constant for $E > M$. For this choice of M we have taken $m_1^A = 1.6$ GeV. We get $r^A > \overline{r}_1^A$ given by Eq. (18). We have numerically computed E_0 such that the right-hand side of Eq. (23) plus $\overline{r}_1^A(E_0)$ becomes positive, ensuring $R^{K^-p} > 0$ for $E > E_0$. The principal result is the same in this case as for the $R^{\bar{p}p}$ case: $R^{K^-p} > 0$ for $E > E_0$, and E_0 is close to the minimum of the total cross section of the crossed reaction, i.e., to the minimum of $\sigma_{\text{tot}}^{K^+p}$. We can also choose[13] M slightly higher than m_2, for instance $M = 14$ or 15 GeV. For these choices we have verified that E_0 is slightly higher, but close to M. However, let us remark that our calculation takes into account neither the singularities below the elastic threshold nor the subtraction constant, and that the minimum of $\sigma_{\text{tot}}^{K^+p}$ occurs at an energy which is low compared to the minimum of σ_{tot}^{pp}. On the other hand, the calculations presented here show that it is not necessary to take into account the whole experimental rise of σ^{pp} or σ^{K^+p} in order to determine E_0. For that reason, we feel that future work which includes what has been neglected here and takes into account the maximum experimental rise will obtain results similar to those presented here.

G. Elastic $\pi^\pm p$ case

The elastic $\pi^\pm p$ data are characterized by two main experimental facts:

(i) A behavior of type A is not yet experimentally observed for $\sigma_{\text{tot}}^{\pi^+p}$ up to 55 GeV.

(ii) Comparing the experimental π^-p data at Serpukhov energies (60 GeV) (Ref. 14) with a new result at FNAL (Ref. 15) does not indicate any rise.

Point (i) illustrates an important result concerning the sign of the real part: If we take into account only the experimental data for σ^{π^+p}, the absence of a rise in σ^{π^+p} up to 55 GeV seems to be tied to the negativity of R^{π^-p} up to that energy. Of course, one can assume as in some models[4] in the literature that σ^{π^+p} has a minimum and rises at energies higher than 60 GeV. If we apply our method using such models for σ^{π^+p}, and assume that after some rise σ^{π^+p} is nondecreasing, then we can find the sufficient rise such that R^{π^-p} becomes and remains positive at energies somewhat beyond the minimum of σ^{π^+p}. However, these models[4] also require a rise for σ^{π^-p} somewhat beyond 60 GeV and those models[16] leading to a positive R^{π^-p} give values for σ^{π^-p} at the same time which are not clearly consistent with the experimental data.[15] In conclusion, we think that we must wait

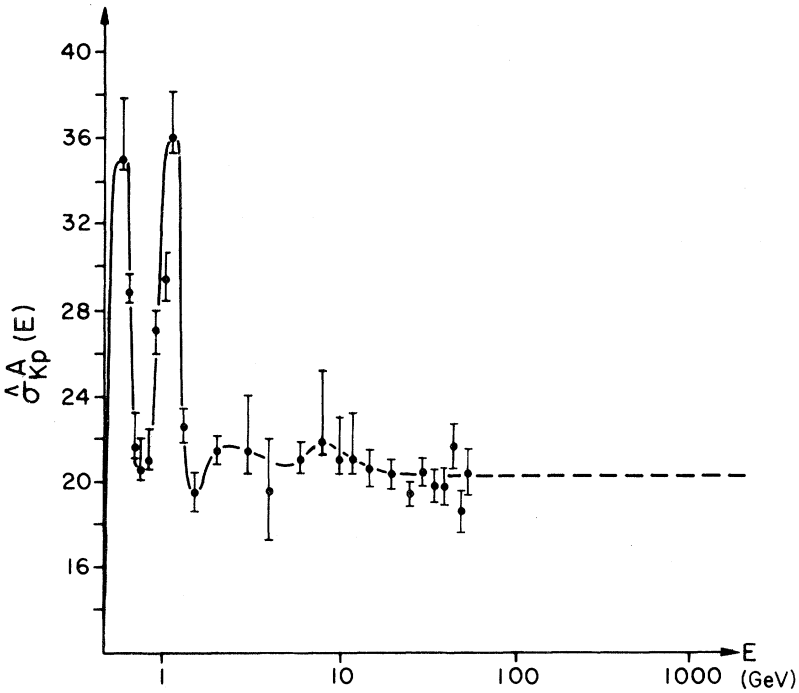

FIG. 10. $\hat{\sigma}^{A}_{Kp}$ data, the solid curve representing the fit used in calculating E_0 for R^{K^-p} given in Fig. 9. The dashed curve is an extrapolation to higher energies from the model of Bourrely and Fischer.

for further experimental data before we apply our method to the $\pi^{\pm}p$ case.

