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Dual four-point functions with no negative residues

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The positivity of residues in a general class of dual four-point functions is studied. A large domain in parameter space [$\alpha(0)$, $\alpha'(0)$, trajectory curvature, external mass] is found in which there are no negative residues at any level. The ghost-free domain of $\alpha(0)$ expands as the positive curvature of the trajectory is increased. This suggests departure from linearity of trajectory as a way of breaking the $\alpha(0)=1$ no-ghost condition for the Veneziano N-point function. For a set of parameters which provides a good fit to the nondiffractive component of proton-proton elastic scattering, it is shown that the amplitude has no negative residues.

A serious difficulty with the Veneziano model[1] is the unrealistic constraint $\alpha(0)=1$ which follows from the requirement of no ghosts in the N-point function.[2] One possible approach to breaking this constraint is to study more general classes of dual models which incorporate extra degrees of freedom characterized by some new parameters. One such class of models[3-5] is already known in which the relevant new parameter is the trajectory curvature. The Veneziano model is the zero-curvature limit of this more general class.[3] The hope here is that the range of acceptable intercepts may be altered in a desirable direction as the curvature of the trajectory is changed. It has been known for some time that the negative-curvature generalization of the Veneziano model has the undesirable feature that ghosts are present for all values of $\alpha(0)$.[3] It appears that the ghost situation continuously worsens as the curvature becomes more negative. The question arises as to whether the ghost situation improves as the curvature turns positive. In the positive-curvature case,[4,5] the N-point function has not yet been found. However, we can examine the signs of the residues of the four-point function as a first step. This alone might be sufficient to rule out positive curvature as a useful degree of freedom. It turns out that the positive-curvature four-point function has *no* negative residue at *any* level in a sizeable domain of intercept. Therefore, the interest in finding the N-point functions is greatly enhanced.

We are also interested in four-point-function phenomenology in which external particles do not necessarily lie on the trajectory and N-point-function constraints are ignored. From this point of view, a four-point function would be acceptable if it has no negative residues, regardless of whether or not ghosts show up in the N-point function. In the Veneziano case it is very difficult to prove definitively that the four-point function has no negative residues[6] for any value of $\alpha(0)\neq 1$, and in phenomenological applications this requirement is usually ignored. However, in the positive-curvature case of the present model, we are able to specify the no-ghost domain. The proof breaks down in the Veneziano limit of zero curvature. Thus, we are able to determine whether or not a four-point-function fit to data satisfies the strict no-ghost requirement. Our analysis shows that the two phenomenologically successful dual four-point functions of Ref. 5 are completely ghost-free.

Further representative results of our analysis are summarized in Figs. 1–3. The presence or absence of ghosts is indicated on two-dimensional plots in which we vary $\alpha(0)$, the intercept, and q, the degree of nonlinearity of the trajectory. The linear trajectory of the usual Veneziano model corresponds to the right vertical edge of the plots where $q=1$. Figures 1–3 differ in the masses of the external particles. They correspond respectively to external pions, protons, and a spin-one particle lying on the trajectory. For convenience, we have set the slope $\alpha'(0)=1$ in all three plots. Each plot is divided into three main regions: region G, where ghosts exist; region NG, where all partial-wave coefficients of residues of all poles have been found to be positive; and a small region U in which we are able to show the nonexistence of negative partial-wave residues only up to some finite level. Much of the Veneziano limit falls in the third region. It is remarkable that except for this small region, in which technical difficulties are known to exist, a definitive conclusion can be made in all other regions of the parameter space. This lack of technical problems is directly related to the fact that the dual amplitude with nonlinear trajectories has an accumulation point of poles at a finite value of s. In the Veneziano limit, the accumulation point is at infinity.

From these plots, we see a general trend that the ghost-free domain of the intercept expands as the nonlinearity of the trajectory is increased (i.e., as we move away from $q = 1$). If this trend persists in the N-point function, there may be a hope of breaking the $\alpha(0) = 1$ constraint in the Veneziano model by resorting to nonlinear trajectories.