V. CONCLUSION

In this paper, for the reactions $a+b\to a+b$ and $a+\bar{b}\to a+\bar{b}$, we have tried to obtain general results concerning the positivity of the real forward-scattering amplitude beyond some finite, computable energy. The originality of our method is that we do not consider a particular model, and we incorporate in our formalism as much experimental knowledge of the total cross sections as possible. For the real part of symmetric amplitudes we have shown that behavior of type A (see Fig. 1) with sufficient rise in the symmetric total cross section is a sufficient condition to ensure the positivity of the symmetric real part at some finite energy. By making the usual decomposition of $R^{a\bar{b}}$ into a sum of symmetric and antisymmetric parts, we find that the results on the positivity of R^{sym} also give the positivity of $R^{a\bar{b}}$. Unfortunately, none of the symmetric total cross sections (for pp, Kp, or πp) have yet been confirmed to have a behavior of type A.

We have thus been led to consider a different decomposition. With this new decomposition and special assumptions about the antisymmetric amplitude we have shown that if σ^{ab}_{tot} has a behavior of type A, then we can compute the energy at which $R^{a\bar{b}}$ becomes positive. Applying this method to the two known cases where σ^{ab}_{tot} exhibits a behavior of type A, we have found numerically that R^{K^-p} and $R^{p\bar{p}}$ must be positive at energy values close to the minima of $\sigma^{K^+p}_{\text{tot}}$ and σ^{pp}_{tot}, respectively. On the other hand, the experimental knowledge of $\sigma^{\pi^+p}_{\text{tot}}$ does not yet indicate behavior of type A.

The validity of our results is subject to three main conditions:

(i) Our assumption about the behavior of σ^{ab}_{tot} beyond the experimentally observed rise must be correct; i.e., that σ^{ab}_{tot} is nondecreasing.

(ii) Our assumptions about the antisymmetric amplitudes must be correct; i.e., that there exists a finite energy beyond which $E^{1/2}(\sigma^{a+\bar{b}}_{\text{tot}}-\sigma^{a+b}_{\text{tot}})$ is either nondecreasing or decreasing, but is going to a constant (under particular conditions). Moreover, we have assumed that there is no change of sign for $\sigma^{a\bar{b}}_{\text{tot}}-\sigma^{ab}_{\text{tot}}$ at high energies.

(iii) In the formalism developed in this paper we have taken into account the forward dispersion relations from the physical threshold up to infinity. However, the subtraction constants, the pole terms, and the unphysical cuts have been neglected.

Experiments in the near future will either partially confirm or invalidate point (i) (especially in the K^+p case) and point (ii) above. If it is experimentally established that assumption (ii) is wrong, namely that $E^{1/2}\sigma^A$ decreases and does not go to a constant, although $\sigma_{\text{tot}}^{a\bar{b}}$ and σ_{tot}^{ab} do not cross, then our formalism can be applied to the determination of the value beyond which R^{ab} remains positive.

Concerning point (iii), we note that a future work should be done including these corrections. The fact that we have found numerically that $R^{\bar{p}p}$ and R^{K^-p} are appreciably positive near the minimum of σ_{tot}^{ab} indicates that these corrections will not affect our results significantly.

ACKNOWLEDGMENTS

We are grateful to Dr. N. N. Khuri, Dr. V. Rittenberg, Dr. R. L. Cool, and Dr. K. Goulianos for valuable discussions. The main part of the calculations has been done at Saclay S. P. T. We thank C. Verneyre for her constant help in the calculations. One of us (H. C.) thanks Professor M. A. B. Bég, Professor N. N. Khuri, and Professor A. Pais for their warm hospitality at the Rockefeller University.