The four-point function under investigation has the form[4,5]

$$B_4(\sigma, \tau) = \tau^{\alpha(s)} \frac{G(q/\sigma\tau)}{G(q/\sigma)G(q/\tau)}\, G(q) \tag{1}$$

where

$$\tau^{\alpha(s)} = \sigma^{\alpha(t)} = q^{\alpha(s)\alpha(t)}\ , \qquad G(x) = \prod_{l=0}^{\infty} (1 - xq^l) \tag{2}$$

and q is a parameter which lies between 0 and 1. Poles in the s channel are located at

$$\sigma = -as + q^{\alpha(0)} = q^k, \quad k = 1, 2, \ldots \tag{3}$$

where

$$a = \alpha'(0) q^{\alpha(0)} \ln q^{-1} \geq 0.$$

The residue at the kth pole is a kth-order polynomial in $\tau = -at + q^{\alpha(0)}$ given by

$$R_k = q_k \prod_{n=1}^{k} (t + t_n)\ , \tag{4}$$

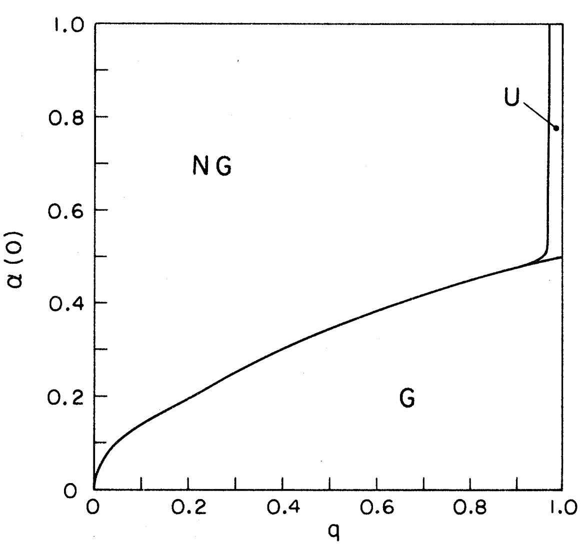

FIG. 1. A plot of the presence of negative residues in the dual four-point amplitude as a function of $\alpha(0)$ and q, the degree of nonlinearity of the trajectory. In region G, ghosts are found. In region NG, there is no negative residue at any level. In region U, there are no negative residues in the low-lying poles (up to about the 50th level). The situation about the higher poles is uncertain, $\alpha'(0) = 1$. External particles have the mass of pions.

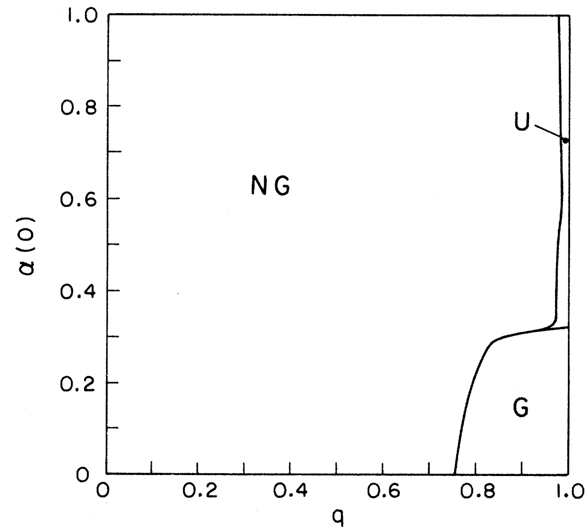

FIG. 2. Same as Fig. 1, except that the external particles have the mass of protons.

where

$$q_k = \frac{q^{k(k+1)/2} a^{k-1}}{\prod_{l=1}^{k-1} (1 - q^l)}$$

and

$$t_n = \frac{q^{-n+1} - q^{\alpha(0)}}{a}\ .$$

Our technique of analysis is based on the observation that at each pole in s of the dual amplitude all the zeros $(-t_n)$ of the polynomial residue lie on

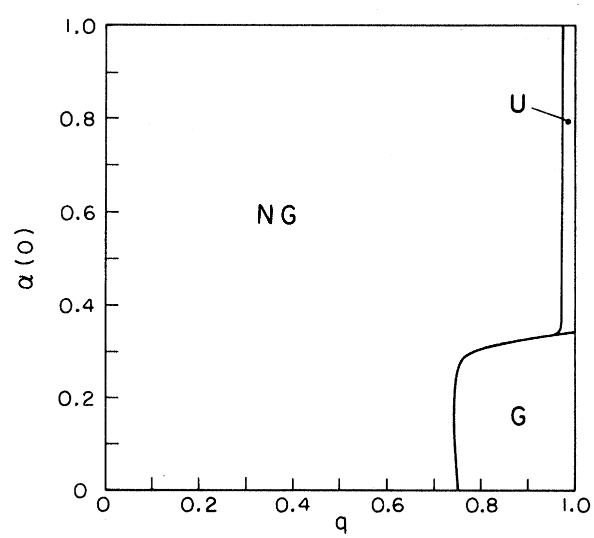

FIG. 3. Same as Fig. 1, except that the external particles correspond to spin-one particles on the trajectory.

the real axis in t. Since t is proportional to $x \equiv \cos\theta_t$, the polynomial expressed as a product of zeros in x would have a similar property. Thus, at the kth pole, the residue has the form

$$R_k = C(k) \prod_{n=1}^{k} (x + x_n) \ , \tag{5}$$

where all x_n's are real. Corresponding to each residue function R_k, we construct k "partial products"

$$Q_m = \prod_{n=1}^{m} (x + x_n) \ , \quad m = 1, 2, \ldots, k \ . \tag{6}$$

The kth partial product Q_k is obviously proportional to R_k. Now, we will make a partial-wave decomposition for *each* partial product

$$Q_m = \sum_{n=0}^{m} C_m^n (s_k - 4M^2)^n P_n(x) \ , \tag{7}$$

where s_k is the position of the kth pole, and M is the external mass. Q_m and C_m^n depend implicitly on k since

$$x_n = 1 + \frac{2t_n}{s_k - 4M^2} \ . \tag{8}$$

The partial-wave coefficients C_k^n of the highest partial product Q_k have direct physical significance, since they are proportional to the coupling constants. The reason for constructing the partial-wave decompositions of the lower partial products Q_m is that the resulting coefficients C_m^n are related to C_k^n by simple recursion relations. They therefore provide a framework for an easy calculational scheme.

To see this, we make use of the well-known recursion relation for Legendre polynomials

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \ . \tag{9}$$

Substituting Eq. (9) into Eqs. (7) and (6), we have the following relation for the coefficients C_m^n:

$$C_{m+1}^n = x_{m+1} C_m^n + a(n) C_m^{n+1} + b(n) C_m^{n-1} \ , \tag{10}$$

where

$$a(n) = \left(\frac{n}{2n-1}\right)\left(\frac{1}{s_k - 4M^2}\right) \ ,$$

$$b(n) = \left(\frac{n+1}{2n+3}\right) (s_k - 4M^2) \ ,$$

$$C_m^n \equiv 0 \quad \text{if } n > m \text{ or } n < 0 \ .$$

The physically relevant coefficients C_k^n may be calculated by direct iteration of Eq. (10), starting from $C_0^0 \equiv 1$. However, the real power of this equation lies with the simplicity of its algebraic structure which makes it possible in most cases to deduce the signs of C_k^n by performing only a few iterations. We will show this by considering separately the poles above and below the threshold.

Above threshold. Consider a pole at $s_k > 4M^2$. It is obvious that $a(n)$ and $b(n)$ are positive. $\{x_m\}$, the zeros of the dual polynomial residue, form a monotonically increasing series with a small number of negative elements if q is away from 1. Applying these properties of $a(n)$, $b(n)$, and x_m in Eq. (10), it is easy to see that if at some Mth iteration of Eq. (10) we find that (i) $x_{M+1} \geqslant 0$ and (ii) $C_M^n \geqslant 0$ for all n, then it is guaranteed that $C_k^n \geqslant 0$ for all n (i.e., no negative residues at the s_k pole). Our computer program was written to iterate Eq. (10), starting from $m = 0$, and stops at the first level which satisfies the above two conditions. As we scan our parameter space away from $q = 1$ region by the above method, we find that either a ghost appears at a low-lying level, or that the above conditions (i) and (ii) are satisfied at a rather low value of M for all poles. Thus, for $q \neq 1$ a finite number of iterations enables us to make a definite statement about any pole even if it has arbitrarily high spin. This would not have been possible by direct evaluation of the partial-wave coefficients because of numerical problems.