APPENDIX A

In this appendix we determine conditions B and C of theorem II. We consider $R = R_a + R_b$,

$$R_a = \int_{E^2/m_1'}^{\infty} \sigma(x) h\,dx\,,$$

$$R_b = \int_{m_1}^{m_1'} \sigma(x) h\,dx\,,$$

$$h = (x^2 - m_0^2)^{1/2}/x(x^2 - E^2) \quad \text{(A1)}$$

for $E^2 \geq E_0^2 = m_2 m'$. We have also $m_0 \leq m_1 \leq m_1' \leq m_2 \leq m'$, and we want to find a condition on (m_1', m') such that $R \geq 0$ for $E > E_0$. First we consider R_b. Let us introduce n_q:

$$m_1 = n_0 \leq n_1 \leq n_2 \cdots n_{p-1} \leq n_p = m_1'\,.$$

Owing to the fact that $\sigma(x)$ is decreasing for $x \in [n_0, n_p]$ we have

$$R_b > \sum_{q=0}^{q=p-1} \sigma(n_q) \int_{n_q}^{n_{q+1}} h\,dx\,. \quad \text{(A2)}$$

Let us define

$$b_{m_0}(x) = \{1 + [\alpha(x)]^{1/2}\}/\{1 - [\alpha(x)]^{1/2}\} \geq 1\,,$$

$$\alpha_{m_0}(x) = (x^2 - m_0^2)/(E^2 - m_0^2) < 1\,, \quad \text{(A3)}$$

$$F_{m_0}(\mu, \nu) = m_0\left(\arccos\frac{m_0}{\mu} - \arccos\frac{m_0}{\nu}\right) > 0 \quad \text{if } \mu > \nu\,.$$

We get (for simplicity we do not write the index m_0)

$$E^2 R_b > X \equiv \sum_{0}^{p-1} \sigma(n_q)\left[F(n_{q+1}, n_q) + \frac{(E^2 - m_0^2)^{1/2}}{2} \ln\frac{b(n_q)}{b(n_{q+1})}\right], \quad \text{(A4)}$$

where

$$1 < b(n_q) < b(n_{q+1}).$$

The last term in the right-hand side of (A4) can be written

$$\tfrac{1}{2}(E^2 - m_0^2)^{1/2}\left[\left(\sum_{q=1}^{q=p-1} \ln b(n_q)[\sigma(n_q) - \sigma(n_{q-1})]\right) + \sigma(n_0)\ln b(n_0) - \sigma(n_{p-1})\ln b(n_p)\right].$$

Let us define

$$c_\lambda = (1 + \sqrt{\beta_\lambda})/(1 - \sqrt{\beta_\lambda})\,,$$

$$\beta_\lambda = (E^2 - m_0^2)\Big/\left[\left(\frac{\lambda E^2}{m_1'}\right)^2 - m_0^2\right]. \quad \text{(A5)}$$

Then we have

$$\frac{2R_b E^2}{(E^2 - m_0^2)^{1/2}\ln c_{\lambda=1}} > \frac{2X}{(E^2 - m_0^2)^{1/2}\ln c_{\lambda=1}}\,. \quad \text{(A6)}$$

In Appendix B it is shown that if $n_{p-1}^2 < m_1'^2 - 3m_0^2$,

(i) $(E^2 - m_0^2)^{1/2}\ln c_{\lambda=1}$ decreases when E increases,

(ii) $\ln b(n_q)/\ln c_{\lambda=1}$ increases when E increases for $q = 0, 1, \ldots, p$; and goes to the limit n_q/m_1' when $E \to \infty$,

(iii) $0 < \ln b(m_1')/\ln c_{\lambda=1} < 1$, and

(iv) $\ln c_\lambda(E)/\ln c_{\lambda=1}(E)$ increases when E increases if $\lambda > 1$.

Then we get

$$\frac{2R_b E^2}{(E^2 - m_0^2)^{1/2}\ln c_{\lambda=1}(E_0)} > X_0 \quad \text{for } E > E_0\,, \quad \text{(A7)}$$

$$X_0 \equiv \sum_{q=0}^{q=p-1} \frac{2\sigma(n_q)F(n_{q+1}, n_q)}{(E_0^2 - m^2)^{1/2}\ln c_{\lambda=1}(E_0)} + \sigma(m_1)\frac{\ln b(m_1, E_0)}{\ln c_{\lambda=1}(E_0)} - \sigma(n_{p-1}) + \sum_{q=1}^{q=p-1}[\sigma(n_q) - \sigma(n_{q-1})]\frac{n_q}{m_1'}\,. \quad \text{(A8)}$$

When $p = 1$, which means that we do not introduce n_q, the final result (A8) is replaced by

$$\frac{2R_b}{(E^2 - m_0^2)^{1/2}\ln c_{\lambda=1}(E_0)} > \sigma(m_1)\left(-1 + \frac{\ln b(m_1, E_0)}{\ln c_{\lambda=1}(E_0)}\right),$$

$$m_1^2 \leq m_1'^2 - 3m_0^2. \quad \text{(A8')}$$

Next, we consider R_a and introduce λ_q: Let us consider $\lambda_0 = 1 \leq \lambda_1 \leq \lambda_2 < \cdots \lambda_{r-1} \leq \lambda_r$. Because $\sigma(E^2/n_1')$ is nondecreasing for $E \geq E_0$ and $h > 0$ in R_a, we get

$$R_a > \sum_{s=0}^{s=r-1} \sigma\left(\lambda_s \frac{E^2}{m_1'}\right) \int_{\lambda_s E^2/m_1'}^{\lambda_{s+1}E^2/m_1'} h\,dx + \sigma\left(\lambda_r \frac{E^2}{m_1'}\right) \int_{\lambda_r E^2/m_1'}^{\infty} h\,dx, \tag{A9}$$

$$R_a > \sigma\left(\frac{E_0^2}{m_1'}\right) \int_{E^2/m_1'}^{\infty} h\,dx + \sum_{s=1}^{s=r}\left[\sigma\left(\lambda_s \frac{E_0^2}{m_1'}\right) - \sigma\left(\lambda_{s-1}\frac{E_0^2}{m_1'}\right)\right] \int_{\lambda_s E^2/m_1'}^{\infty} h\,dx, \tag{A10}$$

and

$$\frac{2R_aE^2}{(E^2-m_0^2)^{1/2}\ln c_{\lambda=1}} > \sigma\left(\frac{E_0^2}{m_1'}\right) + \sum_{s=1}^{s=r}\left[\sigma\left(\lambda_s \frac{E_0^2}{m_1'}\right) - \sigma\left(\lambda_{s-1}\frac{E_0^2}{m_1'}\right)\right]\left(\frac{\ln c_{\lambda_s}(E)}{\ln c_{\lambda=1=\lambda_0}(E_0)}\right). \tag{A11}$$

Applying result (iv) we get that for all $E > E_0$

$$\frac{2R_aE^2}{(E^2-m_0^2)^{1/2}\ln c_{\lambda=1}(E)} > Y_0, \tag{A12}$$

$$Y_0 = \sigma\left(\frac{E_0^2}{m_1'}\right) + \left\{\sum_{s=1}^{s=r}\left[\sigma\left(\frac{\lambda_s E_0^2}{m_1'}\right) - \sigma\left(\lambda_{s-1}\frac{E_0^2}{m_1'}\right)\right]\frac{\ln c_{\lambda_s}(E_0)}{\ln c_{\lambda=1}(E_0)}\right\}, \quad m \le \lambda_r \frac{E_0^2}{m_1'}, \quad \sigma(\lambda_r E_0^2/m_1') \ge \sigma(m). \tag{A13}$$

If $\lambda_r = 1$, then (A12) becomes

$$\frac{2R_aE^2}{(E^2-m_0^2)^{1/2}\ln c_{\lambda=1}(E)} > \sigma(m), \quad m \le \frac{E_0^2}{m_1'}. \tag{A13$'$}$$

Taking into account the simplest formulation [(A8′) and (A13′)], we see that we have a first condition giving $R \ge 0$ for $E > E_0$:

Condition B (theorem II):

$$\sigma(m) > \sigma(m_1)\left(1 - \frac{\ln b(m_1, E_0)}{\ln c_{\lambda=1}(E_0)}\right),$$

$$m_1'^2 \ge m_1^2 + 3m_0^2, \quad m \ge m' m_2/m_1'.$$

If we consider $p=2$, $r=2$, or $p=3$, $r=2$, or $p=2$, $r=3$, or $p=3$, $r=3$, ... we have the possibility of improving condition B as we want. The general *condition C* (theorem II) is the following:

(a) $X_0 + Y_0 > 0$, where X_0 and Y_0 are expressed by Eqs. (A8) and (A13), and

(b) $m_1'^2 \ge n_{p-1}^2 + 3m_0^2$, $m' \le m \le \lambda_r m' m_2/m_1'$, which ensures $R > 0$ for $E \ge E_0 = (m' m_2)^{1/2}$.