In practice, we use the above procedure explicitly only for a finite number of poles immediately above the threshold. However, we may prove that all higher-lying poles are also ghost-free by the following observation: The higher poles have an accumulation point (i.e., $\lim_{k\to\infty} s_k \to s_\infty < \infty$). Similarly, for a fixed value of m, the zeros x_m, which are implicitly dependent on k, also converge to some value x_m^∞ (i.e., $\lim_{k\to\infty} x_m^{(k)} \to x_m^\infty$). It follows from Eq. (10) that the partial-wave coefficients C_m^n also converge to some limiting value as k gets large. Thus, if we iterate Eq. (10) with coefficients corresponding to s_∞ and x_m^∞, and find that conditions (i) and (ii) are satisfied for some M, then we know that an infinite number of poles near the accumulation point are ghost-free. If, in addition, we use the previously mentioned procedure of checking for ghosts at every level up to a level sufficiently near the accumulation point, then we are able to make definitive statements about the existence of ghosts at any level. Only for q very near 1 does this computational procedure become unfeasible. The reason that the procedure fails in the Veneziano limit ($q \to 1$) is that the masses, the zeros, and hence the coefficients in the recursion relation do not approach a limit.

Below threshold. For poles below the threshold, i.e., $s_k < 4M^2$, there is no ghost as long as the intercept is greater than zero. To prove this, we first note that if we make a partial-wave expansion for the monomial t^k, there is no negative expansion coefficient. This is just the well-known fact that the Coulomb amplitude (which has residue t^k) has no ghost. This result may be proved directly by deriving a recursion relation analogous to Eq. (10) for the monomial t^k. Next, we observe that the residue at s_k may be written in the form $\sum_{i=0}^{k} d_i t^i$, where $d_i \geqslant 0$ for every i as long as the intercept is positive. It follows, therefore, that there are no ghosts below the threshold.

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Diffraction scattering and factorization below the asymptotic region

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Methods are described for calculating amplitudes involving the exchange of an "effective" Pomeron pole which do not depend on assumptions about the actual asymptotic behavior with energy of these amplitudes. It is shown that an "effective" Pomeron pole should (at least approximately) factorize, and should not decouple from inelastic channels even in the forward direction.

I. INTRODUCTION

This paper is devoted to the development of techniques for calculating the amplitudes for high-energy hadronic diffraction processes in a way which is independent of specific models. In particular, we will not make any assumption about the asymptotic behavior of these amplitudes with energy, so that our results will not depend upon, for example, the question of whether or not total cross sections are asymptotically constant.

A popular technique for dealing with high-energy amplitudes is to describe them in terms of singularities in the complex angular momentum plane. Indeed, it is well known that powerful results such as factorization and the various decoupling theorems follow from the assumption that the Pomeron is a simple pole in the J plane.[1] However, since it is also well known that this assumption leads to various inconsistencies, and since in any case experiments are never performed by using infinite energy, it has been found useful[2] to describe the Pomeron as an "effective" pole, that is, to observe that diffractive amplitudes at large but finite energy behave as if the leading singularity in the J plane were a simple pole, without making any commitment as to the actual asymptotic behavior with energy (equivalently, as to the actual nature of the J-plane singularity). One drawback of this approach is that one loses much of the calculational power of the J-plane method.[3] For example, although it may be reasonable to suppose that, since the residue of a J-plane pole would factorize, the residue of an "approximate" pole should factorize approximately, this supposition is not supported by J-plane considerations; it is quite simple to construct examples of an effective pole whose residues grossly violate factorization, which is composed of two closely spaced J-plane poles, each of which factorizes precisely.

In this paper we shall derive (after making several approximations) dynamical equations which high-energy diffraction amplitudes should satisfy, whose basic input is the assumption that elastic and quasielastic cross sections are reasonably constant over a large (but perhaps finite) range of energy; that is, we shall derive dynamical equations for effective poles. The derivation of these