Finally, we would like to consider the case where $m_0 = 0$ in h [see Eq. (A1)]. It is straightforward to verify that all formulas (A2)-(A13′) hold, with $m_0 = 0$ the only change in these formulas. In particular, conditions B and C are deduced by putting $m_0 \equiv 0$ in the corresponding formulas.

APPENDIX B

In this appendix we want to prove the properties (i), (ii), (iii), and (iv) assumed in Appendix A and which were necessary in order to derive conditions B and C of theorem II. We recall that $m_0 \le m_1 \le m_1' \le E$.

(i) $B_1(E) = \frac{1}{2}(E^2 - m_0^2)^{1/2}\ln c_{\lambda=1}(E)$ *decreases when E increases*. Differentiating we get

$$\frac{B_1'(E^4 - m_0^2 m_1'^2)}{m_1'^2 E\beta^{-1/2}(E^2-m_0^2)^{1/2}} = \sum_{p=0}^{\infty}\frac{\beta^p}{2p+1}Z_p,$$

where β is $\beta_{\lambda=1}$ defined in (A5),

$$Z_0 = 2m_0^2(E^2 - m_1'^2) > 0,$$

$$Z_p = Z_0 - 2p(E^4 - 2m_0^2E^2 + m_0^2 m_1'^2),$$

$$Z_p = Z_1 - 2(p-1)[(E^2-m_0^2)^2 + m_0^2(m_1'^2 - m_0^2)],$$

$$Z_p < 0 \quad \text{for } p > 1 \text{ if } Z_1 < 0.$$

We put $m_1'^2 = 3m_0^2 + \lambda$ and get

$$Z_1 = -2m_1'^2(m_1'^2 - m_0^2) - 2(E^2 - m_1'^2)(E^2 + \lambda), \quad Z_1 < 0 \text{ if } \lambda > 0$$

$$(Z_0 + \tfrac{1}{3}\beta Z_1)(E^4 - m_0^2 m_1'^2) = -2(E^2 - m_1'^2)[(\lambda - m_0^2)(\tfrac{1}{3}E^4 + \tfrac{1}{3}m_1'^2E^2) + 2m_0^4 m_1'^2 + \tfrac{1}{3}m_0^2(E^4 - m_1'^4)] < 0 \quad \text{if } \lambda > m_0^2.$$

In conclusion

$$B_1' < 0 \quad \text{if } m_1'^2 > 4m_0^2. \tag{B1}$$

(ii) $B_2(E) = \ln b(n, E)/\ln c_{\lambda=1}(E)$, *where* $m_0 \le n \le m_1'$, *is an increasing function when E increases*. First let

$$\beta_{\lambda=1} - \alpha(x) = \left(\frac{m_1'^2 - x^2}{E^2 - m_0^2}\right) + m_0^2\frac{(E^2 - m_1'^2)}{(E^2-m_0^2)(E^4 - m_0^2 m_1'^2)} > 0 \quad \text{if } x \le m_1'.$$

Second, for simplicity let us denote $\beta_{\lambda=1}$, defined in (A5), as β and $\alpha(n)$, defined in (A3), as α. We get for the derivative

$$\frac{B_2'(\ln c_{\lambda=1})^2(\alpha\beta)^{1/2}(1-\alpha)(1-\beta)(E^2-m_0^2)(E^4-m_0^2m_1'^2)}{2E(n^2-m_0^2)m_1'^2}=\mu-1+2\sum_{p=1}\frac{(\beta^p-\mu\alpha^p)}{(2p-1)(2p+1)}\ ,$$

where

$$\mu-1=-\frac{2m_0^2(E^2-m_1'^2)}{E^4-m_0^2m_1'^2}<0$$

and

$$\beta-\alpha\mu=\frac{(E^2-m_0^2)^2(m_1'^2-n^2)+m_0^2[(E^2-m_0^2)^2-(m_1'^2-m_0^2)(n^2-m_0^2)]}{(E^4-m_0^2m_1'^2)(E^2-m_0^2)}>0\,.$$

Thus $\beta>\alpha\mu$, we have also $\beta>\alpha$, and it follows that $\beta^n-\alpha^n\mu>0$, $\forall n\geq 1$. It remains to add $\mu-1$ and the first term $p=1$ of the series and seek a condition in order that the sum be positive:

$$\begin{aligned}[\mu-1+\tfrac{2}{3}(\beta-\alpha\mu)]\,(E^4-m_0^2m_1'^2)(E^2-m_0^2)-\tfrac{2}{3}m_0^2[(E^2-m_0^2)^2-(m_1'^2-m_0^2)(n^2-m_0^2)]&\\ =2(E^2-m_0^2)[-m_0^2(E^2-m_1'^2)+\tfrac{2}{3}(m_1'^2-m^2)(E^2-m_0^2)]&\\ >2m_0^2(E^2-m_0^2)[E^2(\tfrac{1}{3}\gamma-1)+m_1'^2-\tfrac{1}{3}\gamma m_0^2]\quad \text{if } m_1'^2>n^2+\gamma m_0^2&\\ >0\quad\text{if } \gamma\geq 3\,.&\end{aligned}$$

In conclusion

$$B_2'>0\quad\text{if } m_1'^2>n^2+3m_0^2\,. \tag{B2}$$

(iii) $B_3(E)=\ln b(m_1',E)/\ln c_{\lambda=1}(E)<1$. This follows from the fact that $b(m_1')\geq 1$, $c_{\lambda=1}\geq 1$ and

$$\beta_{\lambda=1}-\alpha(m_1')=\frac{m_0^2(E^2-m_1'^2)}{(E^2-m_0^2)^2(E^4-m_0^2m_1'^2)}>0\,.$$

In conclusion $B_3<1$ without restriction.

(iv) $B_4(E)=\ln c_\lambda(E)/\ln c_{\lambda=1}(E)$ increases when E increases if $\lambda>1$. We recall that β_λ and $\beta_{\lambda=1}=\beta$ are defined in (A5). We get for the derivative

$$\frac{B_4'(\ln c_{\lambda=1}(E))^2(\beta\beta_\lambda)^{1/2}(1-\beta)(1-\beta_\lambda)(\lambda^2E^4-m_0^2m_1'^2)(E^4-m_0^2m_1'^2)}{2E(E^2-m_0^2)m_1'^2(E^4-2E^2m_0^2+m_0^2(m_1'/\lambda)^2)}=\mu-1+2\sum_{p=1}\frac{(\beta^p-\beta_\lambda{}^p\mu)}{(2p-1)(2p+1)}\ , \tag{B3}$$

where

$$\mu=\left(\frac{E^4-2E^2m_0^2+m_0^2m_1'^2}{E^4-2E^2m_0^2+m_0^2(m_1'/\lambda)^2}\right)\left(\frac{E^4-m_0^2(m_1'/\lambda)^2}{E^4-m_0^2m_1'^2}\right)>1\quad\text{for } \lambda>1\,,$$

$$\beta>\beta_\lambda\,,$$

and

$$\beta-\beta_\lambda\mu=\frac{(\lambda^2-1)(E^2-m_0^2)m_1'^2[(E^2-m_1'^2)E^2+E^2(m_1'^2-2m_0^2)]}{\lambda^2(E^4-m_0^2m_1'^2)}>0\quad\text{if } \lambda>1 \text{ and } m_1'^2>2m_0^2.$$

In conclusion,

$$B_4'>0\ \text{if } \lambda>1 \text{ and } m_1'^2\geq 2m_0^2\,. \tag{B4}$$

In conclusion we remark that all conditions (i)–(iv) of this appendix are satisfied if

$$m_1'^2\geq n_{p-1}{}^2+3m_0^2\,, \tag{B5}$$

where $n_q\leq n_{p-1}$ for $q\leq p-1$ and $n_{p-1}\leq m_1'$ is defined in (A2). If $p=1$; i.e., condition B of theorem II and appendix A holds, then $n_{p-1}=m_1$; if $p>1$; i.e., condition C of theorem II and appendix A holds, then n_{p-1} is the last intermediate value, the nearest to m_1' introduced in the interval $[m_1,m_1']$. Now what happens if $m_0=0$ in B_1, B_2, B_3, B_4? It is straightforward to verify that $B_1'<0$, $B_2'>0$, if $m_1'^2>n^2$, $B_3=1$, and $B_4'>0$.

*Work supported in part by the U. S. Atomic Energy Commission under Contract Number AT(11-1)-2232.

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